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Abstract

To enact a policy, a leader needs votes from committee members with heterogeneous opposition intensities. She sequentially offers transfers in exchange for votes. The transfers are either promises paid only if the policy passes or paid up front. With transfer promises, a vote costs nearly zero. With up-front payments, a vote can cost significantly more than zero, but the leader is better off with up-front payments. The leader does not necessarily buy the votes of those least opposed. The opposition structure most challenging to the leader involves either a homogeneous committee or a committee with two homogenous groups. Our results provide an explanation for several empirical regularities: lobbying of strongly opposed legislators, the Tullock Paradox and expansion of the whip system in the U.S. House concurrent with ideological homogenization of parties. We also discuss several extensions including private histories and simultaneous offers.

JEL Classification: C78, D72, P16.

Keywords: vote buying; legislative bargaining; coalition building; endogenous sequencing; transfer promise; up-front payment; contracting with externalities.

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1 Introduction

It is common practice for political leaders to use side payments to gain support of specific voting members or voting blocks on general-interest legislation and public policy. As a well-known example, President Lyndon B. Johnson secured the critical support of Charles Halleck, Republican congressman of Indiana, in passing the Civil Rights Act by offering him a NASA research facility at Purdue University. Other examples abound.[1] Depending on the situation, the leader could be the president or some other executive, the party whip, or a lobbyist.[2] Similarly, the side payments could take the form of targeted district spending (pork), commitment to advance other legislation important to the member, valued committee assignments, campaign funds and so on.

We refer to this kind of practice as "vote buying" in this paper. We introduce a simple game-theoretical model to capture certain elements of this process – a resourceful leader strategically using transfers to gain support from various members – that we consider important. In our model, a leader, whose goal is to enact a new policy and needs q votes for it to pass, sequentially approaches members with varying degrees of opposition, making an offer to each member in exchange for his vote. By accepting an offer, a member commits to voting yes when the policy is up for a vote. Since the members are heterogeneous in their preferences, which members' votes should the leader buy? In what sequence should the leader approach them? How much do the votes cost and under what conditions will the policy get successfully enacted?

We address these questions in two variants that differ in terms of the nature of the offer. In the *transfer-promise* game, the offer is a promise to make a transfer if the policy passes in exchange for the member's vote. This is applicable when the final bill that the members vote on bundles the policy and the transfers together. In the *up-front-payment* game, the offer is an up-front payment not contingent on whether the policy passes. This is more appropriate when the offer is separate from the bill involving the policy, for example, it could be a president's offer to campaign on behalf of a congressional member.

In the transfer-promise game, we find that the policy passes if and only if the leader's gain is higher than the sum of the losses of the q members who are least opposed to the policy. Strikingly, whenever the leader is successful in getting the policy passed, the payments are close to zero in equilibrium. These payments are accepted because whenever a member is approached in equilibrium, he is "dispensable" in the sense that if he rejects the offer, the policy still passes (since there are other members whose votes

[1] On the passage of the Civil Rights Act of 1964, see Caro [2012, chapter 23]. Other examples include involvement of President Bill Clinton in the passage of NAFTA [Mayer 1998, chapter 8], and involvement of Dan Rostenkowski, the chair of the U.S. House Ways and Means Committee, in the passage of the Tax Reform Act of 1986 [Merriner 1999, chapter 8]. Evans [2004] studies the use of pork barrel projects for coalition building in the U.S. Congress.

[2] The whip is the official who ensures party discipline by, among other things, making deals with party members. For a comprehensive study of party whips in the U.S. political system, see Evans [2018].

the leader could still buy). This implies that his rejection only delays the passing of the policy, and therefore he accepts an offer close to zero when patient. Moreover, whenever the leader's gain is high enough such that the policy passes in equilibrium, there exists a sequence of members along which everyone is dispensable, resulting in the nearly costless passage of the policy.

Whose votes should the leader try to buy? One immediate response might be that she should buy the votes of those least opposed. This is largely but not always borne out in the transfer-promise game, and the order of vote buying could be important. Specifically, let us order the members by their intensity of opposition so that a higher index indicates higher intensity. Note that whenever the policy passes in equilibrium, any member with an index $(q+1)$ or higher is dispensable at the beginning of the game, but the qth member may or may not be. If the qth member is dispensable at the beginning of the game, then there exists a sequence of approaching the q members who are least opposed such that each is dispensable along the sequence, making it an optimal way of building a winning coalition. If the qth member is indispensable at the beginning of the game, however, this is no longer feasible. In this case, the leader must approach the $(q+1)$th member first. After buying his vote, all members become dispensable and the leader then approaches those $(q-1)$ members who are least opposed in an arbitrary order.

In the up-front-payment game, the structure of the equilibrium is different now that an accepted offer is sunk cost for the leader. The equilibrium now features two phases: in the first phase, each approached member is "indispensable" in the sense that if he rejects the offer, the policy does not pass and therefore he needs to be compensated fully for his loss (the temptation phase); but as soon as some member becomes dispensable, the equilibrium enters the second phase in which any approached member is dispensable until the leader secures enough votes. In this phase, the members are offered payments close to zero, just like in the transfer-promise game (the exploitation phase).[3]

The temptation phase features non-negligible payments, and is shaped by the leader's incentive to minimize the aggregate cost of votes. The basic tradeoff is that the temptation phase is longer when members included in that phase are less opposed to the policy and shorter when members included are more opposed. However, the included members can be tempted in an arbitrary order. Similarly to the transfer-promise game, the exploitation phase might start with the leader approaching a member strongly opposed to the policy because all the less opposed members are indispensable.

Even though the leader may end up paying certain members a significant amount when offers are up-front payments whereas she always pays a negligible amount when offers are transfer promises, the leader is better off if she can offer up-front payments instead of transfer promises. This is because the temptation phase with up-front payments is non-empty only when the policy does not pass with transfer promises. In this case up-front

[3]We borrow the labels of temptation phase and exploitation phase from Genicot and Ray [2006].

payments endow the leader with a commitment to pass the policy eventually, because the payments made during the temptation phase are sunk and hence ignored.

Our analysis provides a number of insights. First, the cost of a member's vote reflects not only his opposition to the policy but also the opposition of other members. Strong opposition makes a member's vote expensive; however, for strategic reasons, strong oppositions of other members make a member indispensable and therefore his vote expensive. These two effects have countervailing forces, and when the strategic effect dominates, the vote of a more opposed member is cheaper than the vote of a less opposed member. These observations explain findings from recent empirical work on lobbying that would otherwise be puzzling.[4] For example, Bertrand, Bombardini, and Trebbi [2014] find systematic evidence of lobbyists meeting with U.S. congressmen of opposite political affiliation. Similarly, You [2019], focusing on foreign lobbying on trade legislation in the U.S., finds that pro-trade lobbyists often approach strong opponents of free trade.

Second, we provide an explanation for the "Tullock Paradox", the observation that special interests buy political favors at costs far below their valuation of the favors [Tullock 1967, 1997, see Rasmusen and Ramseyer 1994 for a number of examples]. Similarly, recent empirical works on political campaigns [Ansolabehere, de Figueiredo, and Snyder 2003] and lobbying [de Figueiredo and Richter 2014] raise a question about "why is there so little money" in U.S. politics. Our model predicts the apparent paradox: in equilibrium the leader buys the votes necessary for the policy to pass at a cost far below her valuation of the policy.[5] A related point is that the cost of votes, reflecting strategic considerations, does not measure well the strength of opposition against a policy. Hence, it would be a mistake to infer from an observed low cost of votes a lack of opposition.

Third, because the structure of opposition mainly determines whether the policy passes or not and to a much less extent the cost, those capable of passing policies irrespective of the cost are usually considered successful political leaders. This lends support to a standard way to measure legislative capacity by counting the number of enacted laws [Binder 2015]. Moreover, our model predicts that opposition structures most challenging to the leader are either those with homogeneous members or those with two homogeneous groups, one strongly opposed and one unopposed. These predictions allow us to link two concurrent changes in the U.S. House of Representatives in the last few decades: first, the ideological homogenization of the parties and emergence of intra-party factions; second,

[4]Not all lobbying is un-opposed as in our model but a significant fraction is. For example: Baumgartner and Leech [2001] find that lobbying is one-sided on about one fifth of the issues they analyze, Leaver and Makris [2006] provide several examples, and Kim [2017] finds that the median number of lobbies active on trade bills introduced in the U.S. Congress between 1999-2014 is one. Kim and Huneeus [2019]; Kim and Kunisky [2020] provide similar large-sample evidence.

[5]This result is related but distinct from the one in Dal Bó [2007], who shows that policies can be bought cheaply by leaders using "pivotal bribes", payments contingent on voting profiles rewarding players who cast decisive votes. In our model similar result arises although the offers take a simpler form and pivotal bribes are not possible.

the significant expansion of the whip system.[6] We can view the whip expansion partly as a response to an increasing demand for party discipline, driven by the changes in the opposition structures in the House.

Fourth, our model shows that the leader prefers up-front payments to transfer promises. An implication of this finding is that majority/minority status of a party in a legislature, which determines its access to transfer promises (side payments bundled together with a legislation), should not have a big impact on the ability of its whip system to induce party discipline, which is what we observe in the U.S. House [Meinke 2016, page 173]. We can also draw lessons in terms of when transfer promises are most likely to be observed by noting that they may have other benefits that are not included in our model. One potentially important benefit is "explainability": as part of the same bill, transfer promises enhance a lawmaker's credibility in explaining a vote contrary to the preferences of his district constituents.[7] Transfer promises that enhance explainability tend to relate directly to the substantive issue of the bill at hand, and therefore we should expect them to be used in bills that have significant distributive implications, such as appropriation, transportation, or tax bills and trade agreements. This provides an explanation for why pork barrel spending is often found in these but not other kinds of bills [Evans 2004].

We should emphasize that even though we have framed our model in the political economy context, it has a diverse set of applications in other areas such as industrial organization and corporate finance. For example, it can be adapted to understand issues in exclusionary contracts or corporate takeovers with heterogeneous buyers or shareholders.[8]

Related literature. Our paper contributes to three literatures. The first studies *vote buying*.[9] Most of this literature builds on Groseclose and Snyder [1996] and Dekel, Jackson, and Wolinsky [2008] who study a model with two vote buyers who move sequentially, making simultaneous offers to all vote sellers, who then simultaneously decide whether to sell their votes.[10] These papers focus on the strategic interaction between the compet-

[6]McCarty, Poole, and Rosenthal [2006, chapter 2] document ideological homogenization of parties in the U.S. Congress from early 1980s. The same trend still holds when the ideological positions are estimated free of party influence [Canen, Kendall, and Trebbi 2020]. Well-known intra-party factions in the House during this period include the Tuesday Group or the Tea Party Republicans and the Blue Dog Democrats [Pearson 2015, chapter 2]. Meinke [2008] documents expansion of the House whip system.

[7]See Evans [2018, chapter 1] for a discussion of "explainability" and how it played an important role when, for example, Republican leadership used side payments to secure the support of U.S. congressman Robin Hayes to grant fast track trade promotion authority to President Bush.

[8]See Whinston [2006] for a discussion on the importance of incorporating heterogenous buyers into antitrust economics that studies vertical contracts.

[9]Vote buying belongs to a larger literature on special interest politics [Grossman and Helpman 2001]. Part of this literature models lobbying differently than we do: as a menu auction [Bernheim and Whinston 1986; Denzau and Munger 1986; Grossman and Helpman 1994; Helpman and Persson 2001; Le Breton and Salanie 2003], or as a first-price auction [Duggan 2018], or as a stochastic Colonel Blotto game [Myerson 1993]. Several vote buying papers have different focus: Snyder [1991] is a single vote buyer version of Groseclose and Snyder [1996], Baron [2006] studies lobbying of agenda-setters, and several papers contrast lobbying under open and secret ballots [Felgenhauer and Grüner 2008; Seidmann 2011; Morgan and Vardy 2012].

[10]Diermeier and Myerson [1999]; Banks [2000]; Dekel, Jackson, and Wolinsky [2009]; Le Breton and

ing vote buyers and any strategic interaction between the vote sellers is absent.[11] The coalition of vote sellers that receives substantial payments in equilibrium is larger than minimal-winning composed of both supporters and opponents of the policy, in the former paper, and is minimal-winning composed of weak opponents of the policy, in the latter paper, when outcome-contingent transfers are considered.[12]

Our model complements the existing vote-buying theories by studying a model with a single vote buyer – providing insights about vote buying by powerful executives, whips and lobbies when unopposed – and by bringing out the strategic interactions among the vote sellers and between the vote sellers and the buyer.[13] The results are complementary as well: we predict that minimal-winning coalitions receive transfers and may include strongly opposed vote sellers when their endogenous bargaining positions are weak. We also answer questions about whose votes are bought and in what sequence, that were not addressed in the previous models.

The second literature is *multi-agent contracting with externalities*. A typical application considers an incumbent firm trying to sign exclusionary contracts with buyers in order to prevent entry by its competitors. Unlike our paper, most of this literature either does not consider sequential contracting [Bernheim and Whinston 1998; Segal 1999, 2003; Bernstein and Winter 2012] or assumes homogeneous buyers [Rasmusen, Ramseyer, and Wiley 1991; Rasmusen and Ramseyer 1994; Segal and Whinston 2000; Fumagalli and Motta 2006; Genicot and Ray 2006; Chen and Shaffer 2014; Iaryczower and Oliveros 2017, 2019].[14] Two exceptions are Möller [2007] and Galasso [2008], but they restrict attention to sequential contracting of two buyers.

Among these papers, Genicot and Ray [2006] is the most closely related. In their model, a principal offers up-front payments to homogenous agents, whose reservation payoffs are increasing in the number of uncontracted agents. Similar to their characterization, the equilibrium in our up-front-payment game also involves two phases: temptation followed by exploitation. Relative to Genicot and Ray [2006], there are two main

Zaporozhets [2010]; Morgan and Vardy [2011, 2012]; Le Breton, Sudholter, and Zaporozhets [2012] all analyze similar models.

[11]Vote sellers in Groseclose and Snyder [1996] derive utility from casting votes, not from policy outcomes, which makes their optimal action independent of the other vote sellers' behavior. Vote sellers in Dekel et al. [2008] are "not modeled as players" (page 354).

[12]With vote-contingent transfers, the equilibria in Dekel et al. [2008] feature nearly zero transfers and make no prediction about which vote sellers are approached in equilibrium. The prediction that only a minimal-winning coalition of weakly opposed vote sellers receives transfers also arises in the special case of Groseclose and Snyder [1996] with a single lobby, as well as in the repeated sequential lobbying extension of Groseclose and Snyder [1996] analyzed by Dekel et al. [2009].

[13]Strategic interaction between vote sellers is important in Neeman [1999]; Dal Bó [2007] and Louis-Sidois and Musolff [2019], but they do not address the question of endogenous sequencing, in Rasmusen and Ramseyer [1994], which is an early application of the literature discussed next, and in Spenkuch, Montagnes, and Magleby [2018], which includes an extension of Dekel et al. [2009] with vote buyers approaching vote sellers sequentially in a predetermined order, and shows that the main predictions of Dekel et al. [2009] continue to hold.

[14]See Whinston [2006] and Rey and Tirole [2007] for surveys of this literature.

departures that allow us to address a new set of questions. First, we introduce member heterogeneity, allowing us to characterize the phase and order in which they are approached; it also allows us to answer questions such as how the committee composition affects the passage of the policy. Second, we consider both transfer promises and up-front payments, allowing us to characterize how the nature of transfers affects the leader's ability to pass policy and the associated costs. We differ from Genicot and Ray [2006] in two other aspects. First, they allow the principal to make simultaneous offers. We discuss the new issues that arise with simultaneous offers and some answers towards addressing them in Section 6.2. Second, in one version of their model, the principal can re-approach agents. We explain what happens if re-approaching is allowed in footnote 19.[15]

The third literature is *bargaining with endogenous sequencing*. A typical model in this literature studies a situation in which one player bargains with other n players sequentially. For tractability, this literature often either works with a set of exogenously given sequences [Horn and Wolinsky 1988; Stole and Zwiebel 1996] or restricts attention to the case of $n = 2$ [Marshall and Merlo 2004; Menezes and Pitchford 2004; Noe and Wang 2004; Marx and Shaffer 2007; Bedre-Defolie 2012; Krasteva and Yildirim 2012a,b, 2019; Göller and Hewer 2015]. Cai [2000, 2003], Li [2010] and Xiao [2018] allow for the bargaining sequence to arise endogenously but focus on unanimity.

2 Model

A leader wants to pass a new policy. She sequentially approaches the members of a committee. With each approached member, she tries to reach a bilateral agreement, offering a transfer in return for the member's support. All actions are observable.

Formally, the game is played by the leader and a set of committee members $N = \{1, \dots, n\}$, where $n \geq 1$. Passing the policy requires $q \in \{1, \dots, n\}$ votes from the committee members. Each player's payoff from the status quo is normalized to be 0. The payoff from the new policy is $y > 0$ for the leader and $-x_i$ for each member $i \in N$. We assume that $x_i > 0$ for each $i \in N$ and index the committee members such that $x_i \leq x_{i+1}$, so a member with a higher index is more strongly opposed to the policy.[16]

The leader approaches the committee members in consecutive periods. Suppose that at the beginning of a period, the set of un-approached members is U and the number of approached members who have accepted the offers is n_a. The leader can choose to *approach* a member in U, or *initiate a vote* or *stop*. If the leader decides to approach

[15] In our vote-buying model, a member's decision to sell his vote can have an effect on the payoff of all other players, including those members who already sold their votes whereas in Genicot and Ray [2006], an agent's contracting decision does not affect the payoff of those agents who have already been contracted. In this sense, Genicot and Ray [2006] cannot be directly mapped into a vote-buying model.

[16] Assuming that $x_i > 0$ is without loss of generality since any member with $x_i \leq 0$ prefers the new policy to the status quo and can thus be ignored in the analysis.

a member $i \in U$, then she offers him a non-negative transfer in exchange for his vote. Member i either accepts the offer, thus giving the leader control of his vote, or rejects the offer, and the game proceeds to the next period. If the leader decides to initiate a vote, the policy passes if $n_a \geq q$ and the status quo is maintained if $n_a < q$, and the game ends. If the leader decides to stop, the policy does not pass and the game ends. We consider two variants of this game which differ in terms of the nature of the leader's offer.

(1) Transfer promises: the offer is a promise to make a transfer if and when the policy passes in exchange for the member's vote.

(2) Up-front payments: the offer is an up-front payment in exchange for the member's vote.

In the transfer-promise game, the payment of a transfer is contingent on the policy passing. This applies to situations, for example, in which the final bill that the members vote on bundles the policy and the transfers together. Note that transfer promises are *non-sunk cost* to the leader as the bargaining process unfolds.

In the up-front-payment game, the payment is not contingent on whether the policy passes. This is more applicable to situations in which transfers are separate from the bill involving the policy. Since the transfers are made irrespective of whether or not the policy passes, the payment is *sunk cost* to the leader.

In either game, we assume that the leader has a sufficiently large budget, specifically, larger than y. We discuss the implications of a smaller budget after Proposition 2.

To describe the players' payoffs at the terminal nodes, let N_a be the set of members who accepted the leader's offers and for each $i \in N_a$, let τ_i be the period in which the offer t_i was accepted. If a vote is held, let τ denote the period. We assume that the players have a common discount factor $\delta \in (0, 1)$.

Consider the transfer-promise game. If no vote is held or a vote is held and the policy does not pass, each player receives a payoff of 0. If the policy passes, the leader receives a payoff of $\delta^{\tau-1}(y - \sum_{i \in N_a} t_i)$, and member i receives a payoff of $\delta^{\tau-1}(t_i - x_i)$ if $i \in N_a$ and $\delta^{\tau-1}(-x_i)$ if $i \notin N_a$.

In the up-front-payment game, if no vote is held or a vote is held and the policy does not pass, the leader receives a payoff of $-\sum_{i \in N_a} \delta^{\tau_i-1} t_i$, and member i receives a payoff of $\delta^{\tau_i-1} t_i$ if $i \in N_a$ and 0 if $i \notin N_a$. If the policy passes, the leader receives a payoff of $\delta^{\tau-1} y - \sum_{i \in N_a} \delta^{\tau_i-1} t_i$, and member i receives a payoff of $-\delta^{\tau-1} x_i + \delta^{\tau_i-1} t_i$ if $i \in N_a$ and $-\delta^{\tau-1} x_i$ if $i \notin N_a$.

In the transfer-promise game, the transfers are paid when the policy passes and hence the payoffs from the policy and transfers are discounted by the same factor. In contrast, in the up-front-payment game, the transfers are paid immediately upon the acceptance while the passage of the policy happens only at the end of the game and hence the payoffs from the policy and from the transfers are discounted by different factors.[17]

[17]We assume that accepted transfers are paid immediately instead of at the end of the game in the

Histories are public and record the identities of the approached members, the transfers offered and members' acceptance decisions. Strategies are maps from histories to available actions. The solution concept we use is subgame perfect equilibrium (in which a member who is indifferent between accepting and rejecting accepts).[18] For the rest of the paper, we simply use the term equilibrium to refer to this solution concept.

In Section 6, we relax some of the assumptions made in the model presented here. We show that some of the results still hold and discuss how some change when histories are not perfectly observable. Similarly, instead of assuming that members care only about policy outcomes (and transfers if they receive any), we discuss what happens if they care about how they cast their votes. We also consider an extension allowing the leader to approach more than one member in each period. Other assumptions, for example, that the leader has all the bargaining power and cannot re-approach members, have been relaxed in related work.[19]

We introduce the notion of states to facilitate the analysis. Recall that a history at the beginning of a period records the set of members who have been previously approached, the transfers offered to them, and the members' acceptance decisions. For the transfer-promise game, a state is (S, r, t), where $S \subseteq N$, $S \neq \varnothing$, $r \in \{1, \ldots, \min\{q, |S|\}\}$ and $t \geq 0$, corresponding to a set of histories such that the set of members who have been previously approached is $N \setminus S$ (and hence the set of un-approached members is S), the number of members who have sold their votes to the leader is $q - r$ (and hence the leader still needs r votes for the policy to pass) and the sum of the promised transfers to these members is t. For the up-front-payment game, a state (S, r) excludes the accepted and hence paid transfers.[20] Note that given any state and any two histories inducing

up-front payment model. This modeling choice make the analysis simpler for the following reason. We introduce a notion of *state* at the end of this section, and having the transfer paid immediately implies that paid transfers need not enter the state since they are irrelevant for the continuation game whereas having the transfers paid at the end of the game implies that paid transfers need to be part of the state since how much they cost the leader now depends on when the game ends.

[18]This refinement restricts the behavior of an indifferent member who is not approached on the equilibrium path. It is not needed in some well-known bargaining games, e.g., the ultimatum game, because the responder is always approached on the equilibrium path. The refinement does not change the set of equilibria that are observationally equivalent because for any subgame perfect equilibrium one can construct an outcome equivalent subgame perfect equilibrium in which indifferent members accept.

[19]Iaryczower and Oliveros [2019] study how the allocation of bargaining power affects contractual outcomes between a principal and multiple agents. Endowing agents with bargaining power leads to a hold-up: the agents approached late extract surplus from the principal and this lowers the principal's incentive to invest in contracting with agents early. In our model endowing members with bargaining power would make the conditions for the passage of policy more stringent. Segal and Whinston [2000] and Genicot and Ray [2006] allow for re-approaching in models of contracting with externalities. Re-approaching has offsetting effects on the cost that the principal incurs, depending on whether the horizon is finite or infinite; it makes agents more demanding because the principal is no longer committed to no renegotiation, and it makes agents less demanding because it intensifies competition among them. In Segal and Whinston [2000] the latter effect dominates and re-approaching benefits the leader, while it has opposite effect in Genicot and Ray [2006]. We conjecture that similar effects would arise in our model if re-approaching was allowed.

[20]States where $r > |S|$ or $r \leq 0$ can arise in the extensive form but are trivial to analyze and hence

that state, the subgames following the two histories are identical and hence have the same set of equilibria. Let $\Gamma(S, r, t)$ denote a subgame starting with state (S, r, t) in the transfer-promise game and $\Gamma(S, r)$ denote a subgame starting with state (S, r) in the up-front-payment game. Then the entire game is $\Gamma(N, q, 0)$ in the transfer-promise game and $\Gamma(N, q)$ in the up-front-payment game.

3 Transfer promises

We begin by studying the model in which the leader offers transfer promises.

3.1 When does the policy pass?

Consider a subgame $\Gamma(S, r, t)$. Proposition 1 below says that whether the policy passes in equilibrium depends on how the leader's gain (net of the transfer already accepted) compares with the sum of the losses of the r members in S least opposed to the policy. Applying this result to the whole game, we immediately have that whether the policy passes in equilibrium depends on whether the leader's gain is strictly higher than the sum of the losses of the q members least opposed to the policy.

Given $S \subseteq N$ and $1 \leq r \leq |S|$, let $S^r \subseteq S$ denote the set of the r members in S who have the lowest losses.

Proposition 1. *Suppose the leader offers transfer promises. Consider a subgame* $\Gamma(S, r, t)$*. In any equilibrium,* (a) *if* $y - t > \sum_{i \in S^r} x_i$*, the policy passes, and* (b) *if* $y - t < \sum_{i \in S^r} x_i$*, the policy does not pass.*

To gain some intuition for part (a), note that a member i accepts an offer greater than his loss. Hence, when the leader needs r votes, if her gain (net of the offers already accepted) is larger than the sum of r least opposed members' losses, she can guarantee a positive payoff by approaching these r members and making each an offer that just compensates for his loss. Since the leader's payoff is 0 when the policy does not pass, she is better off if she buys these votes and therefore the policy passes in any equilibrium.

For part (b), if $|S| = r$ (unanimity), then each member accepts an offer only if it at least compensates for his loss. Hence, if the leader's gain (net of the offers already accepted) is lower than the sum of r least opposed members' losses, then she can only receive a negative payoff by getting the policy passed, whereas she receives a payoff of 0 if the policy does not pass. Given that she can choose to stop, the policy does not pass. If $|S| = r + 1$, then the member approached first will accept an offer only if it at least compensates for his loss since without his vote, the policy will not pass in the continuation game. Since every member can reason like this and thus demands an offer

disregarded until we deal with them in the proofs.

that makes him at least even, the leader cannot buy enough votes without making offers that exceed her gain from the policy. This induction argument shows that the policy does not pass in any equilibrium.

3.2 Equilibrium sequencing with transfer promises

Proposition 1 establishes that if $y > \sum_{i=1}^{q} x_i$, then the policy passes in any equilibrium. In what sequence should the leader approach the members and what offers does she make them? The answer is immediate under unanimity ($q = n$): the leader makes each member i an offer x_i and the sequence of approaching does not matter. In what follows, we consider nonunanimity ($q < n$). It is useful to introduce the notion of "(in)dispensability".

Definition 1. *Suppose the leader offers transfer promises. Consider state (S, r, t) where $r < |S|$. (a) We say that member $i \in S$ is indispensable in (S, r, t) if $\sum_{j \in S^r} x_j < y - t < \sum_{j \in S^r_{-i}} x_j$. (b) We say that member $i \in S$ is dispensable in (S, r, t) if $y - t > \sum_{j \in S^r_{-i}} x_j$.*

Intuitively, a member is indispensable in a state if the policy does not pass in equilibrium without the leader securing his vote whereas a member is dispensable in a state if the policy still passes in equilibrium even without the leader securing his vote. A member's strategic position is stronger when indispensable than when dispensable. When indispensable, he accepts an offer only if it at least compensates for his loss since by rejecting it, the policy will fail to pass. When dispensable, however, the policy still passes even if he rejects the offer. Since his rejection only delays the passage, he is willing to accept an offer that just compensates him for a sooner passage ($(1 - \delta)x_i$ to member i). When δ is sufficiently high, a dispensable member is willing to accept an offer close to 0, lower than any offer needed to secure an indispensable member's vote, irrespective of his loss.

Let A^d denote the set of members who are dispensable at the beginning of the game. Suppose that the policy passes in equilibrium, that is, $y > \sum_{i=1}^{q} x_i$. Note that any member in $\{q+1, q+2, \ldots, n\}$ is in A^d. As Proposition 2 below shows, what members' votes the leader buys depends on whether the qth member is in A^d. For expositional convenience, the proposition assumes that the members' losses are all distinct, and we discuss the case when some members' losses coincide in the text that follows the proposition.

If the qth member is in A^d, then it is optimal for the leader to approach the members with the lowest q losses. She starts by approaching a member in $A^d \cap \{1, \ldots, q\}$ since he is dispensable; after buying this vote, every remaining member $i \in \{1, \ldots, q\}$ becomes dispensable and therefore will accept an offer $(1 - \delta)x_i$, and the leader approaches them in an arbitrary sequence. Since the members being approached in this sequence have the lowest losses among all, the leader cannot improve her payoff by approaching others.

If the qth member is not in A^d, then any member with a lower loss is also not in A^d. Hence, if the leader starts by approaching any of them, she has to make an offer equal to the loss. However, since member $(q+1)$ is in A^d, she only needs to offer $(1-\delta)x_{q+1}$. Moreover, after securing this member's vote with an offer close to 0, any member in $\{1,\ldots,q\}$ becomes dispensable in the continuation game, and therefore it is optimal for the leader to approach the members with the lowest $(q-1)$ losses in the continuation game in an arbitrary sequence. The following proposition formalizes the results.

Proposition 2. *In the transfer-promise game, suppose $q<n$, $y>\sum_{i=1}^{q} x_i$, and $x_i<x_{i+1}$ for all i. There exists $\bar{\delta}<1$ such that for $\delta>\bar{\delta}$ the following results hold.*
(a) If $y>\sum_{i=1}^{q-1} x_i + x_{q+1}$, that is, member q is dispensable at the beginning of the game, then in any equilibrium, the leader starts by approaching a member in $A^d \cap \{1,\ldots,q\}$ and then approaches the remaining members in $\{1,\ldots,q\}$ in an arbitrary order; when she approaches member i, she offers $t_i=(1-\delta)x_i$ and it is accepted.
(b) If $y<\sum_{i=1}^{q-1} x_i + x_{q+1}$, that is, member q is indispensable at the beginning of the game, then in any equilibrium, the leader starts by approaching member $q+1$ and then approaches members in $\{1,\ldots,q-1\}$ in an arbitrary order; when she approaches member i, she offers $t_i=(1-\delta)x_i$ and it is accepted.

Interestingly, under non-unanimity rule, when the leader's gain is below the sum of the lowest q members' losses, the policy does not pass, but when it is above it, then the policy passes with minimal cost. Intuitively, when the leader's gain from the policy is sufficiently high, she is able to endogenously create a sequence of members along which each is dispensable. Note that the budget plays a role here too: even though the equilibrium payment is close to 0, if the budget was below the sum of the lowest q losses, then the implicit threat that makes a member dispensable would not be credible and the equilibrium that we have found would unravel.

When some members' losses coincide, we can extend Proposition 2 as follows: the set of members who are approached in equilibrium has to minimize the sum of the transfers among all sets of q members that include at least one dispensable one at the beginning of the game.

Propositions 1 and 2 also imply some straightforward comparative statics. The threshold for the policy passing as well as the cost of buying the votes decrease when the leader faces a larger committee, or needs to buy fewer votes, or when the members are less opposed, or when she has a higher gain from the policy. The leader is better off when any of these happens. Interestingly, lowering the size of the committee benefits the leader if combined with the same decrease in the number of votes required for the policy passing.

4 Up-front payments

We now turn to the model in which the leader offers an up-front payment in exchange for a member's vote. Under unanimity, since every member i has veto right, he accepts an offer if and only if it compensates for his loss x_i, appropriately discounted.[21] Another special case is when the leader needs only one vote for the policy to pass. Once she buys one vote, she initiates voting immediately, and whether the offer is an up-front payment or a transfer promise does not matter. In contrast, when the leader needs more than one vote for the policy to pass, the nature of the offer has important implications, which we illustrate by the following example.

Example 1. *In the up-front-payment game, suppose $n = 3$ and $q = 2$. When either $y > x_1 + x_2$ or $y < x_2$, the equilibrium outcomes mirror those with transfer promises: in the former case the policy passes at a cost near 0 and in the latter case the policy does not pass.*

Now consider $x_2 < y < x_1 + x_2$. If member i is approached first, he is willing to accept an offer if and only if $t_i \geq \delta^2 x_i$. To see this, note that if member i rejects the offer, then the policy will fail to pass since the leader would need to buy each remaining member's vote, which is too costly given that $y < x_1 + x_2$. But after securing member 1's vote by offering him $t_1 = \delta^2 x_1$ (a "temptation" offer), now the leader can buy member 3's vote by offering him $\delta(1 - \delta)x_3$ (an "exploitation" offer). This is because even if member 3 rejects the offer, the policy still passes (with the leader approaching member 2). Since the leader can buy enough votes at a cost lower than y when δ is sufficiently high, the policy passes in equilibrium.

Note that the same strategy would not work if the offers were transfer promises: since they are non-sunk cost, after securing member 1's vote with a temptation offer, the leader's willingness to pay goes below x_2, implying that if member 3 (or member 2) rejects an offer, the policy will not pass. Hence, member 3 (or member 2) will not accept an exploitation offer, and the whole argument unravels.

4.1 When does the policy pass?

We show in the next proposition that given a state (S, r), if the players are sufficiently patient, the policy passes if and only if the leader's gain y is above a threshold $W(S, r)$, defined recursively as follows. For any $S \subseteq N$, $\max S$ is the member in S with the highest loss and let $S' = S \setminus \{\max S\}$ ($\varnothing' = \varnothing$ by convention). Let

$$W(S, r) = \min_{T \in 2^S} \max \left\{ \sum_{i \in T} x_i, W((S \setminus T)', r - |T|) \right\}. \tag{1}$$

Proposition 3. *Suppose the leader offers up-front payments. Consider a subgame $\Gamma(S, r)$. For generic y, there exists $\bar{\bar{\delta}} < 1$ such that for $\delta > \bar{\bar{\delta}}$, in any equilibrium, the policy passes*

[21] Specifically, in any state (S, r) such that $|S| = r$, member i accepts t_i if and only if $t_i \geq \delta^r x_i$. It follows that the leader's payoff is $\delta^r(y - \sum_{i \in S} x_i)$ if the policy passes, and therefore the policy passes in equilibrium if $y > \sum_{i \in S} x_i$.

if $y > W(S,r)$ and the policy does not pass if $y < W(S,r)$.

To understand why $W(S,r)$ is the threshold that determines whether the policy passes in equilibrium, we classify members in terms of their bargaining positions given a state.

Definition 2. *Suppose the leader offers up-front payments. Consider state (S,r). (a) We say that member $i \in S$ is indispensable in (S,r) if $W(S \setminus \{i\}, r-1) < y < W(S \setminus \{i\}, r)$. (b) We say that member $i \in S$ is dispensable in (S,r) if $y > W(S \setminus \{i\}, r)$.*

The definitions of dispensability and indispensability here parallel those in the transfer-promise game. As implied by Proposition 3, a member is indispensable in a state if the policy does not pass without the leader securing his vote, whereas a member is dispensable in a state if the policy still passes even without the leader securing his vote. As before, when a member is indispensable, he has a strong bargaining position and thus accepts an offer if and only if it at least compensates for his loss (with the appropriate discounting), but when a member is dispensable, he has a weak bargaining position and is therefore willing to accept an offer that just compensates him for a sooner passage of the policy. We refer to these two distinct kinds of offers as temptation and exploitation offers, formalized as follows.

Definition 3. *Suppose the leader offers up-front payments. Fix a strategy profile and consider the resulting sequence of approached members. Suppose member i is offered t_i in state (S,r). Member i is tempted if $t_i = \delta^r x_i$ and is exploited if $t_i = \delta^r(1-\delta)x_i$. A profile is in a temptation phase if the approached member is tempted and is in an exploitation phase if the approached member is exploited.*

In any equilibrium of the up-front-payment game, if member i is approached in a state in which he is indispensable, then he is tempted, and if member i is approached in a state in which he is dispensable, then he is exploited. The next result is key in establishing $W(S,r)$ as the threshold that determines whether a policy passes in equilibrium.

Lemma 1. *Suppose the leader offers up-front payments. Consider state (S,r). If there exists a member $i \in S$ who is dispensable, then (a) member $\max S$ is dispensable in (S,r); (b) any member in $S \setminus \{i\}$ is dispensable in state $(S \setminus \{i\}, r-1)$, which implies that starting in state (S,r), there exists a sequence of r members along which each member is dispensable.*

Lemma 1 implies that once an exploitation phase starts, it remains in that phase until the leader buys all the votes she needs. Moreover, since the payment made to an indispensable member equals his loss and the payment made to a dispensable members equals zero (in the limit as δ goes to 1), the total payment that the leader makes equals the sum of the losses of the members approached in the temptation phase. Thus, the policy passes in equilibrium when y exceeds $W(S,r)$: $y > W(S,r)$ implies that $y > \sum_{i\in T} x_i$ and $y > W((S\setminus T)', r-|T|)$ for some T. The former inequality implies that there exists a set of members who can be tempted at a total payment below y. The latter inequality implies that after the leader buys the votes of those in T, that is, in state $(S\setminus T, r-|T|)$, the member with the largest loss is dispensable and the exploitation phase starts.[22]

4.2 Equilibrium sequencing with up-front payments

Our analysis so far shows that with up-front payments, equilibrium has two phases: temptation followed by exploitation. To determine which members are tempted in equilibrium, consider the following problem for any state (S,r):

$$\Pi(S,r,y) = \min_{T\in 2^S} \sum_{i\in T} x_i \text{ s.t. } y > W((S\setminus T)', r-|T|). \tag{2}$$

The constraint ensures that after the leader buys the votes of members in T, there exists one member in the remaining set who is dispensable. As discussed above, the leader's payments to members who are dispensable are 0 in the limit and therefore she is only concerned about her payment to members in T, which equals the sum of their losses. Hence, the least costly way to buy the necessary votes involves tempting members in T that solves (2). It also follows that the payment that the leader makes in equilibrium is the value of the problem, that is, equals $\Pi(S,r,y)$. Proposition 4 characterizes equilibria in which the policy passes.

Proposition 4. *In the up-front-payment game, suppose $y > W(N,q)$. For generic y and any $\varepsilon > 0$, there exists $\bar{\delta} < 1$ such that for any $\delta > \bar{\delta}$, for any equilibrium, the following results hold.*
(a) The equilibrium consists of two phases (with one possibly empty), a temptation phase followed by an exploitation phase. Let T and E be the sets of members included in the two phases respectively.
(b) The leader's equilibrium payoff is within ε of $y - \Pi(N,q,y)$. Each approached member i is made (with appropriate discounting) a temptation offer $t_i = x_i$ if $i \in T$ and an exploitation offer $t_i = (1-\delta)x_i$ if $i \in E$.
(c) The equilibrium T solves (2) and $T \cup E$ includes q members.
(d) For any order such that the members in T are approached before the members in E and the first member in E is dispensable in $(N\setminus T, q-|T|)$, there exists another equilibrium in which the members are approached in that order.

[22] Proposition 3 assumes generic y for expositional convenience: when $y = W(S,r)$ the policy passes in some equilibria but does not pass in others. The choice of an equilibrium makes a member either indispensable or dispensable, complicating the analysis without providing additional insight.

The functions W and Π are crucial in equilibrium characterization, but it is not possible to derive closed-form expressions for these functions except in special cases, which we cover in Proposition B1 in the Supplementary Appendix.[23] Outside of these special cases, in Proposition B2 we show that $W(N, q) = \sum_{i \in M} x_i$ for some $M \subseteq N$, where M excludes any member strictly more opposed than the qth member. Similarly, the equilibrium temptation phase excludes any such member. Moreover, the following example illustrates that the equilibrium temptation phase involves a basic tradeoff: a shorter temptation phase with more-opposed members vs a longer one with less-opposed members. As a result of this tradeoff, it may happen that a member with a higher opposition is tempted in equilibrium while a member with a lower opposition is not tempted or even approached.

Example 2. *In the up-front-payment game, suppose $n = 4$, $q = 3$, x_is are all distinct and $y \in (\max\{x_1 + x_2, x_3\}, \min\{x_1 + x_3, x_4\})$. Calculations shows that $W(N, q) = \max\{x_1 + x_2, x_3\}$, implying that the policy passes in equilibrium. Also, since $y < W(N \setminus \{4\}, 3) = x_1 + x_2 + x_3$, member 4 is indispensable at the beginning of the game. Hence, the temptation phase is nonempty. If the temptation phase includes only one member, then it cannot be member 1 or 2 since after the leader buys his vote, member 4 is still indispensable, but if the leader starts by tempting member 3, then member 4 does become dispensable after his vote is bought and we reach the exploitation phase. If the temptation phase includes two members 1 and 2, then member 4 becomes dispensable after their votes are bought and we reach the exploitation phase again. Therefore, the cheapest one-member temptation phase includes member 3, while the cheapest two-member temptation phase includes members 1 and 2. When $x_3 < x_1 + x_2$, the leader approaches members 3, 4 and 1 in equilibrium, tempting the first one and exploiting the last two. When $x_1 + x_2 < x_3$, the leader approaches members 1, 2, and 4 in equilibrium, tempting the first two and exploiting the last one.*

The comparative statics are similar to those in the transfer-promise game: the leader benefits (that is, the threshold for the policy passing, $W(N, q)$, and the cost of vote buying, $\Pi(N, q, y)$, are lower) when she needs fewer votes from a larger committee consisting of less opposed members, or when she has a higher gain from the policy.[24]

5 Discussion

Implications of sequential vote buying. Our analysis provides a number of insights.

[23] For any state (S, r) with $|S| - r = 1$, problem (1) defining W is a special case of the Partition problem and problem (2) defining Π is a special case of the Knapsack problem. Both problems are well known in computer science and combinatorial optimization and both are NP-hard. Le Breton and Zaporozhets [2010]; Le Breton et al. [2012] explore similar mathematical connections for the Groseclose and Snyder [1996] model.

[24] We state these results in Proposition B3 in the Supplementary Appendix.

First, the cost of a member's vote is determined by the intensity of her opposition and by her bargaining position, which is determined by the intensity of other members' opposition. When the latter effect dominates the former, the leader can exploit a more opposed member and it is cheaper to buy his vote.

Second, we provide an explanation for the Tullock Paradox, the observation that political favours are often bought at costs far below their valuation. In the transfer-promise game, when the policy passes, it passes at almost zero cost. In the up-front-payment game, the cost of policy passage might be nontrivial, but is still almost zero when the leader's gain is sufficiently large.[25] Our analysis thus highlights the importance of distinguishing the cost of vote buying (observable) and the intensity of opposition (not directly observable). In particular, an observed low cost of buying a vote does not necessarily reflect a low intensity of opposition.

Third, in our model the structure of opposition measured by $(x_1, \ldots, x_n)$ mainly determines whether the policy passes and to a much less extent the cost. The leader's choice of which policies to champion thus drives mainly her success in passing policies, and affects less the cost of its passage. Similarly, lobbying strategy aiming to lower the opposition to a policy will primarily affect its passage. An interesting question is which opposition structures challenge the leader the most. With transfer promises, keeping the total opposition measured by $\sum_{i \in N} x_i$ constant, having homogenous members maximizes the threshold for the policy passing. This is because with homogenous members, it is harder for the leader to pit members against each other and make them dispensable. With up-front payments, homogeneity again poses a challenge for the leader. We find that having two homogeneous groups, one with strongly opposed members and another with almost-unopposed members, maximizes the threshold for the policy passing.[26]

Finally, our results highlight the members whose preferences determine the equilibrium outcomes. In both games, the policy passage is determined by the q members least opposed because both $\sum_{i=1}^{q} x_i$ and $W(N, q)$ include only these members. Preferences of other members might enter the cost (see Proposition 2 and Example 1), but these members are always exploited and their preferences do not significantly affect the costs.

As we explain in the introduction, many of the theoretical predictions are supported empirically and the insights allow us to make sense of otherwise puzzling observations and build novel connections between observed phenomena.

Transfer promises vs up-front payments. The sequential nature of our model highlights the difference between up-front payments and transfer promises in terms of their different commitment values for the leader. Between the two regimes, the leader is better

[25] In the Supplementary Appendix, we prove that $\Pi(N, q, y) = 0$ when $y > \sum_{i=1}^{q} x_i$ and $q < n$.

[26] In the Supplementary Appendix, we derive an upper bound on $W(N, q)$ and show that it can be reached arbitrarily closely by having x_i constant for n_b members, where n_b is any integer multiple of $N - q + 1$, and having $x_i = \varepsilon$ for the remaining members. The bound cannot be reached since we do not allow $x_i = 0$.

off and the members are worse off under up-front payments.

Proposition 5. *For generic y, sufficiently large δ and if $x_i < x_{i+1}$ for all i, the leader's equilibrium payoff is weakly higher and each member i's equilibrium payoff is weakly lower in the up-front-payment game than in the transfer-promise game.*

The proposition follows because whenever the policy passes with transfer promises, it also passes with up-front payments with the same minimal transfers to the same members, and for certain parameters, the policy passes only with up-front payments. As explained in the introduction, we should expect transfer promises to be used mostly in bills with significant distributive implications because of their benefit of "explainability," consistent with empirical observations.[27]

Welfare implications. We next discuss welfare implications of vote buying, using the utilitarian welfare criterion. We first consider the transfer-promise game. When efficiency dictates that the policy should not pass (when $y - \sum_{i \in N} x_i < 0$), vote buying can only lower welfare and does when the policy passes (when $y - \sum_{i \in N^q} x_i > 0$). When efficiency dictates that the policy should pass (when $y - \sum_{i \in N} x_i > 0$), vote buying can only improve welfare and either does when the policy passes or the equilibrium remains utilitarian inefficient. These observations apply in the up-front-payment game after changing $y - \sum_{i \in N^q} x_i$ to $y - \sum_{i \in M} x_i$, where M is some subset of the members. We can extend the results by accounting for the (unmodeled) members in favor of the policy, by adding the sum of their gains from the policy to $y - \sum_{i \in N} x_i$. The inefficiencies arise from the externalities across members: a member deciding to sell his vote ignores the potential impact of his decision on the utility of the other members.

6 Extensions

6.1 Private histories

Since negotiations are sometimes conducted behind closed doors and results kept private, it is interesting to investigate what happens in equilibrium with private histories. To do this, we modify the main model so that the leader's previous actions (except for the offer made to the member) are not directly observable but keep all other assumptions unchanged. To facilitate comparison between the results in the main model and those in this extension, we define an equilibrium outcome to include the sequence of members approached, the offer made to each member, their acceptance/rejection decisions, and

[27] Dekel, Jackson, and Wolinsky [2008] also compare up-front payments and campaign promises, but the vote sellers in their model are not strategic and they focus on the competition between two vote buyers, which resembles an auction. These differences in modeling lead to important differences in results too: they find that with campaign promises, the equilibrium outcome may involve substantial offers to voters whereas with up-front payments, because of their all-pay nature, the voters receive minimal transfers, the opposite of our findings.

whether the policy passes. (For the game with private histories, we use Perfect Bayesian Equilibrium as the solution concept since it has no proper subgame and SPE is too weak.)

In the transfer-promise game, the equilibrium outcome with public histories can be supported with private histories too. Under unanimity, this is immediate since the leader offers each member i a transfer of x_i and any sequence is optimal whether the history is public or private. Under non-unanimity, first suppose member q is dispensable with public histories. As shown in Proposition 2, with public histories, the leader approaches the members in $\{1, \dots, q\}$ in some sequence in equilibrium and offers each $(1-\delta)x_i$. To support this outcome with private histories, consider the profile such that any $i \in N$ accepts t if and only if $t \geq (1-\delta)x_i$ and the leader follows the same sequence as in the equilibrium with public histories. The leader has no profitable deviation since this is the least costly way of buying q votes; and no member has any profitable deviation either since he is indeed dispensable given other players' strategies. Next suppose member q is indispensable with public histories. As shown in Proposition 2, the leader first approaches member $q+1$ and then the members in $\{1, \dots, q-1\}$ in some sequence and offers each $(1-\delta)x_i$. This outcome is supported in equilibrium with private histories by the following strategy profile: any member $i \in N\setminus\{q\}$ accepts t if and only if $t \geq (1-\delta)x_i$ and member q accepts t if and only if $t \geq x_q$ and the leader follows the same sequence as in the equilibrium with public histories.[28]

Proposition 6. *In the transfer-promise game, suppose $y > \sum_{i=1}^{q} x_i$. Any equilibrium outcome with public histories is an equilibrium outcome with private histories.*

With up-front payments, however, an equilibrium outcome achieved with public histories may not be supported with private histories. To illustrate, recall that in Example 1, when $x_2 < y < x_1 + x_2$, the equilibrium involves the leader approaching 1 first, making a temptation offer $\delta^2 x_1$, and then approaching 3, making an exploitation offer $\delta(1-\delta)x_3$. With private histories, however, this cannot be supported in equilibrium. To see why, note that with public histories, member 1 is made a temptation offer because he is indispensable. When his rejection is unobservable, however, he is no longer indispensable – the leader can still get the policy passed by making 3 an exploitation offer and 2 a temptation offer, and therefore the equilibrium collapses. What outcomes can be supported? Under non-unanimity rule, when players' rejections are not observable, it is easier to make them dispensable. Specifically, if $y > x_1$, an equilibrium exists in which the leader exploits members in $\{2, \dots, q+1\}$, using member 1 as a threat point.[29]

Even though equilibrium outcomes achieved with public histories may not be sup-

[28]Since member q is not approached on the equilibrium path, we specify his belief to be such that if he is approached, the leader needs one vote and he is the only member remaining.

[29]This is supported by the following equilibrium acceptance rules: member $i \in \{2, \dots, q+1\}$ accepts a transfer t iff $t \geq \delta^{(q+2-i)}(1-\delta)x_i$, member 1 accepts t iff $t \geq \delta x_1$ and member $i \in \{q+2, \dots, n\}$ accepts t iff $t \geq \delta(1-\delta)x_i$.

ported with private histories in the up-front-payment game, it can still be supported with *semi-public* histories, with observable states.[30] Semi-publicity requires that members observe who have been approached and whether the offers were accepted, but does not require observability of the offers. It applies in situations in which politicians publicly announce support of certain bills without disclosing the deals reached with the leadership.

6.2 Simultaneous offers

In the vote-buying literature, it is common to assume that the leader must make offers to all members simultaneously. Either it does not matter whether the offers are simultaneous or sequential, as in Groseclose and Snyder [1996], or the simultaneous assumption provides analytical tractability. Given historical accounts of the vote-buying process in legislatures, our view is that sequential offers reflect it better.[31] For comparison, we briefly discuss two variants of our model with simultaneous offers.

6.2.1 Simultaneous benchmark

We first consider a benchmark in which the leader must make offers to all members simultaneously. First, the leader either stops, in which case the game ends, or offers a profile of transfer promises in which case the members sequentially (in some predetermined order) decide whether to accept the offers, and the game proceeds to the second period in which the leader either initiates a vote or stops. In all other aspects the game is the same as the sequential transfer-promise game.

Proposition 7. *Consider the simultaneous vote-buying game with transfer promises. If $y < \sum_{i=1}^{q} x_i$, then the policy does not pass in any equilibrium. If $y > \sum_{i=1}^{q} x_i$, then in any equilibrium, the policy passes and q members who have the lowest sum of losses are offered transfers x_i.*

Proposition 7 implies that the condition for the policy passing in equilibrium is the same whether the leader makes transfer promises simultaneously or sequentially. Moreover, the sets of members whose votes are bought are similar too: those who are least opposed to the policy are bought (with the caveat that under sequential vote-buying, sometimes the $(q + 1)$th member needs to be bought). However, the *amount* of transfers they receive are drastically different: when approached sequentially, the members receive exploitation offers (offers close to 0), whereas when approached simultaneously, they receive temptation offers (offers that equal their losses). This is because when the leader has to make simultaneous offers to all members, she can no longer use other members

[30] This is because the strategies in the equilibrium that we characterize with public histories depend only on (S, r), even though a priori we do not make any such restriction.

[31] See, for example, Mayer [1998], Merriner [1999], Caro [2012] and Evans [2018].

as threat points. Without the divide-and-conquer mechanism at her disposal, she now has to buy q votes by compensating the members fully for their losses. Analogous results hold if the offers are up-front payments instead of transfer promises.[32,33]

6.2.2 Simultaneous offers allowed

We now allow the leader to approach multiple members simultaneously (but not necessarily all at the same time). A complete analysis of this game is beyond the scope of our paper, but we provide some ideas as to how to extend the techniques developed in the main sequential model. For space considerations, we focus on up-front payments.

In the extensive form we study, in each period, the leader either makes simultaneous offers to a subset of members who have not been approached before, or initiates a vote, or stops. If the leader makes offers, the members approached then sequentially decide, in some predetermined order, whether to accept the offers. Formally, when the leader in state (S, r) approaches members $X \subseteq S$ and $\hat{X} \subseteq X$ of them accept, the game proceeds to state $(S \setminus X, r - |\hat{X}|)$.

In the benchmark sequential model from Section 2, the leader's main concern is the expenditure she has to pay to buy the necessary votes. Allowing simultaneous offers introduces a new force that shapes the leader's actions: she may now try to pass the policy more quickly. The following example illustrates the effect of this new force.

Example 3. *Consider a game with four members* $\mathbf{x} = (1, 1, 3, 6)$*,* $y = \frac{31}{10}$*,* $q = 3$ *and* δ *close to 1. The offers described below are accepted in equilibrium and the policy passes.*

In the benchmark sequential model, the leader sequentially approaches members 1 *and* 2 *with temptation offers and member* 4 *with an exploitation offer.*

When simultaneous offers are allowed, the leader simultaneously approaches members 1*,* 2 *and* 4 *with temptations offers to* 1 *and* 2 *and an exploitation offer to* 4*.*

The example seems to suggest that allowing simultaneous offers results in the leader buying all the votes she needs in a single period. This is in fact not always the case due to the constraint on the leader's ability to quickly buy votes from the members included in

[32] The condition for the policy passing and the set of members whose vote are bought in equilibrium are identical. The only difference is that the temptation offers with up-front payments are δx_i instead of x_i. This is because the periods in which the transfers are received and the policy passes are different. When the offers are made simultaneously, the difference between sunk and non-sunk cost disappears and hence the conditions for policy passing and the set of members who receive positive transfers in equilibrium are identical. Details of the results for the case of up-front payment when the offers must be simultaneous are available upon request.

[33] We assume that the members' acceptance decisions are sequential. With simultaneous acceptance decisions, for any profile of transfer promises, multiple equilibria exist in the acceptance stage; one where all members accept and one where all members reject (because no one is pivotal). This multiplicity can be used to construct an infinite number of equilibria in the entire game. Assuming that members make acceptance decisions sequentially rules out this multiplicity, which, alternatively, can be ruled out by an equilibrium refinement, as in Genicot and Ray [2006].

the exploitation phase: multiple members might not be *jointly dispensable*. Analogous to the notion of dispensability, a set of members $X \subseteq S$ is jointly dispensable in state (S, r) if the policy passes in state $(S \setminus X, r)$. The following example shows how joint dispensability shapes the equilibrium, in particular, how it may force the leader to approach members one at a time, making it observationally equivalent to the benchmark sequential model.

Example 3 (continued)**.** *Consider a game with four members* $\mathbf{x} = (1, 1, 3, 6)$*,* $y \in \{\frac{31}{10}, \frac{62}{10}\}$*,* $q = 2$ *and* δ *close to 1.*

In the benchmark sequential model, the leader sequentially approaches members 1 *and* 2 *with exploitation offers.*

When simultaneous offers are allowed and $y = \frac{31}{10}$*, the leader sequentially approaches members* 1 *and* 2 *with exploitation offers.*

When simultaneous offers are allowed and $y = \frac{62}{10}$*, the leader simultaneously approaches members* 1 *and* 4 *with exploitation offers.*

When simultaneous offers are allowed and $y = \frac{31}{10}$, the leader exploits the same members as in the benchmark model. To speed up the passage of the policy while still using exploitation offers, the leader has to approach (the only) jointly dispensable pair of members 3 and 4. Doing so is dominated by passing the policy in two periods by making exploitation offers to members 1 and 2. That is, the leader prefers lower expenditure over quicker passage of the policy that requires more expensive offers. When $y = \frac{62}{10}$, more sets of members become jointly dispensable and the leader prefers quicker passage of the policy, associated now with smaller expenditure on exploitation offers to jointly dispensable members 1 and 4.[34]

The discussion above highlights a significant difference between our paper and Genicot and Ray [2006]: their exploitation phase must proceed sequentially as an implication of assuming that an agent's decision to contract affects the reservation but not the contracted payoff of other agents. In contrast, in our model, the exploitation phase need not be sequential. This happens, as in Example 3, when certain members are jointly dispensable, which is more likely when the leader's gain from the policy is high.

6.3 Preferences for casting votes

We relax the assumption that members care only about the outcome of the policy and incorporate preferences for how they cast their votes. Suppose each member incurs a loss of c from the action of casting a vote in support of the policy, then our results regarding who to approach and how much to offer would go through with the modification that the leader has to add c to each offer to compensate for the member's loss. If the members

[34] Joint dispensability limits the number of members the leader can simultaneously exploit in state (S, r) to $|S| - r$; any set $X \subseteq S$ with $|X| > |S| - r$ is not jointly dispensable in (S, r) because the policy cannot pass in state $(S \setminus X, r)$ as $|S| - |X| < r$.

incur heterogeneous losses from the action of casting a supportive vote, then there would be new trade-offs for the leader in terms of who to approach and what offers to make.

6.4 Other extensions for future work

Several extensions discussed above invite intriguing questions that need more developed analysis to be fully addressed, as we have already pointed out. We conclude our paper by discussing a few other promising directions.

Members sometimes form coalitions to bargain with the leader to increase their leverage since they jointly have a higher voting weight. A full analysis of bargaining with coalitions is beyond the scope of this paper, and we have an ongoing project to understand coalitional bargaining in a setting similar to the one in this paper.

Our framework assumes complete information. While this is a good approximation in certain scenarios such as in the case of congressional whips buying the votes of their party members, it may be less realistic in others contexts. There have been some attempts at incorporating incomplete information into multi-dimensional legislative bargaining,[35] but work remains to be done to address questions such as those put forward in our paper.

While a single vote buyer is appropriate for many applications, it would be interesting to investigate the implications of competition. New questions arise in the presence of competition too: Does competition increase or decrease the cost of vote buying? Who benefit most from competition? What happens to social welfare?[36]

Heterogeneity among members is a central feature of our model, and it refers to the different preferences that the members have regarding the policy. Other kinds of heterogeneity could arise as well: we have already pointed out that members could have different voting weights; in the application of exclusionary contracts, buyers could have different market shares; in the application of corporate takeovers, shareholders could have different percentages of shares. How would these heterogeneities affect the contracting relationships and coalition formation process? We leave these and other intriguing questions for future work.

Appendix

Proof of Proposition 1. Fix state (S, r, t). We first prove part (a). Assume $y - t > \sum_{i \in S^r} x_i$. Suppose, towards a contradiction, that there exists an equilibrium in $\Gamma(S, r, t)$ in which the policy does not pass. The leader's payoff in this equilibrium is 0. We next show that the leader has a strictly profitable deviation. Note that it is optimal for any member i to accept $t_i > x_i$. Consider the strategy of approaching members in S in a

[35] See, for example, Chen and Eraslan [2013, 2014].

[36] Iaryczower and Oliveros [2017] address the question of how competition affects the cost of contracting with externalities in a model with homogenous agents.

descending order, offering zero transfer to the first $|S| - r$ members (no matter what the history is) and then offering $x_i + \varepsilon$ to each of the remaining r members (no matter what the history is). Since the last r members accept the offers, policy passes and the leader's payoff is $y - t - \sum_{i \in S^r} x_i - r\varepsilon > 0$ for $\varepsilon > 0$ sufficiently low. Hence, the leader has a profitable deviation, a contradiction.

We next prove part (b) by induction. Assume $y - t < \sum_{i \in S^r} x_i$. First consider $|S| = r$. Suppose, towards a contradiction, that there exists an equilibrium in which the policy passes. Since $|S| = r$, each $i \in S$ accepts the leader's offer in this equilibrium. Since any member's rejection leads to the failure of the policy passing, the equilibrium payoff of any member $i \in S$ is nonnegative. Hence, the leader must offer each member i at least x_i, which implies that the leader's payoff in this equilibrium is no higher than $y - t - \sum_{i \in S} x_i < 0$. Since the leader's payoff is 0 if she stops immediately in $\Gamma(S, r, t)$, it follows that she has a strictly profitable deviation, a contradiction. Hence, the policy doe not pass in any equilibrium.

Next, suppose that part (b) holds for $|S| - r \leq k$ where $0 \leq k < |S|$. We prove that it also holds for $|S| - r = k + 1$. Suppose, towards a contradiction, that there exists an equilibrium in $\Gamma(S, r, t)$ in which the policy passes. Suppose in this equilibrium, the leader approaches member i in the first period of $\Gamma(S, r, t)$. Note that given the induction hypothesis, if member i rejects the leader's offer, then the policy does not pass in any equilibrium in the resulting subgame $\Gamma(S_{-i}, r, t)$ since $y - t < \sum_{j \in S^r_{-i}} x_j$. Given that the policy passes in equilibrium in $\Gamma(S, r, t)$, the transfer t_i offered to i has to be such that $\delta^r(-x_i + t_i) \geq 0$, that is, $t_i \geq x_i$. Note that in the subgame that follows member i's acceptance, $\Gamma(S_{-i}, r - 1, t + t_i)$, we have $y - t - t_i \leq y - t - x_i$. Since $y - t < \sum_{j \in S^r} x_j$, it follows that $y - t - x_i < \sum_{j \in S^{r-1}_{-i}} x_j$ and therefore $y - t - t_i < \sum_{j \in S^{r-1}_{-i}} x_j$. By the induction hypothesis, the policy does not pass in any equilibrium in $\Gamma(S_{-i}, r - 1, t + t_i)$, a contradiction. Hence, part (b) holds. ■

Proof of Proposition 2. Consider state (S, r, t). Suppose the leader offers t_i to member $i \in S$. If i is dispensable, then his payoff equals $t_i - x_i$ by accepting the offer and equals $-\delta x_i$ by rejecting the offer since by Proposition 1 the policy passes in the subgame following the rejection. Hence, he accepts the offer if and only if $t_i - x_i \geq -\delta x_i$, that is, $t_i \geq (1 - \delta)x_i$. If i is indispensable, then his payoff equals $t_i - x_i$ by accepting the offer and equals 0 by rejecting the offer since by Proposition 1 the policy does not pass in the subgame following the rejection. Hence, he accepts the offer if and only if $t_i \geq x_i$. Since $x_{i+1} > x_i$, this immediately implies that if there exists a sequence of members along which each is dispensable, then for δ sufficiently high, the leader would approach only dispensable members in any equilibrium.

To prove part (a), we first show that after approaching a member in $A^d \cap \{1, \ldots, q\}$, any remaining member in $\{1, \ldots, q\}$ becomes dispensable. Suppose the leader approaches member $j \in A^d \cap \{1, \ldots, q\}$ first and offers $(1 - \delta)x_j$. Since $j \in A^d \cap \{1, \ldots, q\}$, we have $y > \sum_{i\in\{1,\ldots,q\}\setminus\{j\}} x_i + x_{q+1}$, and it follows that if δ is sufficiently high, $y - (1-\delta)x_j > \sum_{i\in\{1,\ldots,q\}\setminus\{k,j\}} x_i + x_{q+1}$ for any $k \in \{1,\ldots,q\}$ and $k \neq j$. Hence, in state $(N-1, q-1, (1-\delta)x_j)$, any remaining member in $\{1,\ldots,q\}$ is dispensable. Suppose the leader approaches member $k \in \{1,\ldots,q\}$ next and offers $(1-\delta)x_k$. For δ sufficiently high, $y-(1-\delta)(x_j+x_k) > \sum_{i\in\{1,\ldots,q\}\setminus\{k,j,\ell\}} x_i + x_{q+1}$ for any $\ell \in \{1,\ldots,q\}$ and $\ell \neq k, j$. Hence, in state $(N-2, q-2, (1-\delta)(x_j+x_k))$, any remaining member in $\{1,\ldots,q\}$ is dispensable. Continuing this argument, any remaining member in $\{1,\ldots,q\}$ is dispensable no matter in what sequence the leader chooses to approach them. Since members in $\{1,\ldots,q\}$ have the lowest losses, it follows that in any equilibrium, the leader first approaches a member in $A^d \cap \{1,\ldots,q\}$ and then the remaining members in $\{1,\ldots,q\}$ in an arbitrary order and offers each member i a transfer $(1-\delta)x_i$, which is accepted.

To prove part (b), note that since $\sum_{i=1}^{q} x_i < y < \sum_{i=1}^{q-1} x_i + x_{q+1}$, member $q+1$ is the member with the lowest loss who is dispensable at the beginning of the game. Using an argument similar to that in part (a), after the leader approaches member $(q+1)$ and offers him $(1-\delta)x_{q+1}$, all members in $\{1,\ldots,q-1\}$ are dispensable no matter in what sequence the leader chooses to approach them. Since members in $\{1,\ldots,q-1\}$ have the lowest losses, it follows that in any equilibrium, the leader first approaches member $(q+1)$ and then members in $\{1,\ldots,q-1\}$ in an arbitrary order and offers each member i a transfer $(1-\delta)x_i$, which is accepted. ■

Proofs of Propositions 3 and 4 and of Lemma 1. We state and prove Lemmas A1 and A2 and Proposition A1. Propositions 3 and 4 follow from Proposition A1. Lemma 1 is a consequence of Proposition A1 and of Lemma A1. Several preliminaries follow.

The following notation is used throughout. (Recall $S' = S\setminus\{\max S\}$ for any $S \in 2^N \setminus \varnothing$, and $\varnothing' = \varnothing$ by convention.) First, for any $S \in 2^N\setminus\varnothing$ and any $k \in \{1,\ldots,|S|\}$, let (k) be the member with the k-th smallest index in S. Formally, (k) is defined recursively as $(k) = \min S$ if $k = 1$ and $(k) = \min\{S\setminus\cup_{i=1}^{k-1}(i)\}$ if $k \geq 2$. We use $(k)_S$ instead of (k) only when the underlying S needs to be made explicit. Second, let $\mathcal{D}^W = \{(S,r) \in 2^N\times\mathbb{Z}|r \leq n\}$ and $\mathcal{D}^G = \{(S,r) \in 2^N\times\mathbb{Z}|S \neq \varnothing \wedge 1 \leq r \leq |S|\}$ be two sets of states. Third, let $\mathcal{L} = \{\sum_{i\in S} x_i|S \in 2^N\}\cup\{\infty\}$ and note that $\mathcal{L}$ is finite. Fourth, let $\bar{\delta} = \max\{\bar{\delta}_a, \bar{\delta}_b, \bar{\delta}_c, \bar{\delta}_d\}$, where $\bar{\delta}_a = 1 - \frac{\min_{x\in\mathcal{L},x<y}(y-x)}{nx_n}$, $\bar{\delta}_b = 1 - \frac{x_1}{nx_n}$, $\bar{\delta}_c = 1 - \frac{\min_{x,x'\in\mathcal{L},x'<x}(x-x')}{nx_n}$, and, given $\varepsilon > 0$, $\bar{\delta}_d = \max\{0, 1-\frac{\varepsilon}{y+nx_n}\}^{\frac{1}{n+1}}$. Note that because $\min_{x\in\mathcal{L},x<y}(y-x) > 0$, $x_1 > 0$, $\min_{x,x'\in\mathcal{L},x'<x}(x-x') > 0$, and $\varepsilon > 0$, we have $\bar{\delta} < 1$. Fifth, define $W : \mathcal{D}^W \to \mathbb{R}\cup\{\infty\}$ as follows: $W(S,r) = 0$ for any $(S,r) \in \mathcal{D}^W\setminus\mathcal{D}^G$ with $r \leq 0$, $W(S,r) = \infty$ for any $(S,r) \in \mathcal{D}^W\setminus\mathcal{D}^G$ with $r > |S|$, and $W(S,r)$ is defined by (1) for any $(S,r) \in \mathcal{D}^G$. Sixth, $\Pi : \mathcal{D}^G \times \mathbb{R}_{++} \to \mathbb{R}$ is defined by (2). Seventh, a member $i \in S$ in state $(S,r) \in \mathcal{D}^W$ is dispensable if $y > W(S\setminus\{i\}, r))$, indispensable if $y \in (W(S\setminus\{i\}, r-1), W(S\setminus\{i\}, r))$, and inconsequential if $y < W(S\setminus\{i\}, r-1)$. Definition 2 establishes the former two notions, the last one is new.

Lemma A1. *Consider state $(S,r) \in \mathcal{D}^W$.*

1. *If $(S,r) \in \mathcal{D}^G$, then any T that solves (1) satisfies $|T| \leq r$.*
2. *If $|S| \geq 1$, then $W(S,1) = x_{(1)}$ and $W(S,|S|) = \sum_{i \in S} x_i$.*
3. *$W(S,r) \in \mathcal{L}$. If $(S,r) \in \mathcal{D}^G$, then $W(S,r) \in (0,\infty)$.*
4. *$W(S,r) \geq W(S,r-1)$.*
5. *$W(\hat{S},r) \leq W(S,r)$ for any $\hat{S} \in 2^N$ such that $|\hat{S}| \geq |S|$ and $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1,\ldots,|S|\}$.*
6. *If $|S| \geq 1$, then $W(S,r) \geq W(S \setminus \{i\}, r-1)$ $\forall i \in S$.*

Proof. Part 1: Consider $(S,r) \in \mathcal{D}^G$ and T that solves (1) given (S,r). Suppose, towards a contradiction, $|T| > r$. Because $r - |T| < 0$, we have $W((S \setminus T)', r - |T|) = 0$ and hence $W(S,r) = \sum_{i \in T} x_i$. Now consider T'. We have $|T'| = |T| - 1$ because $|T| > r$ and $(S,r) \in \mathcal{D}^G$ jointly imply $T \neq \varnothing$. Thus $|T'| \geq r$. Hence $\sum_{i \in T'} x_i < \sum_{i \in T} x_i = W(S,r)$ and $W((S \setminus T')', r - |T'|) = 0$, a contradiction because T solves (1) given (S,r).

Part 2: It suffices to prove that, for any $S \in 2^N \setminus \varnothing$, $W(S,1) = x_{(1)}$ and $W(S,|S|) = \sum_{i \in S} x_i$. To see the latter, consider $S \in 2^N \setminus \varnothing$. Because $S \in 2^N \setminus \varnothing$, $(S,|S|) \in \mathcal{D}^G$. The claim thus follows because the objective function in (1) evaluated at $(S,|S|)$ and $T = S$ equals $\sum_{i \in S} x_i$ and evaluated at $(S,|S|)$ and any $T \in 2^S \setminus S$ equals ∞. We prove the former by induction on $|S|$. Note that for any $S \in 2^N \setminus \varnothing$, $(S,1) \in \mathcal{D}^G$. That $W(S,1) = x_{(1)}$ for any $S \in 2^N \setminus \varnothing$ with $|S| = 1$ follows because $W(S,|S|) = \sum_{i \in S} x_i$ for any $S \in 2^N \setminus \varnothing$ with $|S| \geq 1$. Now suppose that $W(S,1) = x_{(1)}$ for any $S \in 2^N \setminus \varnothing$ with $|S| \leq k$, where $k \geq 1$. We need to prove that $W(S,1) = x_{(1)}$ for any $S \in 2^N \setminus \varnothing$ with $|S| = k+1$. To see this, consider T that solves (1) given (S,r). By part 1, $|T| \in \{0,1\}$. If $|T| = 0$, then $W(S,1) = W(S',1) = x_{(1)}$, where the second equality follows from the induction hypothesis. If $|T| = 1$, then, because $W((S \setminus U)', 1 - |U|) = 0$ for any $U \in 2^S$ with $|U| = 1$, we have $T = \{\min S\}$ as well as $W(S,1) = \sum_{i \in T} x_i$.

Part 3: We first prove that $W(S,r) \in \mathcal{L}$ for any $(S,r) \in \mathcal{D}^W$. We proceed by induction on $|S|$. That $W(S,r) \in \mathcal{L}$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = 0$ follows directly from definition of W. Now suppose that $W(S,r) \in \mathcal{L}$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \leq k$, where $k \geq 0$. We need to prove that $W(S,r) \in \mathcal{L}$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = k+1$. To see this, we have either *(i)* $r \leq 0$, in which case $W(S,r) = 0 \in \mathcal{L}$, or *(ii)* $r > |S|$, in which case $W(S,r) = \infty \in \mathcal{L}$, or *(iii)* $r \in \{1,\ldots,|S|\}$, in which case, given T that solves (1) given (S,r), either $W(S,r) = \sum_{i \in T} x_i \in \mathcal{L}$, or, by the induction hypothesis, $W(S,r) = W((S \setminus T)', r - |T|) \in \mathcal{L}$. We now prove that $W(S,r) < \infty$ for any $(S,r) \in \mathcal{D}^G$. This follows because the objective function in (1) evaluated at $(S,r) \in \mathcal{D}^G$ and $T = S$ equals $\sum_{i \in T} x_i < \infty$. We now prove that $W(S,r) > 0$ for any $(S,r) \in \mathcal{D}^G$. We proceed by induction on $|S|$. That $W(S,r) > 0$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$ follows because $(S,r) \in \mathcal{D}^G$ and $|S| = 1$ imply $r = 1$ and hence, by part 2, $W(S,r) = x_{(1)} > 0$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$. Now suppose that $W(S,r) > 0$ for any $(S,r) \in \mathcal{D}^G$ with $|S| \leq k$, where $k \geq 1$. We need to prove that $W(S,r) > 0$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$. To see this, consider T that solves (1) given (S,r). We have either *(i)*

$T = \varnothing$, in which case $W(S,r) = W(S',r) > 0$ either by the induction hypothesis when $(S',r) \in \mathcal{D}^G$ or directly from definition of W when $(S',r) \notin \mathcal{D}^G$ and hence $r > |S'|$, or (ii) $T \neq \varnothing$, in which case $W(S,r) \geq \sum_{i \in T} x_i > 0$.

Part 4: We proceed by induction on $|S|$. That $W(S,r) \geq W(S,r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = 0$ follows directly from definition of W. Now suppose that $W(S,r) \geq W(S,r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \leq k$, where $k \geq 0$. We need to prove that $W(S,r) \geq W(S,r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = k+1$. To see this, we have either (i) $r \leq 1$, in which case $W(S,r) \geq \min \mathcal{L} = 0$ by part 3 and $W(S,r-1) = 0$, or (ii) $r > |S|$, in which case $W(S,r) = \infty \geq W(S,r-1)$, or (iii) $r \in \{2, \ldots, |S|\}$, in which case, given T that solves (1) given (S,r), $W(S,r) = \max\{\sum_{i \in T} x_i, W((S \setminus T)', r - |T|)\} \geq \max\{\sum_{i \in T} x_i, W((S \setminus T)', r - 1 - |T|)\} \geq W(S, r-1)$, where the first inequality follows from the induction hypothesis.

Part 5: We proceed by induction on $|S|$. That $W(\hat{S}, r) \leq W(S,r)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = 0$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| \geq |S|$ and $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1, \ldots, |S|\}$ follows because given $S = \varnothing$ and any $\hat{S} \in 2^N$ we have either (i) $r \leq 0$, in which case $W(S,r) = W(\hat{S},r) = 0$, or (ii) $r \geq 1$, in which case $W(S,r) = \infty \geq W(\hat{S},r)$. Now suppose that $W(\hat{S},r) \leq W(S,r)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \leq k$, where $k \geq 0$, and for any $\hat{S} \in 2^N$ such that $|\hat{S}| \geq |S|$ and $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1, \ldots, |S|\}$. We need to prove that $W(\hat{S},r) \leq W(S,r)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = k+1$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| \geq |S|$ and $\hat{x}_{(i)_{\hat{S}}} \leq x_{(i)_S}$ $\forall i \in \{1, \ldots, |S|\}$. To see this, we have either (i) $r \leq 0$, in which case $W(S,r) = W(\hat{S},r) = 0$, or (ii) $r > |S|$, in which case $W(S,r) = \infty \geq W(\hat{S},r)$, or (iii) $r \in \{1, \ldots, |S|\}$, in which case, given T that solves (1) given (S,r) and $\hat{T} = \{(i)_{\hat{S}} | i \in \{1, \ldots, |S|\}, (i)_S \in T\}$, $W(\hat{S},r) \leq \max\{\sum_{i \in \hat{T}} \hat{x}_i, W((\hat{S} \setminus \hat{T})', r - \hat{T})\} \leq \max\{\sum_{i \in T} x_i, W((S \setminus T)', r - |T|)\} = W(S,r)$, where the second inequality follows from the construction of $\hat{T}$ and from the induction hypothesis.

Part 6: Let $S^* = S \setminus \{\min S\}$ for any $S \in 2^N \setminus \varnothing$. Note that for any $S \in 2^N$ with $|S| \geq 2$, $(S^*)' = (S')^*$. By part 5, it suffices to prove that $W(S,r) \geq W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \geq 1$, which we prove by induction on $|S|$. That $W(S,r) \geq W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = 1$ follows either directly from definition of W when $(S,r) \notin \mathcal{D}^G$ or from $W(S,r) = x_{(1)} > 0$ shown in part 2 and $W(S^*, r-1) = 0$ when $(S,r) \in \mathcal{D}^G$. Now suppose that $W(S,r) \geq W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| \leq k$, where $k \geq 1$. We need to prove that $W(S,r) \geq W(S^*, r-1)$ for any $(S,r) \in \mathcal{D}^W$ with $|S| = k+1$. To see this, we have either (i) $r \leq 1$, in which case $W(S,r) \geq \min \mathcal{L} = 0$ by part 3 and $W(S^*, r-1) = 0$, or (ii) $r > |S|$, in which case $W(S,r) = W(S^*, r-1) = \infty$, or (iii) $r \in \{2, \ldots, |S|\}$, in which case, given T that solves (1) given (S,r), either (a) $\min S \in T$, in which case $S \setminus T = S^* \setminus T^*$ and $T^* \in 2^{S^*}$ and hence $W(S,r) = \max\{\sum_{i \in T} x_i, W((S \setminus T)', r - |T|)\} \geq \max\{\sum_{i \in T^*} x_i, W((S^* \setminus T^*)', r - 1 - |T^*|)\} \geq W(S^*, r-1)$, or (b) $\min S \notin T$ and $|T| \geq r-1$, in which case $W((S^* \setminus T)', r - 1 - |T|) = 0$ and $T \in 2^{S^*}$ and hence $W(S,r) = \max\{\sum_{i \in T} x_i, W((S \setminus$

$T)', r-|T|)\} \geq \sum_{i \in T} x_i = \max\{\sum_{i\in T} x_i, W((S^* \setminus T)', r-1-|T|)\} \geq W(S^*, r-1)$, or (c) $\min S \notin T$ and $|T| \leq r-2$, in which case $|T| \leq |S|-2$, $(S\setminus T)^* = S^* \setminus T$ and $T \in 2^{S^*}$ and hence $W(S,r) = \max\{\sum_{i\in T} x_i, W((S\setminus T)', r-|T|)\} \geq \max\{\sum_{i\in T} x_i, W(((S\setminus T)')^*, r-1-|T|)\} = \max\{\sum_{i\in T} x_i, W((S^* \setminus T)', r-1-|T|)\} \geq W(S^*, r-1)$, where the first inequality follows from the induction hypothesis. ■

Lemma A2. *Consider state $(S,r) \in \mathcal{D}^G$ and $y > 0$. If $r = |S|$, then $T = S$ is the unique solution to (2).*

Proof. Fix $(S,r) \in \mathcal{D}^G$ and $y > 0$. Because $r = |S|$ we have, for any $T \in 2^S \setminus S$, $|(S\setminus T)'| = |S| - |T| - 1 = r - |T| - 1 < r - |T| \geq 1$ and hence $W((S\setminus T)', r-|T|) = \infty$. For $T = S$, we have $W((S \setminus T)', r-|T|) = W(\varnothing, 0) = 0$ and hence S is the unique solution to (2) given (S,r). ■

Proposition A1. *Suppose the leader offers up-front payments. Fix state $(S,r) \in \mathcal{D}^G$ and $\varepsilon > 0$ and suppose that $y \notin \mathcal{L}$ and $\delta > \bar{\delta}$. In $\Gamma(S,r)$:*

1. An equilibrium exists. If multiple equilibria exist, then the payoff is constant across equilibria (a) for the leader, and (b) for any member $i \in N \setminus S$.

2. In any equilibrium, if member $i \in S$ is approached in state (S,r), then he accepts a transfer if and only if it is weakly greater than a cutoff. In state (S,r), if i is dispensable his cutoff is $\delta^r(1-\delta)x_i$, if i is indispensable his cutoff is $\delta^r x_i$, and if i is inconsequential his cutoff is 0.

3. In any equilibrium, the policy (a) passes if $y > W(S,r)$, and (b) does not pass if $y < W(S,r)$.

4. If the policy passes in an equilibrium, then

(a) it passes in $r+1$ rounds,

(b) the equilibrium consists of a temptation phase followed by an exploitation phase,

(c) the set of members included in the temptation phase solves (2),

(d) the leader's payoff is within ε of $y - \Pi(S,r,y)$,

(e) for any order that places the members from the temptation phase before the members from the exploitation phase, and that starts the latter phase with a dispensable member, there exists another equilibrium in which the members are approached in that order.

Proof. We prove Proposition A1 by induction on the size of $|S|$. Before the induction argument, we deal with several preliminaries. First, we introduce two functions l and c that define a space of strategies Σ and during the proof we (inductively) specify the l and c functions such that a profile σ constitutes an equilibrium if and only if $\sigma \in \Sigma$. For any $(S,r) \in \mathcal{D}^W$, let $l(S,r) \subseteq \{\text{initiate a vote}, \text{stop}\} \cup (S \times \mathbb{R}_+)$ and for any state $(S,r) \in \mathcal{D}^W$ and any $i \in S$, let $c(S,r,i) \in \mathbb{R}_+$. Let $H = \cup_{r\in\mathbb{Z}, r\leq n} H_r$, where H_r is the set of possible histories in $\Gamma(N,r)$. At any $h \in H$, either the leader moves or $h = (h_l, (i,t))$, that is, a member i responds to the leader's offer t made at history h_l. In the former case, let

(S^h, r^h) be the state that corresponds to h. In the latter case, let (S^h, r^h) be the state that corresponds to h_l. Let $\tilde{\sigma}_l$ be a strategy of the leader that satisfies $\tilde{\sigma}_l(h) \in l(S^h, r^h)$ for any $h \in H$ at which the leader moves and let Σ_l be the space of all $\tilde{\sigma}_l$ strategies. Let $\tilde{\sigma}_i$ be the strategy of member $i \in N$ such that, for each history $h \in H$ at which the leader approaches i with an offer t, $\tilde{\sigma}_i(h) =$ accept if and only if $t \geq c(S^h, r^h, i)$. Let $\tilde{\sigma} = (\tilde{\sigma}_l, (\tilde{\sigma}_i)_{i\in N})$ be a profile of strategies constructed from the l and c functions and $\Sigma = \Sigma_l \times (\times_{i\in N}\{\tilde{\sigma}_i\})$ be the space of all $\tilde{\sigma}$ profiles.

States $(S, r) \in \mathcal{D}^W$ with $r \leq 0$. For any $(S, r) \in \mathcal{D}^W$ with $r \leq 0$ and any $i \in S$, set $l(S, r) = \{\text{initiate a vote}\}$ and $c(S, r, i) = 0$. Fix $(S, r) \in \mathcal{D}^W$ with $r \leq 0$. $\Gamma(S, r)$ admits unique equilibrium in which the principal initiates a vote at any history in which she moves and any member accepts any transfer at any history in which he moves. Therefore, a profile σ constitutes an equilibrium in $\Gamma(S, r)$ only if $\sigma \in \Sigma$. Moreover, any $\tilde{\sigma} \in \Sigma$ constitutes an equilibrium in $\Gamma(S, r)$ and thus the equilibrium payoff from (S, r) is y for the principal and $-x_i$ for any member $i \in N$.

States $(S, r) \in \mathcal{D}^W$ with $r > |S|$. For any $(S, r) \in \mathcal{D}^W$ with $r > |S|$ and any $i \in S$, set $l(S, r) = \{\text{initiate a vote}, \text{stop}\} \cup (S \times \{0\})$ and $c(S, r, i) = 0$. Fix $(S, r) \in \mathcal{D}^W$ with $r > |S|$. In any equilibrium of $\Gamma(S, r)$ the policy does not pass, hence any approached member accepts any offered transfer and thus the leader never offers strictly positive transfer to any member. Therefore, a profile σ constitutes an equilibrium in $\Gamma(S, r)$ only if $\sigma \in \Sigma$. Moreover, any $\tilde{\sigma} \in \Sigma$ constitutes an equilibrium in $\Gamma(S, r)$ and thus the equilibrium payoff from (S, r) is 0 for the leader and for any member $i \in N$.

Initial induction step. We now prove Proposition A1 for any $(S, r) \in \mathcal{D}^G$ with $|S| = 1$. Notice that $(S, r) \in \mathcal{D}^G$ and $|S| = 1$ implies $r = 1$ and thus $W(S, r) = \Pi(S, r, y) = x_{(1)}$. For any $(S, r) \in \mathcal{D}^G$ with $|S| = 1$ and any $i \in S$, set $l(S, r) = \{(i, \delta x_i)\}$ if $y > x_i$, $l(S, r) = \{\text{initiate a vote}, \text{stop}, (i, 0)\}$ if $y < x_i$ and $c(S, r, i) = \delta x_i$. Fix $(S, r) \in \mathcal{D}^G$ with $|S| = 1$ and let $S = \{i\}$. In any equilibrium of $\Gamma(S, r)$, if the leader in (S, r) approaches i with transfer t, then i's payoff from rejection is 0, because the game moves to state $(\varnothing, 1)$, and i's payoff from accepting is $t - \delta x_i$, because the game moves to state $(\varnothing, 0)$. Hence, if approached in (S, r), i accepts t if and only if $t \geq \delta x_i$. Note that i is indispensable in (S, r) because $W(\varnothing, 0) = 0 < y < W(\varnothing, 1) = \infty$. For the leader, thus, the payoff in (S, r) from initiating a vote or stoping is 0 and the payoff from approaching member i with an offer t is 0 if $t \in [0, \delta x_i)$ and is $\delta y - t$ if $t \geq \delta x_i$. Hence, the leader in (S, r) approaches i with offer δx_i if $y > x_i$ and either initiates a vote or stops or approaches i with offer 0 if $y < x_i$. Therefore, a profile σ constitutes an equilibrium in $\Gamma(S, r)$ only if $\sigma \in \Sigma$. Moreover, any $\tilde{\sigma} \in \Sigma$ constitutes an equilibrium in $\Gamma(S, r)$ and thus the equilibrium payoff from (S, r) is $\delta(y - x_i)$ for the leader, 0 for member i and $-x_j$ for any member $j \in N \setminus \{i\}$ if $y > x_i$ and is 0 for the leader and for any member $j \in N$ if $y < x_i$. Because $y \notin \mathcal{L}$ and $x_i \in \mathcal{L}$, we have $y \neq x_i$ and this concludes the proof of Proposition A1 for all states $(S, r) \in \mathcal{D}^G$ with $|S| = 1$.

Induction step from k to $k+1$. Assume Proposition A1 holds for all states $(S,r) \in \mathcal{D}^G$ with $|S| \leq k$, where $k \geq 1$. We now prove Proposition A1 for any $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$. Fix $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$.

Part 2: In any equilibrium of $\Gamma(S,r)$, suppose the leader in (S,r) approaches $i \in S$ with transfer t. Let v_a be i's payoff from accepting and v_r be i's payoff from rejecting. There are three cases to consider.

Case 1: $r = 1$. If $r = 1$, we have $v_a = t - \delta x_i$, because the game proceeds to state $(S \setminus \{i\}, r-1)$, and, by the induction hypothesis, $v_r = 0$ if $y < W(S \setminus \{i\}, r)$ and $v_r = -\delta^2 x_i$ if $y > W(S \setminus \{i\}, r)$. In the former case, i accepts the leader's offer t if and only if $t \geq \delta x_i$ and in the latter case i accepts the leader's offer t if and only if $t \geq \delta(1-\delta)x_i$. Note that, because $W(S \setminus \{i\}, r-1) = 0$ when $r = 1$, in the former case i is indispensable in (S,r) and in the latter case i is dispensable in (S,r).

Case 2: $r = |S|$. If $r = |S|$, we have $v_r = 0$, because the game proceeds to state $(S \setminus \{i\}, r)$, and, by the induction hypothesis, $v_a = t$ if $y < W(S \setminus \{i\}, r-1)$ and $v_a = t - \delta^r x_i$ if $y > W(S \setminus \{i\}, r-1)$. In the former case i accepts the leader's offer t if and only if $t \geq 0$ and in the latter case i accepts the leader's offer t if and only if $t \geq \delta^r x_i$. Note that, because $W(S \setminus \{i\}, r) = \infty$ when $r = |S|$, in the former case i is inconsequential in (S,r) and in the latter case i is indispensable in (S,r).

Case 3: $r \in \{2, \ldots, |S|-1\}$. Because $y \notin \mathcal{L}$, because $W(S \setminus \{i\}, r-1), W(S \setminus \{i\}, r) \in \mathcal{L}$ by Lemma A1 part 3 and because $W(S \setminus \{i\}, r-1) \leq W(S \setminus \{i\}, r)$ by Lemma A1 part 4, there are three cases to consider: in (S,r), i is either inconsequential, when $y < W(S \setminus \{i\}, r-1)$, or indispensable, when $y \in (W(S \setminus \{i\}, r-1), W(S \setminus \{i\}, r))$, or dispensable, when $y > W(S \setminus \{i\}, r)$. By the induction hypothesis, if i is inconsequential we have $v_a = t$ and $v_r = 0$ and i accepts the leader's offer t if and only if $t \geq 0$, if i is indispensable we have $v_a = t - \delta^r x_i$ and $v_r = 0$ and i accepts the leader's offer t if and only if $t \geq \delta^r x_i$, and if i is dispensable we have $v_a = t - \delta^r x_i$ and $v_r = -\delta^{r+1} x_i$ and i accepts the leader's offer t if and only if $t \geq \delta^r(1-\delta)x_i$.

For any $i \in S$, set $c(S,r,i) = 0$ if i is inconsequential in (S,r), set $c(S,r,i) = \delta^r x_i$ if i is indispensable in (S,r) and set $c(S,r,i) = \delta^r(1-\delta)x_i$ if i is dispensable in (S,r).

Parts 1 and 1a (part 1b follows from parts 4a, 3a and 3b we prove below): By construction, a profile σ constitutes an equilibrium in any proper subgame of $\Gamma(S,r)$ if and only if $\sigma \in \Sigma$. It thus suffices to prove that, for any $\tilde{\sigma} \in \Sigma$, the leader has an optimal action at the initial history of $\Gamma(S,r)$ when her payoff in proper subgames of $\Gamma(S,r)$ is determined by $\tilde{\sigma}$, and that the payoff the optimal action provides to the leader is independent of $\tilde{\sigma}$. That the payoff the optimal action provides to the leader is independent of $\tilde{\sigma}$ follows from the induction hypothesis; the leader's payoff from states $(S \setminus \{i\}, r)$ and $(S \setminus \{i\}, r-1)$ is, for any $i \in S$, constant across equilibria and the payoff from initiating a vote or stopping is 0.

To shows that the leader has an optimal action at the initial history of $\Gamma(S,r)$, fix $\tilde{\sigma} \in$

Σ and, $\forall i \in S$, let a_i be the leader's equilibrium payoff from $(S\setminus\{i\}, r-1)$ and let r_i be the leader's equilibrium payoff from $(S\setminus\{i\}, r)$. Let $A = \{\text{initiate a vote}, \text{stop}\} \cup (S \times \mathbb{R}_+)$ be the leader's action space at the initial history of $\Gamma(S, r)$ and let $v : A \to \mathbb{R}$ be the leader's payoff function at the initial history of $\Gamma(S, r)$. Clearly, $v(\text{initiate a vote}) = v(\text{stop}) = 0$. For any $(i, t) \in S \times \mathbb{R}_+$, we have

$$v(i,t) = \begin{cases} \delta r_i & \text{if } t < c(S,r,i) \\ \delta a_i - t & \text{if } t \geq c(S,r,i) \end{cases}. \tag{A1}$$

The leader's payoff maximization problem reads $\max_{a\in A} v(a)$. For any $i \in S$, $\max_{t\in\mathbb{R}_+} v(i,t)$ has a solution and we can set $v_i = \max_{t\in\mathbb{R}_+} v(i,t)$. Because $\max_{a\in A} v(a)$ is equivalent to $\max\{0, \max_{i\in S}\{v_i\}\}$ and because the latter problem is finite and hence admits a solution, the former problem admits a solution as well. Set $l(S,r) = \arg\max_{a\in A} v(a)$. By construction, σ constitutes an equilibrium in $\Gamma(S,r)$ if and only if $\sigma \in \Sigma$.

Part 4a: Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes and suppose, towards a contradiction, that on the equilibrium path the leader approaches $r+1$ or more members. If the member i approached in (S,r) accepts the leader's offer then the game proceeds into state $(S\setminus\{i\}, r-1)$ in which, by the induction hypothesis, the leader on the equilibrium path approaches $r-1$ members. Hence, it must be the case that the leader in (S,r) approaches $i_0 \in S$ with an offer $t_0 < c(S,r,i_0)$, the game proceeds to $(S\setminus\{i_0\}, r)$ and starting from $(S\setminus\{i_0\}, r)$ the leader's equilibrium sequence of actions is $(i_a, t_a)_{a=1}^r$ along which all approached members accept. For $a \in \{1,\ldots,r\}$, let $S_a = S \setminus \cup_{c=0}^{a-1}\{i_c\}$. Then the equilibrium sequence of states is $(S_a, r+1-a)_{a=1}^r$ and, $\forall a \in \{1,\ldots,r\}$, i_a is offered t_a in state $(S_a, r+1-a)$ and hence, because i_a accepts t_a, we have $t_a = c(S_a, r+1-a, i_a)$. Because the policy passes in $\tilde{\sigma}$, all members in $(i_a)_{a=1}^r$, when approached, are either indispensable or dispensable. Let $T = \{i_a | a \in \{1,\ldots,r\}, y < W(S_a\setminus\{i_a\}, r+1-a)\}$ be the set of indispensable members approached and let $E = \{i_a | a \in \{1,\ldots,r\}, y > W(S_a\setminus\{i_a\}, r+1-a)\}$ be the set of dispensable members approached. By construction, $\forall a \in \{1,\ldots,r\}$, we have $c(S_a, r+1-a, i_a) = \delta^{r+1-a}x_{i_a}$ if $i_a \in T$ and $c(S_a, r+1-a, i_a) = \delta^{r+1-a}(1-\delta)x_{i_a}$ if $i_a \in E$. The leader's equilibrium payoff from (S,r) under $\tilde{\sigma}$ is thus

$$\begin{aligned}&\delta^{r+1}y - \sum_{a\in\{1,\ldots,r\}, i_a\in T} \delta^a\delta^{r+1-a}x_{i_a} - \sum_{a\in\{1,\ldots,r\}, i_a\in E} \delta^a\delta^{r+1-a}(1-\delta)x_{i_a} = \\ &\delta^{r+1}\left(y - \sum\nolimits_{i\in T} x_i - (1-\delta)\sum\nolimits_{i\in E} x_i\right).\end{aligned} \tag{A2}$$

Because $y \notin \mathcal{L}$, we have $y \neq \sum_{i\in T} x_i$ and hence, because $\tilde{\sigma}$ constitutes an equilibrium, $y > \sum_{i\in T} x_i$. Because $\delta > \bar{\delta} \geq \bar{\delta}_a$, we thus have $y - \sum_{i\in T} x_i - (1-\delta)\sum_{i\in E} x_i > 0$.

We now construct $\tilde{\sigma}'$ that constitutes a profitable deviation for the leader. For $a \in \{1,\ldots,r\}$, let $S_a^\circ = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and note that $S_a^\circ \setminus \{i_0\} = S_a$. Let $\tilde{\sigma}'$ be identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1, \ldots, r\}$, approaches member i_a with an offer t_a at any history that corresponds to state $(S_a^\circ, r+1-a)$. We now argue that, $\forall a \in \{1, \ldots, r\}$, member i_a offered t_a in state $(S_a^\circ, r+1-a)$ accepts. For any $i_a \in T$ this is immediate because we have $c(S_a, r+1-a, i_a) = \delta^{r+1-a} x_{i_a} \geq c(S_a^\circ, r+1-a, i_a)$. For any $i_a \in E$, we have $y > W(S_a \setminus \{i_a\}, r+1-a) = W(S_a^\circ \setminus \{i_0, i_a\}, r+1-a) \geq W(S_a^\circ \setminus \{i_a\}, r+1-a)$, where the weak inequality follows by Lemma A1 part 5, and hence $c(S_a^\circ, r+1-a, i_a) = \delta^{r+1-a}(1-\delta) x_{i_a} = c(S_a, r+1-a, i_a)$. Because, $\forall a \in \{1, \ldots, r\}$, member i_a offered t_a in state $(S_a^\circ, r+1-a)$ accepts, the leader's payoff from (S, r) under $\tilde{\sigma}'$ is $\delta^r(y - \sum_{i \in T} x_i - (1-\delta)\sum_{i \in E} x_i) > \delta^{r+1}(y - \sum_{i \in T} x_i - (1-\delta)\sum_{i \in E} x_i)$, which establishes the desired contradiction.

Part 4b: it suffices to prove that given any $\tilde{\sigma} \in \Sigma$ in which the policy passes, if there is a dispensable member in (S, r), then the leader at the initial history of $\Gamma(S, r)$ approaches a dispensable member i with an offer that i accepts and that there is a dispensable member in the state the game proceeds to, in $(S \setminus \{i\}, r-1)$. The see the latter claim, if $i \in S$ is dispensable in (S, r) we have $y > W(S \setminus \{i\}, r)$, by Lemma A1 part 6 we have, $\forall j \in S \setminus \{i\}$, $W(S \setminus \{i\}, r) \geq W(S \setminus \{i, j\}, r-1)$, and thus any $j \in S \setminus \{i\}$ is dispensable in $(S \setminus \{i\}, r-1)$. To see the former claim, fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Let $E = \{i \in S | y > W(S \setminus \{i\}, r)\}$, $T = \{i \in S | y \in (W(S \setminus \{i\}, r-1), W(S \setminus \{i\}, r))\}$ and $Z = \{i \in S | y < W(S \setminus \{i\}, r-1)\}$ be, respectively, the set of dispensable, indispensable and inconsequential members in (S, r). The leader in equilibrium does not approach $i \in Z$ at the initial history of $\Gamma(S, r)$ because the policy passes in $\tilde{\sigma}$. Suppose, towards a contradiction, that the leader in equilibrium at the initial history of $\Gamma(S, r)$ approaches $i_0 \in T$ and $E \neq \varnothing$. By part 4a, the leader in equilibrium approaches i_0 with offer t_0 that i_0 accepts, that is, with $t_0 = c(S, r, i_0) = \delta^r x_{i_0}$, where the second equality follows from $i_0 \in T$, and hence the leader's equilibrium payoff from (S, r) is at most $\delta^r(y - x_{i_0})$.

Let $(i_a)_{a=1}^r$ be a sequence of members and, $\forall a \in \{1, \ldots, r\}$, let $S_a = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and $t_a = \delta^{r+1-a}(1-\delta) x_{i_a}$. Suppose that the sequence of members $(i_a)_{a=1}^r$ is such that $i_1 \in E$ and $i_a \in S_a$ for any $a \in \{2, \ldots, r\}$. Consider $\tilde{\sigma}'$ identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1, \ldots, r\}$, approaches member i_a with an offer t_a at any history that corresponds to state $(S_a, r+1-a)$. We now argue that, $\forall a \in \{1, \ldots, r\}$, member i_a offered t_a in state $(S_a, r+1-a)$ accepts. To see this, it suffices to argue that, $\forall a \in \{1, \ldots, r\}$, i_a is dispensable in state $(S_a, r+1-a)$. By construction, $i_1 \in E$ and hence i_1 is dispensable in $(S, r) = (S_1, r)$. Note that because i_1 is dispensable in (S, r), we have $y > W(S \setminus \{i_1\}, r)$. By Lemma A1 part 6, we have, $\forall a \in \{2, \ldots, r\}$, $W(S \setminus \{i_1\}, r) \geq W(S \setminus \cup_{c=1}^a \{i_c\}, r+1-a) = W(S_a \setminus \{i_a\}, r+1-a)$. Therefore, $\forall a \in \{2, \ldots, r\}$, i_a is dispensable in $(S_a, r+1-a)$. Because the leader approaches all members in the $(i_a)_{a=1}^r$ sequence with offers the members accept, her payoff from $\tilde{\sigma}'$ is $\delta^r y - \sum_{a \in \{1, \ldots, r\}} \delta^{a-1} \delta^{r+1-a}(1-\delta) x_{i_a} = \delta^r(y - (1-\delta)\sum_{a \in \{1, \ldots, r\}} x_{i_a})$. Because $\delta > \bar{\delta} \geq \bar{\delta}_b$, we have $x_{i_0} \geq x_1 > n(1-\delta) x_n \geq (1-\delta)\sum_{a \in \{1, \ldots, r\}} x_{i_a}$ and thus $y - x_{i_0} < y - (1-\delta)\sum_{a \in \{1, \ldots, r\}} x_{i_a}$, which establishes the desired contradiction.

Part 4c: Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Let $(i_a)_{a=1}^r$ be the equilibrium sequence of approached members. By part 4a, this sequence consists of r members. For $a \in \{1, \ldots, r\}$, let $S_a = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and $t_a = c(S_a, r+1-a, i_a)$. On the equilibrium path, $\forall a \in \{1, \ldots, r\}$, the leader approaches i_a with an offer t_a in state $(S_a, r+1-a)$ and i_a accepts t_a. Let $T = \{i_a | a \in \{1, \ldots, r\}, y < W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set indispensable members approached and let $E = \{i_a | a \in \{1, \ldots, r\}, y > W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set of dispensable members approached. By part 2, $t_a = \delta^{r+1-a} x_{i_a}$ for any $i_a \in T$ and $t_a = \delta^{r+1-a}(1-\delta) x_{i_a}$ for any $i_a \in E$. The leader's equilibrium payoff from $\tilde{\sigma}$ is thus $\delta^r(y - \sum_{i \in T} x_i - (1-\delta)\sum_{i \in E} x_i)$.

When $r = |S|$, we have $T = S$ because, $\forall a \in \{1, \ldots, r\}$, $W(S_a \setminus \{i_a\}, r+1-a) = \infty$. By Lemma A2, S is a solution to (2) given (S, r). Hence, suppose $r < |S|$. Clearly, $T \in 2^S$ and we argue that $y > W((S \setminus T)', r - |T|)$. We have either $|T| = r$ or $|T| < r$. In the former case, $W((S \setminus T)', r - |T|) = 0$. In the latter case, for the first dispensable member approached, $i_{|T|+1}$, we have $y > W(S_{|T|+1} \setminus \{i_{|T|+1}\}, r - |T|)$. By part 4b, all indispensable members are approached before any dispensable member is approached and hence $S_{|T|+1} = S \setminus T$. Because $i_{|T|+1}$ is approached in state $(S_{|T|+1}, r - |T|) = (S \setminus T, r - |T|)$, we have $i_{|T|+1} \leq \max S \setminus T$ and hence, by Lemma A1 part 5, $W((S \setminus T) \setminus \{i_{|T|+1}\}, r - |T|) \geq W((S \setminus T) \setminus \{\max S \setminus T\}, r - |T|)$. Thus $y > W((S \setminus T)', r - |T|)$. Because $T \in 2^S$ and $y > W((S \setminus T)', r - |T|)$, it suffices to prove that $T_\circ \in 2^S$ such that $\sum_{i \in T} x_i > \sum_{i \in T_\circ} x_i$ and $y > W((S \setminus T_\circ)', r - |T_\circ|))$ does not exist.

Suppose, towards a contradiction, that $T_\circ \in 2^S$ such that $\sum_{i \in T} x_i > \sum_{i \in T_\circ} x_i$ and $y > W((S \setminus T_\circ)', r - |T_\circ|))$ exists. If $|T_\circ| > r$, then any $T_b \subseteq T_\circ$ such that $|T_b| = r$ satisfies $\sum_{i \in T_\circ} x_i > \sum_{i \in T_b} x_i$ and $W((S \setminus T_\circ)', r - |T_\circ|)) = W((S \setminus T_b)', r - |T_b|))$ and hence it is without loss of generality to assume that $|T_\circ| \leq r$. Let $(i_a^\circ)_{a=1}^r$ be a sequence of members and, $\forall a \in \{1, \ldots, r\}$, let $S_a^\circ = S \setminus \cup_{c=1}^{a-1}\{i_c^\circ\}$. Suppose that the sequence of members $(i_a^\circ)_{a=1}^r$ is such that $i_a^\circ \in T_\circ$ $\forall a \in \{1, \ldots, |T_\circ|\}$, $i_{|T_\circ|+1}^\circ = \max S \setminus T_\circ$, and $i_a^\circ \in S_a^\circ$ $\forall a \in \{|T_\circ|+2, \ldots, r\}$. Let $E_\circ = \{i_a^\circ | a \in \{|T_\circ|+1, \ldots, r\}\}$. Let $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}$ for any $i_a^\circ \in T_\circ$ and $t_a^\circ = \delta^{r+1-a}(1-\delta) x_{i_a^\circ}$ for any $i_a^\circ \in E_\circ$. Consider $\tilde{\sigma}'$ identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1, \ldots, r\}$, approaches member i_a° with an offer t_a° at any history that corresponds to state $(S_a^\circ, r+1-a)$. We now argue that, $\forall a \in \{1, \ldots, r\}$, member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$ accepts. To see this, for any $i_a^\circ \in T_\circ$ we have $t_a^\circ = \delta^{r+1-a} x_{i_a^\circ}$ and hence member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$ accepts. To see that any $i_a^\circ \in E_\circ$ accepts t_a° in $(S_a^\circ, r+1-a)$, it suffices to argue that any $i_a^\circ \in E_\circ$ is dispensable in $(S_a^\circ, r+1-a)$. Member $i_{|T_\circ|+1}^\circ = \max S \setminus T_\circ$ is approached in $(S_{|T_\circ|+1}^\circ, r - |T_\circ|) = (S \setminus T_\circ, r - |T_\circ|)$ and we have $y > W((S \setminus T_\circ)', r - |T_\circ|)$. Thus $i_{|T_\circ|+1}^\circ$ is dispensable in $(S_{|T_\circ|+1}^\circ, r - |T_\circ|)$. Because $i_{|T_\circ|+1}^\circ$ is dispensable in $(S_{|T_\circ|+1}^\circ, r - |T_\circ|)$, we have $y > W(S \setminus \cup_{c=1}^{|T_\circ|+1}\{i_c^\circ\}, r+1-(|T_\circ|+1))$. By Lemma A1 part 6, we have, $\forall a \in \{|T_\circ|+2, \ldots, r\}$, $W(S \setminus \cup_{c=1}^{|T_\circ|+1}\{i_c^\circ\}, r+1-(|T_\circ|+1)) \geq W(S \setminus \cup_{c=1}^{a}\{i_c^\circ\}, r+1-a)$. Thus, $\forall a \in \{|T_\circ|+2, \ldots, r\}$,

$y > W(S \setminus \cup_{c=1}^{a}\{i_c^\circ\}, r+1-a)$ and thus i_a° is dispensable in $(S_a^\circ, r+1-a)$. Because the leader approaches all members in the $(i_a^\circ)_{a=1}^r$ sequence with offers the members accept, her payoff from $\tilde{\sigma}'$ is $\delta^r(y - \sum_{i\in T_\circ} x_i - (1-\delta)\sum_{i\in E_\circ} x_i)$. Because $\delta > \bar{\delta} \geq \bar{\delta}_c$, we have $\sum_{i\in T_\circ} x_i + (1-\delta)\sum_{i\in E_\circ} x_i \leq \sum_{i\in T_\circ} x_i + n(1-\delta)x_n < \sum_{i\in T} x_i \leq \sum_{i\in T} x_i + (1-\delta)\sum_{i\in E} x_i$ and thus $y - \sum_{i\in T} x_i - (1-\delta)\sum_{i\in E} x_i < y - \sum_{i\in T_\circ} x_i - (1-\delta)\sum_{i\in E_\circ} x_i$, which establishes the desired contradiction.

Part 4d: Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Let $(i_a)_{a=1}^r$ be the equilibrium sequence of approached members. By part 4a, this sequence consists of r members. For $a \in \{1,\ldots,r\}$, let $S_a = S \setminus \cup_{c=1}^{a-1}\{i_c\}$ and $t_a = c(S_a, r+1-a, i_a)$. On the equilibrium path, $\forall a \in \{1,\ldots,r\}$, the leader approaches i_a with an offer t_a in state $(S_a, r+1-a)$ and i_a accepts t_a. Let $T = \{i_a | a \in \{1,\ldots,r\}, y < W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set indispensable members approached and let $E = \{i_a | a \in \{1,\ldots,r\}, y > W(S_a \setminus \{i_a\}, r+1-a)\}$ be the set of dispensable members approached. By part 2, $t_a = \delta^{r+1-a} x_{i_a}$ for any $i_a \in T$ and $t_a = \delta^{r+1-a}(1-\delta)x_{i_a}$ for any $i_a \in E$. The leader's equilibrium payoff from $\tilde{\sigma}$ is thus $\delta^r(y - \sum_{i\in T} x_i - (1-\delta)\sum_{i\in E} x_i)$. By part 4c, T is a solution to (2) given (S,r) and hence $\sum_{i\in T} x_i = \Pi(S,r,y)$. We thus have $l(\delta) \leq \delta^r(y - \sum_{i\in T} x_i - (1-\delta)\sum_{i\in E} x_i) \leq u(\delta)$, where

$$\begin{aligned} l(\delta) &= \delta^r(y - \Pi(S,r,y) - r(1-\delta)x_n) \\ u(\delta) &= y - \Pi(S,r,y). \end{aligned} \tag{A3}$$

The result now follows because $\delta > \bar{\delta} \geq \bar{\delta}_d$ implies that $u(\delta) - l(\delta) \leq \epsilon$.

Part 4e: Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Suppose the leader in the equilibrium approaches members i_1 and i_2 in the first two periods with offers t_1 and t_2 respectively. It suffices to prove two claims. First, we claim that if $t_1 = \delta^r x_{i_1}$ and $t_2 = \delta^{r-1}x_{i_2}$, then $\tilde{\sigma}' \in \Sigma$, where $\tilde{\sigma}'$ is identical to $\tilde{\sigma}$ except that the leader approaches member i_2 with $t_2' = \delta^r x_{i_2}$ at any history that corresponds to state (S,r) and approaches member i_1 with $t_1' = \delta^{r-1}x_{i_1}$ at any history that corresponds to state $(S \setminus \{i_2\}, r-1)$. To see that $\tilde{\sigma}' \in \Sigma$, note that $t_2' \geq c(S,r,i_2)$ and $t_1' \geq c(S \setminus \{i_2\}, r-1, i_1)$. Thus the leader's payoff from $\tilde{\sigma}'$ is the same as her payoff from $\tilde{\sigma}$ and, hence, $(i_2, t_2') \in l(S,r)$. The second claim is identical in that it swaps members i_1 and i_2, except that it replaces $t_1 = \delta^r(1-\delta)x_{i_1}$ and $t_2 = \delta^{r-1}(1-\delta)x_{i_2}$ by $t_1' = \delta^{r-1}(1-\delta)x_{i_1}$ and $t_2' = \delta^r(1-\delta)x_{i_2}$, and adds an additional constraint that member i_2 is dispensable in (S,r). The proof of this claim is similar and omitted.

Part 3a: Suppose, towards a contradiction, that $y > W(S,r)$ and the policy does not pass in some $\tilde{\sigma} \in \Sigma$. Fix $\tilde{\sigma} \in \Sigma$ in which the policy does not pass. The leader's payoff from $\tilde{\sigma}$ is thus 0. If $r = |S|$, set $T = S$. Because $W(S,r) = \sum_{i\in S} x_i$ when $r = |S|$, $y > W(S,r)$ implies $y > \sum_{i\in S} x_i$. If $r < |S|$, let $T \in 2^S$ be a solution to (1) given (S,r). Thus $W(S,r) = \max\{\sum_{i\in T} x_i, W((S\setminus T)', r - |T|)\}$ so that $y > W(S,r)$ implies $y > \sum_{i\in T} x_i$ and $y > W((S \setminus T)', r - |T|)$.

Let $(i_a^\circ)_{a=1}^r$ be a sequence of members and, $\forall a \in \{1,\dots,r\}$, let $S_a^\circ = S \setminus \cup_{c=1}^{a-1}\{i_c^\circ\}$. Suppose that the sequence of members $(i_a^\circ)_{a=1}^r$ is such that $i_a^\circ \in T$ $\forall a \in \{1,\dots,|T|\}$, $i_{|T|+1}^\circ = \max S \setminus T$, and $i_a^\circ \in S_a^\circ$ $\forall a \in \{|T|+2,\dots,r\}$. Let $E = \{i_a^\circ | a \in \{|T|+1,\dots,r\}\}$. Let $t_a^\circ = \delta^{r+1-a}x_{i_a^\circ}$ for any $i_a^\circ \in T$ and $t_a^\circ = \delta^{r+1-a}(1-\delta)x_{i_a^\circ}$ for any $i_a^\circ \in E$. Consider $\tilde{\sigma}'$ identical to $\tilde{\sigma}$ except that the leader, $\forall a \in \{1,\dots,r\}$, approaches member i_a° with an offer t_a° at any history that corresponds to state $(S_a^\circ, r+1-a)$. We now argue that, $\forall a \in \{1,\dots,r\}$, member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$ accepts. To see this, for any $i_a^\circ \in T$ we have $t_a^\circ = \delta^{r+1-a}x_{i_a^\circ}$ and hence member i_a° offered t_a° in state $(S_a^\circ, r+1-a)$ accepts. To see that any $i_a^\circ \in E$ accepts t_a° in $(S_a^\circ, r+1-a)$, it suffices to argue that any $i_a^\circ \in E$ is dispensable in $(S_a^\circ, r+1-a)$. Member $i_{|T|+1}^\circ = \max S \setminus T$ is approached in $(S_{|T|+1}^\circ, r-|T|) = (S\setminus T, r-|T|)$ when, by $y > W((S\setminus T)', r-|T|)$, dispensable. Because $i_{|T|+1}^\circ$ is dispensable in $(S_{|T|+1}^\circ, r-|T|)$, we have $y > W(S\setminus \cup_{c=1}^{|T|+1}\{i_c^\circ\}, r+1-(|T|+1))$. By Lemma A1 part 6, we have, $\forall a \in \{|T|+2,\dots,r\}$, $W(S\setminus \cup_{c=1}^{|T|+1}\{i_c^\circ\}, r+1-(|T|+1)) \geq W(S\setminus \cup_{c=1}^{a}\{i_c^\circ\}, r+1-a)$. Thus, $\forall a \in \{|T|+2,\dots,r\}$, $y > W(S\setminus \cup_{c=1}^{a}\{i_c^\circ\}, r+1-a)$ and thus i_a° is dispensable in $(S_a^\circ, r+1-a)$. Because the leader approaches all members in the $(i_a^\circ)_{a=1}^r$ sequence with offers the members accept, her payoff from $\tilde{\sigma}'$ is $\delta^r(y - \sum_{i\in T} x_i - (1-\delta)\sum_{i\in E} x_i)$. Because $\delta > \bar{\delta} \geq \bar{\delta}_a$ and $y > \sum_{i\in T} x_i$, we have $0 < y - \sum_{i\in T} x_i - n(1-\delta)x_n \leq y - \sum_{i\in T} x_i - (1-\delta)\sum_{i\in E} x_i$, which establishes the desired contradiction.

Part 3b: Suppose, towards a contradiction, that $y < W(S,r)$ and the policy passes in some $\tilde{\sigma} \in \Sigma$. Fix $\tilde{\sigma} \in \Sigma$ in which the policy passes. Suppose $r = |S|$. By part 4c and Lemma A2, all members in S are approached on the equilibrium path and each member is approached in a state in which he is indispensable. The leader's payoff from $\tilde{\sigma}$ is thus $\delta^r(y - \sum_{i\in S} x_i) < 0$, where the inequality follows from $y < W(S,r)$, because $W(S,r) = \sum_{i\in S} x_i$ when $r = |S|$. The leader thus has a profitable deviation to stop at the initial history of $\Gamma(S,r)$, a contradiction.

Suppose $r < |S|$ and let T be the set of members approached on the equilibrium path when indispensable. By parts 4a and 2, the leader's equilibrium payoff is at most $\delta^r(y - \sum_{i\in T} x_i)$. From $y \notin \mathcal{L}$, we have $y \neq \sum_{i\in T} x_i$ and from $\tilde{\sigma} \in \Sigma$, we have $y - \sum_{i\in T} x_i \geq 0$. Thus $y - \sum_{i\in T} x_i > 0$. Moreover, by part 4c, T solves (2) given (S,r) and hence $y > W((S\setminus T)', r-|T|)$. Thus $y > \max\{\sum_{i\in T} x_i, W((S\setminus T)', r-|T|)\} \geq W(S,r)$, which establishes the desired contradiction.

This concludes the proof of the induction step from k to $k+1$, for a given $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$. Repeating the same step for all states $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$ completes the specification of the l and c functions that define the space of strategies Σ. ∎

Proof of Proposition 5. We first prove the following lemma.

Lemma A3. *Consider state $(S,r) \in \mathcal{D}^G$. $W(S,r) \leq \sum_{i=1}^r x_{(i)}$.*

Proof. We proceed by induction on $|S|$. That $W(S,r) \leq \sum_{i=1}^{r} x_{(i)}$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$ follows because, as shown in Lemma A1 part 2, $W(S,r) = x_{(1)}$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$. Now suppose that $W(S,r) \leq \sum_{i=1}^{r} x_{(i)}$ for any $(S,r) \in \mathcal{D}^G$ with $|S| \leq k$, where $k \geq 1$. We need to prove that $W(S,r) \leq \sum_{i=1}^{r} x_{(i)}$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$. To see this, we have either (i) $r = |S|$, in which case $W(S,r) = \sum_{i\in S} x_i$ by Lemma A1 part 2, or (ii) $r < |S|$, in which case $W(S,r) \leq W(S',r) \leq \sum_{i=1}^{r} x_{(i)}$, where the first inequality follows from (1) evaluated at $T = \varnothing$, and the second inequality follows from the induction hypothesis. ∎

Consider generic y, sufficiently large δ and assume that $x_i < x_{i+1}$ for all i. Note that $W(N,q) \leq \sum_{i\in N^q} x_i$ by Lemma A3. Thus there are three cases to consider.

Case 1: $y < W(N,q)$. Because $y < W(N,q)$, the policy does not pass in any equilibrium of the up-front-payment game by Proposition 3, and because $y < \sum_{i\in N^q} x_i$, the policy does not pass in any equilibrium of the transfer-promise game by Proposition 1. Thus the equilibrium payoff of all players is zero in both games.

Case 2: $y \in (W(N,q), \sum_{i\in N^q} x_i)$. By Propositions 1 and 3, the policy passes in any equilibrium of the up-front-payment game and does not pass in any equilibrium of the transfer-promise game. By Proposition 4, the equilibrium payoff with up-front payments is non-negative for the leader and is non-positive for each member. The equilibrium payoff with transfer promises is zero for all players.

Case 3: $y > \sum_{i\in N^q} x_i$. By Propositions 1 and 3, the policy passes in any equilibrium for both kinds of transfers. If $q = n$, then the equilibrium payoff, for both kinds of transfers, is $\delta^n(y - \sum_{i\in N} x_i)$ for the leader and is zero for each member. Hence suppose $q < n$.

When $q < n$, we have either $y \in (\sum_{i=1}^{q-1} x_i + x_q, \sum_{i=1}^{q-1} x_i + x_{q+1})$ or $y > \sum_{i=1}^{q-1} x_i + x_{q+1}$. In the latter case, by Proposition 2, the equilibrium payoff with transfer promises is $\delta^q(y - (1-\delta)\sum_{i=1}^{q} x_i)$ for the leader, is $\delta^q(-x_i + (1-\delta)x_i) = -\delta^{q+1}x_i$ for each member $i \in \{1,\ldots,q\}$, and is $-\delta^q x_i$ for each member $i \in \{q+1,\ldots,n\}$. We claim that all players receive the same equilibrium payoffs in the up-front-payment game. To see this, we show that member q is dispensable in state (N,q), which implies, by Proposition 4, that in any equilibrium of the up-front-payment game no member is tempted and members $\{1,\ldots,q\}$ are exploited. Member q is dispensable in state (N,q) if $y > W(N\setminus\{q\},q)$, which holds because $y > \sum_{i=1}^{q-1} x_i + x_{q+1}$ and, by Lemma A3, $\sum_{i=1}^{q-1} x_i + x_{q+1} \geq W(N\setminus\{q\},q)$. The former case is similar and we omit the details, noting that the set of members exploited in equilibrium is $\{1,\ldots,q-1,q+1\}$ and that member $q+1$ is dispensable in state (N,q) because $y > \sum_{i=1}^{q-1} x_i + x_q$ and, by Lemma A3, $\sum_{i=1}^{q-1} x_i + x_q \geq W(N\setminus\{q+1\},q)$. ∎

Proof of Proposition 7. Let $\mathbf{t} = (t_1,\ldots,t_n) \in \mathbb{R}_+^n$ be the profile of transfer promises that the leader makes and $\mathbf{a} = (a_1,\ldots,a_n)$ be the members' action profile where $a_i = 0/1$ indicates rejection/acceptance.

Consider the leader's decision in the second period, that is, in a subgame starting with $\mathbf{t} \in \mathbb{R}^n_+$ and $\mathbf{a} \in \{0,1\}^n$. When $\sum_{i \in N} a_i \geq q$ and $\sum_{i \in N} a_i t_i < y$, the leader's unique equilibrium action is to initiate a vote, after which the policy passes. Otherwise, the policy either passes or does not pass as a result of the leader's equilibrium action.

Consider the members' decision in the first period given $\mathbf{t} \in \mathbb{R}^n_+$. Let $\Gamma(\mathbf{t})$ be the subgame that starts at the history in which the leader's offer is $\mathbf{t}$. Assume the sequence in which members move in $\Gamma(\mathbf{t})$ is $(1, 2, \ldots, n)$. This assumption is without loss of generality because the argument below does not invoke that x_i is weakly increasing in i. Let $H_i(\mathbf{t})$ be the set of histories in $\Gamma(\mathbf{t})$ in which member $i \in N$ moves. For any history $h \in H_i(\mathbf{t})$ in which member $i \in N$ moves, let $\mathbf{a}(h) = (a_1(h), \ldots, a_{i-1}(h))$ be the profile of actions of the members moving before i and let $\#h = \sum_{j \in \{1,\ldots,i-1\}} a_j(h) + \sum_{j \in \{i+1,\ldots,n\}} \mathbb{I}(t_j \geq x_j)$ be the number of members who either move before i and accepted at history h or move after i and have been offered transfers weakly above their loss. Note that for any member $i \in N \setminus \{n\}$ and any history $h_i \in H_i(\mathbf{t})$, the game proceeds from h_i to h_{i+1} as a results of i's action at h_i such that (i) $\#h_{i+1} = \#h_i$ if either i accepts and $t_{i+1} \geq x_{i+1}$ or i rejects and $t_{i+1} < x_{i+1}$, (ii) $\#h_{i+1} = \#h_i + 1$ if i accepts and $t_{i+1} < x_{i+1}$, and (iii) $\#h_{i+1} = \#h_i - 1$ if i rejects and $t_{i+1} \geq x_{i+1}$.

Lemma A4. *Given $\mathbf{t} \in \mathbb{R}^n_+$, let $\sigma(\mathbf{t})$ be an equilibrium of $\Gamma(\mathbf{t})$ and let $\mathbf{a} \in \{0,1\}^n$ be the equilibrium members' action profile. Consider $h_i \in H_i(\mathbf{t})$ at which member $i \in N$ moves.*
1. Suppose that $y > \sum_{j \in N} t_j$. If either $\#h_i = q-1$ and $t_i \geq x_i$ or $\#h_i \geq q$, then the policy passes in $\sigma(\mathbf{t})$ starting from h_i.
2. If either $\#h_i \leq q-2$ or $\#h_i = q-1$ and $t_i < x_i$, then the policy does not pass in $\sigma(\mathbf{t})$ starting from h_i.
3. If the policy passes in $\sigma(\mathbf{t})$, then $\sum_{j \in N} a_j t_j = \sum_{j \in N} t_j$.

Proof. Fix $\mathbf{t} \in \mathbb{R}^n_+$, equilibrium $\sigma(\mathbf{t})$ of $\Gamma(\mathbf{t})$, and the equilibrium members' action profile $\mathbf{a} \in \{0,1\}^n$.

We prove parts 1 and 2 by induction. Consider $h_n \in H_n(\mathbf{t})$ at which member n moves. Irrespective of n's action, in $\sigma(\mathbf{t})$ starting from h_n, the policy does not pass if $\#h_n \leq q-2$ and the policy passes if $y > \sum_{i \in N} t_i$ and $\#h_n \geq q$. If $y > \sum_{i \in N} t_i$, $\#h_n = q-1$ and $t_n \geq x_n$, the policy passes if member n accepts and does not pass if n rejects. The payoff from the two actions is $\delta(-x_n + t_n)$ and 0 respectively. Because $\sigma(\mathbf{t})$ constitutes an equilibrium, in $\sigma(\mathbf{t})$ starting from h_n, n accepts and the policy passes because $t_n \geq x_n$. If $\#h = q-1$ and $t_n < x_n$, the policy does not pass if member n rejects and either does not pass or passes if member n accepts. In the former case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_n. In the latter case, because the payoff from rejection is 0 while the payoff from acceptance is $\delta(-x_n + t_n)$, because $t_n < x_n$, and because $\sigma(\mathbf{t})$ constitutes an equilibrium, member n rejects and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_n.

Suppose the lemma holds for all histories in $\cup_{i \in \{k+1,\ldots,n\}} H_i(\mathbf{t})$, where $k \in \{1, \ldots, n-1\}$.

We need to prove that it holds for each history $h_k \in H_k(\mathbf{t})$. Consider $h_k \in H_k(\mathbf{t})$ at which member k moves.

If $y > \sum_{i\in N} t_i$ and $\#h_k \geq q$, then the game proceeds to h_{k+1} either with $\#h_{k+1} \geq q$ or with $\#h_{k+1} = q-1$ and $t_{k+1} \geq x_{k+1}$. In either case the policy passes in $\sigma(\mathbf{t})$ starting from h_k by the induction hypothesis.

If $y > \sum_{i\in N} t_i$, $\#h_k = q-1$, and $t_k \geq x_k$, when member k accepts, then the game proceeds to h_{k+1} either with $\#h_{k+1} = q-1$ and $t_{k+1} \geq x_{k+1}$ or with $\#h_{k+1} = q$, and when member k rejects, then the game proceeds to h_{k+1} either with $\#h_{k+1} = q-1$ and $t_{k+1} < x_{k+1}$ or with $\#h_{k+1} = q-2$. By the induction hypothesis, in $\sigma(\mathbf{t})$ starting from h_k, the policy passes when k accepts and does not pass when k rejects. The payoff from the two actions is $\delta(-x_k + t_k)$ and 0 respectively. Because $\sigma(\mathbf{t})$ constitutes an equilibrium, in $\sigma(\mathbf{t})$ starting from h_k, member k accepts and the policy passes because $t_k \geq x_k$.

If $\#h_k = q-1$, and $t_k < x_k$, when member k rejects, then the game proceeds to h_{k+1} either with $\#h = q-1$ and $t_{k+1} < x_{k+1}$ or with $\#h_{k+1} = q-2$. By the induction hypothesis, in $\sigma(\mathbf{t})$ starting from h_k, the policy does not pass when k rejects. When k accepts, the policy, in $\sigma(\mathbf{t})$ starting from h_k, either does not pass or passes. In the former case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k. In the latter case, because the payoff from rejection is 0 while the payoff from acceptance is $\delta(-x_k+t_k)$, because $t_k < x_k$, and because $\sigma(\mathbf{t})$ constitutes an equilibrium, member k rejects and the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k.

If $\#h_k \leq q-2$, then the game proceeds to h_{k+1} either with $\#h_{k+1} \leq q-2$ or with $\#h_{k+1} = q-1$ and $t_{k+1} < x_{k+1}$. In either case the policy does not pass in $\sigma(\mathbf{t})$ starting from h_k by the induction hypothesis.

Part 3: It suffices to prove that $t_i > 0$ implies $a_i = 1$ for any $i \in N$. Suppose, towards a contradiction, that $t_i > 0$ and $a_i = 0$ for some $i \in N$. The payoff of member i from $a_i = 0$ is $\delta(-x_i)$ because the policy passes in $\sigma(\mathbf{t})$. Consider i's deviation to $a_i' = 1$. Following the deviation, the policy either does not pass, in which case i's payoff is at least 0, or passes, in which case i's payoff is $\delta(-x_i + t_i)$. In either case, the deviation to a_i' is profitable. ■

We now prove Proposition 7. Let σ be an equilibrium of the simultaneous vote-buying game with transfer promises. Let $\mathbf{t}$ be the equilibrium profile of transfers, let $\sigma(\mathbf{t})$ be the equilibrium in $\Gamma(\mathbf{t})$ induced by σ and let $\mathbf{a}$ be the equilibrium members' action profile. Note that if the policy passes in $\sigma(\mathbf{t})$, then $|\{i \in N | t_i \geq x_i\}| \geq q$ and thus $\sum_{i\in N} t_i \geq \sum_{i=1}^{q} x_i$ by Lemma A4 part 2.

To show that if $y < \sum_{i=1}^{q} x_i$ then the policy does not pass in σ, suppose, towards a contradiction, that $y < \sum_{i=1}^{q} x_i$ and the policy passes in σ. Then $\sum_{i\in N} a_i t_i = \sum_{i\in N} t_i \geq \sum_{i=1}^{q} x_i$ by Lemma A4 parts 2 and 3. Because $\sum_{i=1}^{q} x_i > y$, the leaders equilibrium payoff $\delta(y - \sum_{i\in N} a_i t_i) < 0$, and hence stoping in the first period is a profitable deviation.

To show that if $y > \sum_{i=1}^{q} x_i$ then the policy passes in σ, suppose, towards a contradiction, that $y > \sum_{i=1}^{q} x_i$ and the policy does not pass in σ. Then the leader's equilibrium payoff is at most 0. Consider a deviation for the leader to $\mathbf{t}' = (x_1, \ldots, x_q, 0, \ldots, 0)$. Let $\sigma(\mathbf{t}')$ be an equilibrium of $\Gamma(\mathbf{t}')$, which exists by standard backward induction argument. Because $\sum_{i \in N} t'_i = \sum_{i=1}^{q} x_i < y$, and because the initial history h_1 of $\Gamma(\mathbf{t}')$ satisfies $\# h_1 = q - 1$ and $t_1 \geq x_1$, Lemma A4 part 1 implies that the policy passes in $\sigma(\mathbf{t}')$. Hence, by Lemma A4 part 3, the leader's payoff from the deviation is $\delta(y - \sum_{i \in N} t'_i) = \delta(y - \sum_{i=1}^{q} x_i) > 0$.

To conclude the proof, suppose $y > \sum_{i=1}^{q} x_i$, so that the policy passes in σ. It suffices to show that $\sum_{i \in N} t_i = \sum_{i=1}^{q} x_i$. Because the policy passes in $\sigma(\mathbf{t})$, by Lemma A4 part 2, $\sum_{i \in N} t_i \geq \sum_{i=1}^{q} x_i$. If $\sum_{i \in N} t_i > \sum_{i=1}^{q} x_i$, then $\mathbf{t}'$ is a profitable deviation for the leader. Thus $\sum_{i \in N} t_i = \sum_{i=1}^{q} x_i$. ■

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Abstrakt

Předseda komise se snaží prosadit opatření, k jehož schválení potřebuje podporu části komise. Členové komise jsou více či méně v opozici, protože opatření snižuje jejich užitek. Předseda postupně jednotlivým členům nabízí odměnu za jejich hlas. Odměna může být slib zaplatit v případě, že opatření bude přijato, nebo to může být platba předem. V prvním případě hlasy nestojí téměř nic. V druhém případě mohou hlasy stát více, ale předseda stejně preferuje platby předem. Koupit hlasy členů nejméně v opozici nemusí být pro předsedu optimální. Opozice, která pro předsedu představuje největší překážku, je homogenní, nebo složena ze dvou homogenních skupin členů. Výsledky vysvětlují řadu empirických pozorování: lobování zákonodárců silně v opozici, Tullockův paradox, nebo expanzi orgánů dohlížejících na stranickou disciplínu v americké Sněmovně reprezentantů.

Supplementary Appendix

In this appendix we provide further results for the up-front-payment game, as mentioned in Sections 4 and 5. We continue to use the notation from the proofs of Propositions 3 and 4. Recall $\mathcal{D}^G = \{(S,r) \in 2^N \times \mathbb{Z} | S \neq \varnothing \wedge 1 \leq r \leq |S|\}$, so that state $(N,q) \in \mathcal{D}^G$.

W **and** Π **in special cases.** Proposition B1 provides closed-form expressions for W and Π in several special cases, as mentioned after Proposition 4.

Proposition B1. *Consider state $(S,r) \in \mathcal{D}^G$ and suppose that $y > W(S,r)$.*

1. *(one vote needed) $W(S,r) = x_{(1)}$ and $\Pi(S,r,y) = 0$ if $r = 1$ and $|S| \geq 2$.*
2. *(unanimity) $W(S,r) = \sum_{i \in S} x_i$ and $\Pi(S,r,y) = \sum_{i \in S} x_i$ if $r = |S|$.*
3. *(simple majority and less) $W(S,r) = x_{(r)}$ if $r \leq \frac{|S|+1}{2}$ and $\Pi(S,r,y) = 0$ if $r < \frac{|S|+1}{2}$.*
4. *(homogeneous losses) $W(S,r) = \lceil \frac{r}{|S|-r+1} \rceil x = \lfloor \frac{|S|}{|S|-r+1} \rfloor x$ and $\Pi(S,r,y) = tx$, where t is the smallest non-negative integer such that $y > \lceil \frac{r-t}{|S|-r} \rceil$, if $x_i = x$ for all $i \in S$ and $r < |S|$.*

Proof. We first prove all claims about W. Parts 1 and 2 follow by Lemma A1 part 2.

Part 3: We proceed by induction on r. That $W(S,r) = x_{(r)}$ for any $(S,r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r = 1$ follows from Lemma A1 part 2. Now suppose that $W(S,r) = x_{(r)}$ for any $(S,r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r \leq k$, where $k \geq 1$. We need to prove that $W(S,r) = x_{(r)}$ for any $(S,r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r = k+1$. To see this, consider $(S,r) \in \mathcal{D}^G$ with $r \leq \frac{|S|+1}{2}$ and with $r = k+1$. We claim that $T = \{\min S\}$ solves (1) given (S,r) and thus $W(S,r) = \max\{x_{(1)}, W((S \setminus \{\min S\})', r-1)\} = \max\{x_{(1)}, x_{(r)}\} = x_{(r)}$, where the second equality follows from the induction hypothesis. To see that $\max\{\sum_{i \in T} x_i, W((S \setminus T)', r - |T|)\} \geq x_{(r)}$ for any $T \in 2^S$, given $T \in 2^S$ we have either (i) $|T| \geq r$, in which case $\sum_{i \in T} x_i \geq x_{(r)}$, or (ii) $|T| = 0$, in which case, because $r \geq 2$ implies $|S| \geq 3$ and hence $S' \setminus \{\min S\} = (S \setminus \{\min S\})'$, we have $W((S \setminus T)', r - |T|) = W(S', r) \geq W(S' \setminus \{\min S\}, r-1) = W((S \setminus \{\min S\})', r-1) = x_{(r)}$, where the inequality follows from Lemma A1 part 6, or (iii) $|T| \in \{1, \ldots, r-1\}$, in which case either (a) $(i) \in T$ for some $i \in \{r, \ldots, |S|\}$, in which case $\sum_{j \in T} x_j \geq x_{(r)}$, or (b) $(i) \notin T$ $\forall i \in \{r, \ldots, |S|\}$, in which case $W((S \setminus T)', r - |T|) = x_{(r)}$, where the equality follows from the induction hypothesis, which we can invoke because for any $T \in 2^S$ with $|T| \in \{1, \ldots, r-1\}$, we have $((S \setminus T)', r - |T|) \in \mathcal{D}^G$ and $r - |T| \leq \frac{|(S \setminus T)'|+1}{2}$. The former follows from $|(S \setminus T)'| = |S| - 1 - |T| \geq (2r-1) - 1 - |T| \geq r - |T| \geq 1$, where the second inequality uses $r = k+1 \geq 2$. The latter is equivalent to $r + \frac{1-|T|}{2} \leq \frac{|S|+1}{2}$ and follows from $|T| \geq 1$.

Part 4: First, we show that for any $(S,r) \in \mathcal{D}^G$, we have $\left\lceil \frac{r}{|S|-r+1} \right\rceil = \left\lfloor \frac{|S|}{|S|-r+1} \right\rfloor$. To see this, $(S,r) \in \mathcal{D}^G$ implies $|S| \geq 1$ and $r \in \{1, \ldots, |S|\}$. Thus $|S| - r + 1 \geq 1$ and $0 < \frac{r}{|S|-r+1} \leq \frac{|S|}{|S|-r+1}$. It thus suffices to prove that there exists unique $m \in \mathbb{N}$ such that $\frac{r}{|S|-r+1} \leq m \leq \frac{|S|}{|S|-r+1}$. To see that m exists, if $\frac{r}{|S|-r+1} \notin \mathbb{N}$, then for the smallest integer larger than $\frac{r}{|S|-r+1}$, m', we have $\frac{r+i}{|S|-r+1}=m'$, where $i\in\{1,\ldots,|S|-r\}$. Thus $\frac{|S|}{|S|-r+1}=\frac{|S|-r-i}{|S|-r+1}+m'\geq m'$. That m is unique follows from $\frac{|S|}{|S|-r+1}-\frac{r}{|S|-r+1}<1$.

We now prove that if $(S,r)\in\mathcal{D}^G$ and $x_i=x\ \forall i\in S$, then $W(S,r)=\left\lceil\frac{r}{|S|-r+1}\right\rceil x=\left\lfloor\frac{|S|}{|S|-r+1}\right\rfloor x$. We proceed by induction on $|S|-r$. Suppose $x_i=x\ \forall i\in S$. That $W(S,r)=\left\lceil\frac{r}{|S|-r+1}\right\rceil x=\left\lfloor\frac{|S|}{|S|-r+1}\right\rfloor x$ for any $(S,r)\in\mathcal{D}^G$ with $|S|-r=0$ follows because, by Lemma A1 part 2, $W(S,|S|)=\sum_{i\in S}x_i$ for any $S\in 2^N\setminus\varnothing$. Now suppose that $W(S,r)=\left\lceil\frac{r}{|S|-r+1}\right\rceil x=\left\lfloor\frac{|S|}{|S|-r+1}\right\rfloor x$ for any $(S,r)\in\mathcal{D}^G$ with $|S|-r\leq k$, where $k\geq 0$. We need to prove that $W(S,r)=\left\lceil\frac{r}{|S|-r+1}\right\rceil x=\left\lfloor\frac{|S|}{|S|-r+1}\right\rfloor x$ for any $(S,r)\in\mathcal{D}^G$ with $|S|-r=k+1$. To see this, because $x_i=x\ \forall i\in S$ and by Lemma A1 part 1, (1) simplifies to $W(S,r)=\min_{t\in\{0,\ldots,r\}}\max\{tx,W(\cup_{i=1}^{|S|-t-1}\{i\},r-t)\}=x\min_{t\in\{0,\ldots,r\}}\max\{t,\lceil\frac{r-t}{|S|-r}\rceil\}$, where the second equality follows from the induction hypothesis when $r-t\geq 1$ and directly from definition of W when $r-t=0$. It thus suffices to prove that $\min_{t\in\{0,\ldots,r\}}\max\{t,\lceil\frac{r-t}{|S|-r}\rceil\}=\lceil\frac{r}{|S|-r+1}\rceil=\lfloor\frac{|S|}{|S|-r+1}\rfloor$. The structure of the problem implies that $\min_{t\in\{0,\ldots,r\}}\max\{t,\lceil\frac{r-t}{|S|-r}\rceil\}=t^*$, where $t^*\in\mathbb{N}$ is the largest solution to the problem. In order to derive t^* we need to consider two cases: either $t=\lceil\frac{r-t}{|S|-r}\rceil$ for some $t\in\{0,\ldots,r\}$ or $t\neq\lceil\frac{r-t}{|S|-r}\rceil\ \forall t\in\{0,\ldots,r\}$. In the former case $t^*=\lceil\frac{r-t^*}{|S|-r}\rceil=\frac{r-t^*+i}{|S|-r}$, where $i\in\{0,\ldots,|S|-r-1\}$, and thus $t^*=\frac{r+i}{|S|-r+1}=\lceil\frac{r}{|S|-r+1}\rceil$, where the second equality follows from $t^*\in\mathbb{N}$ and $i\leq|S|-r-1$. In the latter case $t^*=\frac{r-t^*}{|S|-r}+1$, and thus $t^*=\frac{|S|}{|S|-r+1}=\lfloor\frac{|S|}{|S|-r+1}\rfloor$.

We now prove all claims about Π. Fix $(S,r)\in\mathcal{D}^G$ and $y>W(S,r)$. Part 1: To see that $\Pi(S,r,y)=0$ if $r=1$ and $|S|\geq 2$, we have $y>W(S,r)=x_{(1)}=W(S',r)$, where the last equality follows because $W(S',r)=x_{(1)}$ by Lemma A1 part 2. Hence $T=\varnothing$ is admissible in (2).

Part 2: To see that $\Pi(S,r,y)=\sum_{i\in S}x_i$ if $r=|S|$, we have $W((S\setminus T)',r-|T|)=\infty$ for any $T\in 2^S\setminus S$ and $W((S\setminus T)',r-|T|)=0$ when $T=S$. Hence $T=S$ is the only element of 2^S admissible in (2).

Part 3: To see that $\Pi(S,r,y)=0$ if $r<\frac{|S|+1}{2}$, we have $y>W(S,r)=x_{(r)}=W(S',r)$, where the last equality follows because $r<\frac{|S|+1}{2}$ implies that $r\leq\frac{|S|}{2}=\frac{|S'|+1}{2}$. Hence $T=\varnothing$ is admissible in (2).

Part 4: To see that $\Pi(S,r,y)=tx$, where t is the smallest non-negative integer such that $y>\lceil\frac{r-t}{|S|-r}\rceil$, if $x_i=x$ for all $i\in S$ and $r<|S|$, note that in this case (2) writes $\min_{t\in\{0,\ldots,r\}}tx$ s.t. $y>\lceil\frac{r-t}{|S|-r}\rceil$. To see this, we first claim that $T\leq r$ for any T that solves (2) given (S,r). Suppose, towards a contradiction, that $|T|>r$. Consider any $T_a\subseteq T$ such that $|T_a|=r$. Because $|T|>r$, such T_a exists. We have $T_a\in 2^S$, $\sum_{i\in T}x_i>\sum_{i\in T_a}x_i$ and $W((S\setminus T_a)',r-|T_a|)=0$, which is a contradiction because T solves (2) given (S,r). Thus $|T|\leq r$. Second, $W((S\setminus T)',r-|T|)$ in the constraint of (2) becomes $\lceil\frac{r-t}{|S|-r}\rceil$ because $x_i=x\ \forall i\in S$ and $r<|S|$ when $r-t\geq 1$ and directly from definition of W when $r-t=0$. ■

W **and any** T **that solves** (2)**.** Proposition B2 shows that $W(N,q) = \sum_{i\in M} x_i$ for some $M \subseteq N$, and that M and any T that solves (2) exclude members strictly more opposed than member q, as mentioned after Proposition 4.

Proposition B2. *Consider state* $(S,r) \in \mathcal{D}^G$ *and any* $y > 0$.

1. $W(S,r) = \sum_{i\in M} x_i$ *for some* $M \subseteq N$.
2. $W(S,r) = W(\hat{S},r)$ *for any* $\hat{S} \in 2^N$ *such that* $|\hat{S}| = |S|$ *and* $\hat{x}_{(i)_{\hat{S}}} = x_{(i)_S}$ $\forall i \in \{1,\ldots,r\}$.
3. *For all* $i \in \{r+1,\ldots,|S|\}$, *if* $x_{(r)} < x_{(i)}$, *then* $(i) \notin T$ *for any* T *that solves* (2).

Proof. Part 1: That $W(S,r) = \sum_{i\in M} x_i$ for some $M \subseteq N$ for any state $(S,r) \in \mathcal{D}^G$ follows from Lemma A1 part 3.

Part 2: We proceed by induction on $|S|$. That $W(S,r) = W(\hat{S},r)$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = 1$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| = |S|$ and $\hat{x}_{(i)_{\hat{S}}} = x_{(i)_S}$ $\forall i \in \{1,\ldots,r\}$ follows because, as shown in Lemma A1 part 2, $W(S,1) = x_{(1)_S}$ for any $S \in 2^N \setminus \varnothing$. Now suppose that $W(S,r) = W(\hat{S},r)$ for any $(S,r) \in \mathcal{D}^G$ with $|S| \leq k$, where $k \geq 1$, and for any $\hat{S} \in 2^N$ such that $|\hat{S}| = |S|$ and $\hat{x}_{(i)_{\hat{S}}} = x_{(i)_S}$ $\forall i \in \{1,\ldots,r\}$. We need to prove that $W(S,r) = W(\hat{S},r)$ for any $(S,r) \in \mathcal{D}^G$ with $|S| = k+1$ and for any $\hat{S} \in 2^N$ such that $|\hat{S}| = |S|$ and $\hat{x}_{(i)_{\hat{S}}} = x_{(i)_S}$ $\forall i \in \{1,\ldots,r\}$. To see this, we have either (i) $r = 1$, in which case $W(S,r) = x_{(1)_S} = \hat{x}_{(1)_{\hat{S}}} = W(\hat{S},r)$ by Lemma A1 part 2, or (ii) $r = |S|$, in which case $W(S,r) = \sum_{i\in S} x_i = \sum_{i\in\hat{S}} \hat{x}_i = W(\hat{S},r)$ by Lemma A1 part 2, or (iii) $r \in \{2,\ldots,|S|-1\}$, in which case, given T that solves (1) given (S,r), either (a) $|T| = 0$, in which case $W(S,r) = W(S',r) = W(\hat{S}',r) \geq W(\hat{S},r)$, where the second equality follows from the induction hypothesis, or (b) $|T| = r$, in which case $T = \{(i)_S | i \in \{1,\ldots,r\}\}$ because $W((S \backslash T)', r-|T|) = 0$ and thus $W(S,r) = \sum_{i=1}^r x_{(i)_S} = \sum_{i=1}^r \hat{x}_{(i)_{\hat{S}}} \geq W(\hat{S},r)$, or (c) $|T| \in \{1,\ldots,r-1\}$, in which case, given $\hat{T} = \{(i)_{\hat{S}} | i \in \{1,\ldots,r\}, (i)_S \in T\}$, $W(S,r) = \max\{\sum_{i\in T} x_i, W((S\backslash T)', r-|T|)\} = \max\{\sum_{i\in\hat{T}} \hat{x}_i, W((\hat{S}\backslash\hat{T})', r-|\hat{T}|)\} \geq W(\hat{S},r)$, where the second equality follows because, as we show below, it is without loss of generality to assume that $T \subseteq \{(i)_S | i \in \{1,\ldots,r\}\}$, and hence we have $\sum_{i\in T} x_i = \sum_{i\in\hat{T}} \hat{x}_i$ as well as $((S\backslash T)', r-|T|) \in \mathcal{D}^G$ implied by $|T| \in \{1,\ldots,r-1\}$ and $r \leq |S|-1$, $|(S\backslash T)'| = |(\hat{S}\backslash\hat{T})'|$ and $x_{(i)_{(S\backslash T)'}} = \hat{x}_{(i)_{(\hat{S}\backslash\hat{T})'}}$ $\forall i \in \{1,\ldots,r-|T|\}$ and thus, by the induction hypothesis, $W((S\setminus T)', r-|T|) = W((\hat{S}\setminus\hat{T})', r-|\hat{T}|)$. All three subcases of the (iii) case show that $W(S,r) \geq W(\hat{S},r)$. Swapping S and $\hat{S}$ in the argument shows that $W(\hat{S},r) \geq W(S,r)$ and hence that $W(S,r) = W(\hat{S},r)$.

What remains is to show that if T solves (1) given (S,r) with $|S| = k+1$, where $k \geq 1$, and $r \in \{2,\ldots,|S|-1\}$ and if $|T| \in \{1,\ldots,r-1\}$, then there exists another solution of (1), T_a, such that $T_a \subseteq \{(i)_S | i \in \{1,\ldots,r\}\}$ and $|T_a| = |T|$. To see this, let $T_r = \{(i)_S | i \in \{1,\ldots,r\}, (i)_S \in T\}$, $T_{r+1} = \{(i)_S | i \in \{r+1,\ldots,|S|\}, (i)_S \in T\}$, $U_r = \{(i)_S | i \in \{1,\ldots,r\}, (i)_S \in S\backslash T\}$ and $U_{r+1} = \{(i)_S | i \in \{r+1,\ldots,|S|\}, (i)_S \in S\backslash T\}$. Note that because $T \in 2^S$, $S = T_r \cup T_{r+1} \cup U_r \cup U_{r+1}$. If $|T_{r+1}| = 0$ the claim follows by setting $T_a = T$. Hence, suppose that $|T_{r+1}| \geq 1$. Because $|T| = |T_r| + |T_{r+1}|$ and $|T_r| + |U_r| = r$, we have $|U_r| = r-|T|+|T_{r+1}|$ and hence there exists a partition of U_r into U_s and U_m such that $|U_s| = r-|T| \geq 1$, $|U_m| = |T_{r+1}| \geq 1$ and $i < j$ for any $i \in U_s$ and $j \in U_m$. Now construct $T_a = T_r \cup U_m$. By construction $T_a \subseteq \{(i)_S | i \in \{1,\dots,r\}\}$ and $|T_a| = |T|$. Moreover, we have $\max\{\sum_{i\in T} x_i, W((S\setminus T)', r-|T|)\} \geq \max\{\sum_{i\in T_a} x_i, W((S\setminus T_a)', r-|T_a|)\}$, where the inequality follows from $\sum_{i\in T} x_i \geq \sum_{i\in T_a} x_i$, which holds because $T = T_r \cup T_{r+1}$, $T_a = T_r \cup U_m$ and $U_m \subset U_r$, and from $W((S\setminus T)', r-|T|) = W((S\setminus T_a)', r-|T_a|)$, which holds by the induction hypothesis because $((S\setminus T)', r-|T|) \in \mathcal{D}^G$, $S\setminus T = U_s \cup U_m \cup U_{r+1}$, $S \setminus T_a = U_s \cup T_{r+1} \cup U_{r+1}$, $|U_m| = |T_{r+1}|$, $i < j$ for any $i \in U_s$ and $j \in U_m \cup U_{r+1} \cup T_{r+1}$, and $|U_s| = r - |T| = r - |T_a|$.

Part 3: Suppose $r < |S|$ and consider T that solves (2) given (S, r). Suppose, towards a contradiction, that for some $i \in \{r+1,\dots,|S|\}$ we have $x_{(r)} < x_{(i)}$ and $(i) \in T$.

We have $|T| \leq r$ by the argument made in the proof of Proposition B1 part 4. $|T| \leq r$ and $r < |S|$ imply $S \setminus T \neq \varnothing$. There are two cases to consider. In each case we construct $T_a \in 2^S$ such that $\sum_{j\in T} x_j > \sum_{j\in T_a} x_j$ and $W((S\setminus T)', r-|T|) = W((S\setminus T_a)', r-|T_a|)$ establishing a contradiction to T solving (2).

Case 1: $\max S\setminus T \leq (r)$. Because $\max S\setminus T \leq (r)$, we have $\max S\setminus T \leq (r) < (i)$ and hence $x_{\max S\setminus T} \leq x_{(r)} < x_{(i)}$. Let $T_a = (T \cup \{\max S\setminus T\}) \setminus \{(i)\} \in 2^S$. By construction, $\sum_{j\in T} x_j > \sum_{j\in T_a} x_j$. Moreover, $|T| = |T_a|$ and $(S\setminus T_a)' = (((S\setminus T)\setminus \{\max S\setminus T\}) \cup \{(i)\})' = (S\setminus T)'$ and hence $W((S\setminus T)', r-|T|) = W((S\setminus T_a)', r-|T_a|)$.

Case 2: $\max S\setminus T > (r)$. When $|T| = r$, $T_a = \{(j)_S | j \in \{1,\dots,r\}\}$ has the desired properties. When $|T| \in \{1,\dots,r-1\}$, let $T_r = \{(j)_S | j \in \{1,\dots,r\}, (j)_S \in T\}$ and $U_r = \{(j)_S | j \in \{1,\dots,r\}, (j)_S \in S\setminus T\}$. By construction, $|T_r| + |U_r| = r$. Moreover, because $(i) \in T$ and $i \geq r+1$, $|T| \geq |T_r| + 1$ and hence $|U_r| = r - |T_r| \geq r - |T| + 1$. Thus, there exists a partition of U_r into U_s and U_m such that $|U_s| = r - |T| \geq 1$, $|U_m| \geq 1$ and $x_j \leq x_{j'}$ for any $j \in U_s$ and $j' \in U_m$. Let $T_a = (T \cup \{i'\}) \setminus \{(i)\}$ for some $i' \in U_m$. By construction, $T_a \in 2^S$. Moreover, because $x_{(i)} > x_{(r)}$ and $(r) \geq i'$, we have $x_{(i)} > x_{(r)} \geq x_{i'}$ and thus $\sum_{j\in T} x_j > \sum_{j\in T_a} x_j$. Finally, $((S\setminus T)', r-|T|) \in \mathcal{D}^G$, $S\setminus T = U_s \cup U_m \cup Z \cup \{\max S\setminus T\}$, $S\setminus T_a = U_s \cup (U_m \setminus \{i'\}) \cup Z \cup \{\max S\setminus T, (i)\}$, $j < j'$ for any $j \in U_s$ and $j' \in U_m \cup Z \cup \{\max S\setminus T, (i)\}$, and $|U_s| = r - |T| = r - |T_a|$. Thus, by part 2, $W((S\setminus T)', r-|T|) = W((S\setminus T_a)', r-|T_a|)$. ■

Comparative statics in the up-front-payment game. Proposition B3 provides the comparative statics results for the up-front-payment game mentioned in footnote 24.

Proposition B3. *Consider state $(S,r) \in \mathcal{D}^G$ and any $y > 0$.*

1. $W(S,r) \geq W(S,r-1)$ *and* $\Pi(S,r,y) \geq \Pi(S,r-1,y)$ *if* $r \geq 2$.
2. $W(S,r) \geq W(\hat{S},r)$ *and* $\Pi(S,r,y) \geq \Pi(\hat{S},r,y)$ *if* $|S| \leq |\hat{S}|$ *and* $x_{(i)_S} \geq \hat{x}_{(i)_{\hat{S}}}$ *for all* $i \in \{1,\dots,|S|\}$.
3. $W(S,r)$ *is independent of* y *and* $\Pi(S,r,y) \geq \Pi(S,r,y')$ *if* $y' > y$.
4. $W(S,r) \geq W(S\setminus\{i\},r-1)$ *and* $\Pi(S,r,y) \geq \Pi(S\setminus\{i\},r-1,y)$ *for any* $i \in S$ *if* $r \geq 2$.

Proof. The claims about W follow either directly from its definition or by Lemma A1 parts 4 through 6. To see the claims about Π, fix $(S, r) \in \mathcal{D}^G$ and $y > 0$, and consider T that solves (2) given (S, r). We have $y > W((S \setminus T)', r - |T|)$.

Part 1: From Lemma A1 part 4, we have $W((S \setminus T)', r - |T|) \geq W((S \setminus T)', r - 1 - |T|)$ and hence $y > W((S \setminus T)', r - 1 - |T|)$. Because $(S, r - 1) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(S, r - 1, y)$.

Part 2: Consider $\hat{S} \in 2^N$ such that $|S| \leq |\hat{S}|$ and $x_{(i)_S} \geq \hat{x}_{(i)_{\hat{S}}}$ $\forall i \in \{1, \ldots, |S|\}$. Let $\hat{T} = \{(i)_{\hat{S}} | i \in \{1, \ldots, |S|\}, (i)_S \in T\} \in 2^{\hat{S}}$. From Lemma A1 part 5, we have $W((S \setminus T)', r - |T|) \geq W((\hat{S} \setminus \hat{T})', r - |\hat{T}|)$ and hence $y > W((\hat{S} \setminus \hat{T})', r - |\hat{T}|)$. Moreover, because $x_{(i)_S} \geq \hat{x}_{(i)_{\hat{S}}}$ $\forall i \in \{1, \ldots, |S|\}$, we have $\sum_{i \in T} x_i \geq \sum_{i \in \hat{T}} \hat{x}_i$. Because $(\hat{S}, r) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(\hat{S}, r, y)$.

Part 3: Because $y' > y$, we have $y' > W((S \setminus T)', r - |T|)$ and thus $\Pi(S, r, y) \geq \Pi(S, r, y')$.

Part 4: Consider $i \in S$. We have either $i \in T$ or $i \notin T$. When $i \in T$, set $T_a = T \setminus \{i\} \in 2^{S \setminus \{i\}}$. We have $|T| = |T_a| + 1$ and $S \setminus T = (S \setminus \{i\}) \setminus T_a$. Thus $W((S \setminus T)', r - |T|) = W(((S \setminus \{i\}) \setminus T_a)', r - 1 - |T_a|)$ and hence $y > W(((S \setminus \{i\}) \setminus T_a)', r - 1 - |T_a|)$. Moreover, $\sum_{j \in T} x_j \geq \sum_{j \in T_a} x_j$. Because $(S \setminus \{i\}, r - 1) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(S \setminus \{i\}, r - 1, y)$. When $i \notin T$, we have either $|S \setminus T| = 1$ or $|S \setminus T| \geq 2$. In the former case $T \cup \{i\} = S$ so that $(S \setminus T)' = ((S \setminus \{i\}) \setminus T)' = \varnothing$ and Lemma A1 part 4 imply $W((S \setminus T)', r - |T|) = W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$ and hence $y > W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$. In the latter case either $((S \setminus \{i\}) \setminus T)' = (S \setminus T)' \setminus \{\max(S \setminus T)'\}$ when $i = \max S \setminus T$ or $((S \setminus \{i\}) \setminus T)' = (S \setminus T)' \setminus \{i\}$ when $i < \max S \setminus T$ and Lemma A1 part 6 implies $W((S \setminus T)', r - |T|) \geq W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$ and hence $y > W(((S \setminus \{i\}) \setminus T)', r - 1 - |T|)$. Moreover, because $i \notin T$, $T \in 2^{S \setminus \{i\}}$. Because $(S \setminus \{i\}, r - 1) \in \mathcal{D}^G$, $\Pi(S, r, y) \geq \Pi(S \setminus \{i\}, r - 1, y)$. ■

Tullock paradox in the up-front-payment game. To show that $\Pi(N, q, y) = 0$ when $y > \sum_{i=1}^{q} x_i$ and $q < n$, as mentioned in footnote 25, $q < n$ implies $q \leq n - 1$ and hence, by Lemma A3, we have $W(N', q) \leq \sum_{i=1}^{q} x_i$. Thus $y > \sum_{i=1}^{q} x_i \geq W(N', q)$, which implies that $T = \varnothing$ is admissible in (2) given (N, q) and hence $\Pi(N, q, y) = 0$.

Opposition structures maximizing W. We now derive opposition structures, that is, profiles of $(x_1, \ldots, x_n)$, that maximize $W(N, q)$ subject to $\sum_{i \in N} x_i = c$, as mentioned in footnote 26. Lemma B1 part 1 implies that $W(N, q) \leq \frac{\sum_{i \in N} x_i}{n - q + 1}$. Hence the upper bound on $W(N, q)$ subject to $\sum_{i \in N} x_i = c$ is $\frac{c}{n - q + 1}$.

Now consider the profile $\tilde{\mathbf{x}} = (\tilde{x}_i)_{i \in N}$ such that $\tilde{x}_i = \varepsilon$ for $i \in \{1, \ldots, n - n_b\}$ and $\tilde{x}_i = \frac{c - \varepsilon(n - n_b)}{n_b}$ for $i \in \{n - n_b + 1, \ldots, n\}$, where $c > 0$ and $n_b = k(n - q + 1)$ for some $k \in \{1, \ldots, \lfloor \frac{n}{n - q + 1} \rfloor\}$, where $\{1, \ldots, \lfloor \frac{n}{n - q + 1} \rfloor\} \neq \varnothing$ because $\frac{n}{n - q + 1} \geq 1$. By construction, $\sum_{i \in N} \tilde{x}_i = c$. Moreover, there exists $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$, $\tilde{\mathbf{x}} \in \mathbb{R}^n_{++}$ and $x_1 < x_n$. Thus, $\forall \varepsilon \in (0, \bar{\varepsilon})$, by Lemma B1 part 2, we have $W(N, q) \geq \frac{c - \varepsilon(n - n_b)}{n_b} \lceil \frac{n_b - (n - q)}{n - q + 1} \rceil = \frac{c - \varepsilon(n - n_b)}{k(n - q + 1)} \lceil k - \frac{n - q}{n - q + 1} \rceil = \frac{c - \varepsilon(n - n_b)}{n - q + 1}$.

Lemma B1. *Consider state* $(S,r) \in \mathcal{D}^G$.
1. $W(S,r) \leq \frac{\sum_{i\in S} x_i}{|S|-r+1}$.
2. Suppose $x_{(i)} = a$ *for* $i \in \{1,\ldots,n_a\}$ *and* $x_{(i)} = b$ *for* $i \in \{n_a+1,\ldots,n_a+n_b\}$, *where* $b > a > 0$, $n_a, n_b \in \{0,\ldots,|S|\}$, *and* $n_a + n_b = |S|$. *Then* $W(S,r) \geq b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$.

Proof. Part 1: Consider $(S,r) \in \mathcal{D}^G$. We proceed by induction on $|S|-r$. That $W(S,r) \leq \frac{\sum_{i\in S} x_i}{|S|-r+1}$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = 0$ follows from Lemma A1 part 2. Now suppose that $W(S,r) \leq \frac{\sum_{i\in S} x_i}{|S|-r+1}$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = k$, where $k \geq 0$. We need to prove that $W(S,r) \leq \frac{\sum_{i\in S} x_i}{|S|-r+1}$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = k+1$. Proof of this claim uses the following lemma.

Lemma B2. *Given* $n \geq 2$ *and* $r \in \{1,\ldots,n-1\}$, *for any* $(x_1,\ldots,x_n) \in \mathbb{R}^n_{++}$ *such that* $\sum_{i=1}^n x_i = c > 0$ *and* $x_i \leq x_{i+1}$ *for any* $i \in \{1,\ldots,n-1\}$, *a partition of* $\{1,\ldots,n-1\}$ *into two subsets* A *and* B *exists such that* $\sum_{i\in A} x_i \leq \frac{1}{n-r+1}c$ *and* $\sum_{i\in B} x_i \leq \frac{n-r}{n-r+1}c$.

Proof. Fix $n \geq 2$, $r \in \{1,\ldots,n-1\}$, $\mathbf{x} \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n x_i = c > 0$ and $x_i \leq x_{i+1}$ $\forall i \in \{1,\ldots,n-1\}$. Let s be the largest integer such that $\sum_{i=1}^s x_i \leq \frac{c}{n-r+1}$. Because $x_1 \leq \frac{c}{n} \leq \frac{c}{n-r+1}$, $s \geq 1$, and because $\sum_{i=1}^n x_i = c > \frac{c}{n-r+1}$, $s \leq n-1$. Let $A = \{1,\ldots,s\}$ and $B = \{s+1,\ldots,n-1\}$. By construction $\sum_{i\in A} x_i \leq \frac{1}{n-r+1}c$. It thus suffices to prove that $\sum_{i\in B} x_i \leq \frac{n-r}{n-r+1}c$. If $B = \varnothing$, this is immediate. If $B \neq \varnothing$, suppose, towards a contradiction, that $\sum_{i\in B} x_i = \sum_{i=s+1}^{n-1} x_i > \frac{n-r}{n-r+1}c$. Because $B \neq \varnothing$, we have $s \leq n-2$ and hence, by definition of s, $\sum_{i=1}^{s+1} x_i > \frac{1}{n-r+1}c$. Thus $\sum_{i=1}^{s+1} x_i + \sum_{i=s+1}^{n-1} x_i = x_{s+1} + \sum_{i=1}^{n-1} x_i > c = x_n + \sum_{i=1}^{n-1} x_i$ and hence $x_{s+1} > x_n$, which is a contradiction because $s+1 \leq n-1 < n$ implies $x_{s+1} \leq x_n$. ■

Because $(S,r) \in \mathcal{D}^G$ and $|S|-r = k+1 \geq 1$, we have $|S| \geq 2$ and $r \in \{1,\ldots,|S|-1\}$. Therefore, by Lemma B2, there exists a partition of S' into sets A and B such that $\sum_{i\in A} x_i \leq \frac{\sum_{i\in S} x_i}{|S|-r+1}$ and $\sum_{i\in B} x_i \leq \frac{(|S|-r)\sum_{i\in S} x_i}{|S|-r+1}$. Because $A \in 2^S$, (1) implies that $W(S,r) \leq \max\{\sum_{i\in A} x_i, W((S\setminus A)', r-|A|)\}$. By construction of A and B, $\sum_{i\in A} x_i \leq \frac{\sum_{i\in S} x_i}{|S|-r+1}$. Moreover, we have either (i) $|A| \geq r$, in which case $W((S\setminus A)', r-|A|) = 0$, or (ii) $|A| \leq r-1$, in which case $((S\setminus A)', r-|A|) \in \mathcal{D}^G$ and $(S\setminus A)' = (B \cup \{\max S\})' = B$ and hence $W((S\setminus A)', r-|A|) \leq \frac{\sum_{i\in B} x_i}{(|S|-|A|-1)-(r-|A|)+1} \leq \frac{(|S|-r)\sum_{i\in S} x_i}{|S|-r+1}\frac{1}{|S|-r} = \frac{\sum_{i\in S} x_i}{|S|-r+1}$, where the first inequality follows from the induction hypothesis.

Part 2: Suppose that $x_{(i)} = a$ for $i \in \{1,\ldots,n_a\}$ and $x_{(i)} = b$ for $i \in \{n_a+1,\ldots,n_a+n_b\}$, where $b > a > 0$, $n_a, n_b \in \{0,\ldots,|S|\}$, and $n_a + n_b = |S|$. We proceed by induction on $|S|-r$. That $W(S,r) \geq b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = 0$ follows because, by Lemma A1 part 2, $W(S,r) = an_a + bn_b \geq bn_b = b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$.

Now suppose that $W(S,r) \geq b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = k$, where $k \geq 0$. We need to prove that $W(S,r) \geq b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$ for any $(S,r) \in \mathcal{D}^G$ with $|S|-r = k+1$. To see this, consider $(S,r) \in \mathcal{D}^G$ with $|S|-r = k+1$, where $k \geq 0$.

If $n_b = 0$, then $\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil = \lceil -1 + \frac{1}{k+2} \rceil = 0$, and $W(S,r) \geq b\lceil \frac{n_b-(|S|-r)}{|S|-r+1} \rceil$ follows from Lemma A1 part 3. Hence, suppose that $n_b \geq 1$.

We first claim that (1) given (S,r) admits solution T with $|\{i \in T | x_i = b\}| \leq n_b - 1$. Suppose that T solves (1) given (S,r) and that $|\{i \in T | x_i = b\}| = n_b$. By Lemma A1 part 1, $|T| \leq r \leq |S| - 1$ and hence $|S \setminus T| \geq 1$ so that, because $|\{i \in T | x_i = b\}| = n_b$, $x_{\max S \setminus T} = a$. Let $i_b \in T$ be such that $x_{i_b} = b$. Now consider $T_b = (T \setminus \{i_b\}) \cup \{\max S \setminus T\}$. By construction $\sum_{i \in T} x_i > \sum_{i \in T_b} x_i$. Moreover, $|T| = |T_b|$ and $(S \setminus T)' = (S \setminus T_b)'$ and thus $W((S \setminus T)', r - |T|) = W((S \setminus T_b)', r - |T_b|)$ and thus T_b solves (1) given (S,r). Hence, adding an additional constraint $|\{i \in T | x_i = b\}| \leq n_b - 1$ to the optimization problem in (1) does not change the value of the problem.

Second, by Lemma A1 part 5, the value of the objective function of the optimization problem in (1) evaluated at any two $T_1, T_2 \in 2^S$ such that $|\{i \in T_1 | x_i = a\}| = |\{i \in T_2 | x_i = a\}|$ and $|\{i \in T_1 | x_i = b\}| = |\{i \in T_2 | x_i = b\}|$ is the same. A lower bound on $W(S,r)$ is thus

$$\min_{n'_a \in \{0,\ldots,n_a\}, n'_b \in \{0,\ldots,n_b-1\}} \max \left\{ a n'_a + b n'_b, b \lceil \frac{r - n_a - n'_b}{n_a + n_b - r} \rceil \right\} \tag{B1}$$

where the second term in the max operator is a lower bound on $W((S \setminus T)', r - |T|)$, with T composed of n'_a members with $x_i = a$ and n'_b members with $x_i = b$, which, by the induction hypothesis, because $|S \setminus T| = n_a - n'_a + n_b - n'_b \geq 1$, because $b > a$, and because $((S \setminus T)', r - |T|) \notin \mathcal{D}^G$ if and only if $n'_a + n'_b \geq r$, in which case $W((S \setminus T)', r - |T|) = 0$ by definition of W and $\lceil \frac{r - n_a - n'_b}{n_a + n_b - r} \rceil = 0$ as $n'_a \leq n_a$ and $n'_b \leq n_b - 1$, equals $\lceil \frac{n_b - n'_b - 1 - (n_a - n'_a + n_b - n'_b - 1 - (r - n'_a - n'_b))}{n_a - n'_a + n_b - n'_b - 1 - (r - n'_a - n'_b) + 1} \rceil$.

Thus, it suffices to show that if $n'_b < \lceil \frac{n_b - (|S| - r)}{|S| - r + 1} \rceil = \lceil \frac{r - n_a}{n_a + n_b - r + 1} \rceil$, then $\lceil \frac{r - n_a - n'_b}{n_a + n_b - r} \rceil \geq \lceil \frac{r - n_a}{n_a + n_b - r + 1} \rceil$. This implication holds because $\frac{r - n_a - n'_b}{n_a + n_b - r} \geq \frac{r - n_a}{n_a + n_b - r + 1}$ is equivalent to $n'_b \leq \frac{r - n_a}{n_a + n_b - r + 1}$, and because n'_b is an integer and hence $n'_b < \lceil \frac{r - n_a}{n_a + n_b - r + 1} \rceil$ implies that $n'_b \leq \lfloor \frac{r - n_a}{n_a + n_b - r + 1} \rfloor \leq \frac{r - n_a}{n_a + n_b - r + 1}$. ■

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