CALIBRATION OF INTEREST RATE MODELS - TRANSITION MARKET CASE

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Abstract

A methodology to calibrate multifactor interest rate models for transition countries is proposed. Usual methodology of calibration to an implied volatility cannot be used, as there are no regularly traded derivatives. The estimated parameters are used to the analysis of short-term interest rate markets in the Visegrad 4 countries (Slovak Republic, Czech Republic, Poland, Hungary).

Keywords: interest rate, interest rate models, calibration, transition countries

Classification code: C13, C32, C82, E43, G14

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1 Introduction

Theory about the modelling of the term structure of interest rates has evolved over the last 30 years and since then a number of different approaches have been developed. It represents one of the most dynamic parts of the study of finance, where a lot of research is still going on, with interesting practical applications, and therefore is widely used by both academics and practitioners.

The term structure of interest rates concerns the relationship among the yields of default-free zero-coupon bonds that differ only with respect to maturity. Historically, three competing theories have attracted the attention. These are known as the expectations, liquidity preference, and hedging-pressure of preferred habitat theories of the term structure. According to the expectations theory, the shape of the yield curve can be explained by investors’ expectations about future interest rates. The liquidity preference theory argues that short-term bonds are more desirable than long-term bonds, because the former are more liquid. The last theory explains the shape of the term structure by the assumption that if investor is risk-averse, he can be tempted out of his preferred habitats only with the promise of a higher yield on a bond of any other maturity.

The main use of these interest rates models is their application to the pricing of derivatives of interest rates. To correctly determine the price of derivatives one needs to have the model calibrated to the market. A lot of work has been done in this field to calibrate interest rate models to the markets of developed countries. However, such work for the markets of transition countries is very rare and has been done only in the commercial sphere. Calibration methodology used for developed countries’ market cannot be used (reasons for this claim are described below).

One can assume, that the number of various derivatives and the volume of trades in these derivatives will increase sharply in the coming years in the transition economies, as the need for hedging of various risks is increasing with the development of transition economies. Exact pricing methods are thus life-important, not only for business and traders, but also for the regulators of markets to avoid market failures.

The interest rate markets in the transition region have existed only for a certain short period time and therefore the natural question arises: To what extent are these markets efficient? If they are not efficient, the possibility of arbitrage closes in and this can be utilized. Therefore it is important to determine how efficiently the markets work. The models of interest rates might help to answer fully that question, because the main assumption for those models is the efficiency of market. So, hedging against interest rate
risk using calibrated model should be precise in a sense that the prices of derivatives are consistent with the market and the arbitrage is not possible. For example, if a simple option on a bond (caplet) is replicated through the portfolio of cash and bonds, processes for both prices (the price of caplet calculated from the model and the price of portfolio) should be the same. If this is not the case, the market is not working properly.

In this paper we analyze the interest rates markets in transition countries from the point of view of interest rate model of Brace, Gatarek and Musiela (19997). We propose the calibration methodology for this model for transition countries, which suits the specifics of these markets. We also analyze the issues connected with estimating the parameters of mentioned interest rate model. The analysis is done on daily data of 4 Visegrad countries (Poland, Slovakia, Hungary and Czech Republic), i.e. transition countries, where the institutional reforms of economies are the most advanced.

In what follows we will depict the motivation for the research in this area, the aim of the research, main literature dealing with the area of research, proposed methodology and the estimation results and their analysis. The reader not familiar with the models of interest rates can find their short description in the appendix.

2 Motivation for study of models of the term structure

Since the eighties and nineties the number and the volume of traded derivatives has increased sharply. The number of products to satisfy the needs of users were developed. The general reason for the formation of new derivatives has been the need to hedge the interest rate risks of investors. Investors needed to insure against excessive increases or decrease in these rates and against twists in the shape of yield structure. So, the main goals in modelling interest rates (IR) are to find a robust, credible model for the pricing of derivatives and a methodology of calibrating this model to the market data.

When introducing a new class of derivatives, there is no benchmark price on the market. The model of interest rates can help to retrieve this price from the market data only if it correctly describes the evolution of interest rates and if it is not misspecified in the sense that it is built on bad parameters. Therefore both of these roles are very important and solid pricing methodology has to cover both of them.

There is no single interest rate for the economy and in addition the structure of interest rates is interdependent. The interest rate is in general affected
by a lot of factors, and one of those, which impacts the interest rate for a particular security and which is fundamental in the modelling of interest rates, is the maturity. The relation between yield on zero-coupon bonds and maturity is referred to as the term structure of interest rates. In the following text we will refer to the models of interest rates sometimes also as models of term structure as these terminology is fully equivalent. The analysis of term structure is crucial in the analysis of interest-rate-dependent derivatives and among its uses belong the following:

- The analysis of fixed income contracts with varying maturity-it has the biggest influence on the profitability of a portfolio in an environment with volatile interest rates
- The forecasting of future interest rates
- The pricing of derivatives and other contracts with fixed payments- in pricing financial bonds, it is important to look at yields of alternative investing opportunities with a similar length of commitment, and the term structure (or yield curve) gives information about these alternative yields
- The pricing of options on assets with fixed income. This pricing requires the modelling the development of the term structure of interest rates
- The arbitrage possibility between various maturity bonds- the analysis of term structure can be useful in the comparisons of yields of these bonds
- The expectations about the economy - it seems that the shape of the term structure curve has an influence on future economic activity, including investment and consumption and can incorporate useful information about future inflation

There are a number of theories explaining the dynamics of the term structure of interest rates. In recent years, a new way of modelling the term structure has evolved entitled the stochastic process modelling of the term structure. This approach requires several assumptions: the term structure and bond prices are related to certain stochastic factors, these underlying factors are assumed to evolve over time according to a particular hypothesized stochastic process, and the interest rates and bond prices that result must satisfy no-arbitrage conditions. In addition, the valuation of options on fixed income securities all require some assumptions about the term structure
generating process. Much of the research on term structure field has been stimulated by the need to value such contingent claims.

The next step after setting up the model is to calibrate this model to the market data. This means determining the parameters of the model (usually the volatilities of interest rates and the correlations between various maturity interest rates). A lot of research has been done in the field of calibrating various models of IR to market data of developed countries (see next section), however there a gap in the field of calibrating models of IR to the transition countries’ markets. There are a few reasons why this work has not been done yet: the models of IR usually contains strong assumptions about the efficiency of IR markets, there is a problem with access to the data (due to frequent changes in the recording of statistics), and so on.

The main reason for the failure of the calibration of more complex and complicated models of IR to transition markets is that these models are usually calibrated to exactly match the prices of some frequently traded derivatives, as for example swaptions or caps and floors. But there is no market for these derivatives in transition countries, or at least these derivatives are traded very rarely and thus their prices are not reliable and the researchers cannot take them as benchmark prices. Therefore the alternative techniques are needed.

As previously explained, the practitioners and the policymakers in transition countries have no access to the more complex pricing methods of IR derivatives. We can agree that the number of various derivatives and the volume of trade in these derivatives will increase sharply in the coming years. This insight is based on the fact that hedging of opened position is almost necessity and with the development of markets various firms will demand such possibility. Exact pricing methods is thus life-important, not only for business and traders, but also for the regulators of markets to avoid market failures. My research would help to accomplish the goal of the precise calculation of derivatives prices. Also, the research should answer the question to what extent are these models reliable for pricing derivatives in transition markets. To achieve this, we propose a method for calibrating multi-factors models of term structure for transition markets.

In the existing research are two types of models of IR: models of short-rate and models of whole term structure. The former models are older and are based on the modelling of the interest rate over the smallest possible time interval. The whole term-structure one can obtain from predicted future paths of the short-rate. The latter models are modelling the whole term-structure at one time and thus are more complex and have better pricing implications (Rebonato (1998)). They also allow various twists in the shape of the term structure, while the models of short-rate allow only increases
or decreases of the whole term structure curve. The most used models of this class are Heath-Jarrow-Morton (Heath, Jarrow and Morton (1992)) and Brace-Gatarek-Musiela (Brace, Gatarek and Musiela (1997)) models. As described in the next section, the usual method of calibrating the BGM model is to calibrate it to the prices of caps. The BGM model offers a closed solution for the price of caps, where the parameters are volatilities of some forward rate. By inverting this formula, one can obtain the implied volatility from the price of the cap.

If however the researcher has no access to the prices of caps, the volatilities and correlations of various interest rates have to be estimated from other sources of data, but they have to be consistent. So, the goal is to find the methodology to obtain consistent parameters from market data available.

3 Literature review

The main streams are the (general equilibrium) models of short rate, the stochastic volatility modelling and the no-arbitrage models of term structure.

The first to use a general equilibrium approach was Merton (1973) to derive a model of discount bond prices. His model was simply a Brownian motion with drift. The next to use a model of IR was Vasicek (1977), and his model belongs to the most used models of IR. Vasicek made the following assumptions: (A.1) The instantaneous (spot) interest rate follows a diffusion process; (A.2) the price of a discount bond depends only on the spot rate over its term; and (A.3) the market is efficient. Under these assumptions, he showed by means of an arbitrage argument that the expected rate of return on any bond in excess of the spot rate is proportional to its standard deviation. This property is then used to derive a partial differential equation for bond prices. The solution to that equation is given in the form of a stochastic integral representation.

This general equilibrium model has a big disadvantage in that it allows for negative interest rates due to constant coefficient for volatility. This setting was changed by Cox, Ingersoll and Ross (1985), who use an inter-temporal general equilibrium asset pricing model to study the term structure of interest rates. In this model, anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption all play a role in determining bond prices. The volatility of the short rate depends on its value. Many of the factors traditionally mentioned as influencing the term structure are thus included in a way which is fully consistent with maximizing behavior and rational expectations. The model leads to specific formulas for bond prices which are well suited to empirical testing.
Calibration methodologies for these models are known and widely used. One of the approaches is the estimation of parameters of models using the Generalized Method of Moments, in this context pioneered by Chan, Karolyi, Longstaff and Sanders (1992). They found for U.S. treasury data that the most successful models in capturing the dynamics of the short rate are those that allow the volatility of IR changes to be highly sensitive to the level of these riskless rates. It is clear, that these results have important implications for the use of the different term structure models in valuing interest rate dependent derivatives. The problem is, that the GMM method can give imprecise results. That was the motivation of Nowman (1997), who proposed a method of estimation based on the Gaussian estimation method of continuous time dynamic models (that means the method based on using the maximum likelihood technique). He found that for U.K. data the findings of Chan et al. (1992) are not valid and that the volatility of short rate is not sensitive to the level of rate in this case; for the U.S. data these findings are similar to Chan et al. (1992). Nowman (1997) uses better method of estimation, as his model allows to use an exact maximum likelihood estimator, which can help to reduce some of the temporal aggregation bias.

The next approach to modelling the IR is called no-arbitrage pricing. It evolved from the previous approach, one of the differences is, that this approach describes the whole term structure, not only one of its point (as is short rate in the previous text). It has two purposes in relation to the term structure of interest rates. The first, is to price all zero coupon bonds of varying maturities from a finite number of economic fundamentals, called state variables. The second, is to price all interest rate sensitive contingent claims, taking as given the prices of the zero coupon bonds. Heath et al. (1992) presented a unifying theory for valuing contingent claims under a stochastic term structure of interest rates. This methodology, based on the equivalent martingale measure technique, takes as given an initial forward rate curve and a family of potential stochastic processes for its subsequent movements. A no-arbitrage condition restricts this family of processes, yielding valuation formulae for interest rate sensitive contingent claims, which do not explicitly depend on the market prices of risk.

In most developed markets, caps and floors are the most traded derivatives. A cap is a strip of caplets each of which is a call option on a forward rate. Market practice is to price the option assuming that the underlying forward rate process is lognormally distributed with zero drift. Consequently, the option price is given by the Black-Scholes formula. (Black and Scholes (1973)). In an arbitrage free framework, however, forward rates over consecutive time intervals are related to one another and cannot all be lognormal under one measure. Brace et al. (1997) show that mentioned market practice
can be made consistent with an arbitrage-free term structure model.

Calibration of these models to the market data is much more problematic than in the case of short rate models. These models are driven by more independent factors and each forward rate has its own volatility parameters for these factors, which are interdependent. The correlation matrix of forward rates is also important in these models. All these parameters (volatilities and correlation matrix) have to be estimated from the market data consistently, in order to preserve all relationships. The calibration methodology based on the prices of caps is described by Rebonato (1999). It can be extended to the numerical simulation for determining the prices of path-dependent derivatives, sensitive to interest rate.

However, no literature about the calibration of the models of IR to transition markets is available. The pricing approach of Rebonato (1999) cannot be used for transition countries, as it is based on prices of caps and the volatility implied by these caps and as mentioned in the previous section, these products are either not traded, or their prices have no explanatory power (if agreed without economic reasoning). The methodology of the calibration of short-rate models can be used without any exception to calibrate the models for transition countries’ data, because they are based only on levels of interest rates.

The last approach to the modelling of term structure is based on the so-called stochastic volatility assumption. It means that the volatility of stochastic process itself follows a stochastic process. This method allows one to estimate a short rate process without loss of efficiency and consistency and uses the quasi-maximum likelihood method. The first to apply this approach is Ait-Sahalia (1996). Ball and Torous (1999) estimate a stochastic volatility model of short-term riskless interest rate dynamics. Estimated interest rate dynamics are broadly similar across a number of countries and reliable evidence of stochastic volatility is found throughout. In contrast to stock returns, interest rate volatility exhibits faster mean-reverting behavior and innovations in interest rate volatility are negligibly correlated with innovations in interest rates. The less persistent behavior of interest rate volatility reflects the fact that interest rate dynamics are impacted by transient economic shocks such as central bank announcements and other macroeconomic news.

4 Methodology

The first step is to get data about the interest rates from various transition countries and calculation of term structure over certain time periods. The
parameters of BGM model (volatilities and correlations of interest rates with various maturities) is possible to obtain either using the prices of traded derivatives (so to calibrate on implied, or historical volatilities) or using the information on conditional volatilities extracted using some model of conditional volatility (as various types of GARCH models). In this work we propose the use of (G)O-GARCH model and describe the way how it can be used to achieve the calibration of parameters of BGM model.

After obtaining the volatilities from market data for various forward rates, numerical methods may be needed to get the volatilities consistent with the theory, as volatilities of various forward rates are not independent. The next part of the research will be to design an algorithm to model the evolution of interest rates into the future to get future shapes of term structure. The knowledge of future shapes of term structure is crucial in determining the price of interest-rate sensitive derivatives.

The next subsections describe the basic theory of interest rate modelling and the calibration of the BGM model using the (G)O-GARCH model.

4.1 Definitions and relationships

The most basic contract based on the interest rate is an agreement to borrowing a particular amount now in exchange for a promise to repay a bigger amount later. In general, the value of such agreement depends on the credibility of the debtor and other factors different from the time value of money. However, in this paper is not of our interest to find answers to these questions, so we will assume, that there is no possibility of default. Let now define basic concepts:

Let $T^* > 0$ is fixed time horizon for all activities in the market. Under discount bond with maturity $T \leq T^*$ we will understand the contract, which pays out his owner a unit of cash in the fixed time $T$ in the future. The price of discount bond we will denote as $P(t, T)$. Clearly, $P(T, T) = 1$. For every maturity $T$ we will assume, that the price of bond $P(., T)$ follows stochastic, strictly positive process.

The curve $P(t, .)$ describes the price of whole spectrum of bonds with various maturities. We define process $R(t, T)$, called yield to maturity. Formally,

$$R(t, T) = -\frac{1}{T - t} \ln P(t, T) \quad \forall t \in (0, T).$$

Under term structure we will understand a relationship, which expresses yield $R(t, T)$ as a function of the maturity $T$. A forward contract is an agreement negotiated in time $t$ about paying out cash at some later time $T_1$ and receiving the payment back in time $T_2 > T_1$. This claim can be replicated
in time $t$ by buying $T_2$ bond and selling $k$ units of $T_1$ bonds. The initial costs are $P(t, T_2) - kP(t, T_1)$ in time $t$, we pay $k$ in time $T_1$ and we receive the 1$ payment in $T_2$. To give this contract a zero value, $k$ has to be equal

$$k = \frac{P(t, T_2)}{P(t, T_1)}.$$  

The adequate payoff we will call a forward rate covering the period $\langle T_1, T_2 \rangle$ and will denote it as $f(t, T_1, T_2)$. So, we have

$$\frac{P(t, T_2)}{P(t, T_1)} = e^{-f(t, T_1, T_2)(T_2 - T_1)} \quad \forall t \leq T_1 \leq T_2,$$

or

$$f(t, T_1, T_2) = -\frac{\ln P(t, T_2) - \ln P(t, T_1)}{T_2 - T_1}.$$  

If we let $T_2 \to T_1$, we get an instantaneous forward rate

$$f(t, T) = -\frac{\partial}{\partial T} \ln P(t, T),$$

or equivalently

$$P(t, T) = \exp \left( -\int_t^T f(t, u)du \right) \quad \forall t \in \langle 0, T \rangle. \quad (1)$$

Let $r_t$ be instantaneous interest rate over interval $\langle t, t + dt \rangle$. Then the process $B_t$ defined

$$B_t = \exp \left( \int_0^t r_u du \right) \quad \forall t \in \langle 0, T^* \rangle$$

will be called a savings account.

4.2 The models of interest rates

The Heath, Jarrow, Morton (1992) model

The earlier models of term structure were based on the explicit modelling of short rate evolution. This approach have arisen from the need to price simple derivatives of term structure, as for example options or swaps, which depend on one underlying bond. The approach by Heath et al. (1992), to the modelling of term structure evolution is on the other hand based on the explicit specification of the dynamic of instant forward rates $f(t, T)$. This
method is the generalization of the simple models, as shown in Baxter and Rennie (1996).

Let \( W \) be \( d \)-dimensional Brownian motion defined on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\). With the dot symbol (\( \cdot \)) we will denote the standard product of vectors. Under the assumption which are stated in the appendix, the HJM model is characterized by the following theorem (the full treatment as well as proofs can be found in the appendix):

**Theorem 1** For the arbitrary maturity \( T \leq T^* \), under the assumption of non-existence of arbitrage, the dynamics of the price of bond \( P(t, T) \) under the risk-neutral measure \( \mathbb{P}^* \) is

\[
dP(t, T) = P(t, T) (r_t dt - b(t, T)) \cdot dW^*_t
\]

and the forward rate \( f(t, T) \) satisfies

\[
df(t, T) = \sigma(t, T) b(t, T) dt + \sigma(t, T) \cdot dW^*_t,
\]

where \( b(t, T) = -\int_t^T \sigma(t, u) du \).

The Brace, Gatarek, Musiela (1997) model

The common feature of earlier models of interest rates (up to the HJM model) is the fact that (explicitly or implicitly) they include a specification of the stochastic behavior of non-observable financial quantities, as for example instantaneous forward rates. The calibration of these models to the set of market data thus needs some transformation of these data through the “black-box” of model to the dynamics of non-observable quantities.

This picture has radically changed with the introduction of the BGM (Brace et al. (1997)) model, which describes directly observable market quantities, as discrete LIBOR forward rates.

Let us fix a positive real number \( \delta \). Following the definition, the forward \( \delta \)-LIBOR rate \( L(t, T) \) is a discrete forward rate over the interval \( \langle T, T + \delta \rangle \) and is given by relationship

\[
1 + \delta L(t, T) = \frac{P(t, T)}{P(t, T + \delta)} \quad \forall t \in (0, T).
\]

It is possible to find in the appendix the derivation of the dynamics of \( L(t, T) \) under the risk-neutral measure. The advantage of the BGM is that the \( L(t, T) \) rates can be modelled as lognormal.
4.3 The calibration of BGM model using (G)O-GARCH model

Let us discretize the BGM model in the following way (full derivation of this model can be found in the appendix):

\[ y_t^i = \frac{L(t+1, T_i) - L(t, T_i)}{L(t, T_i)} = \mu_i(t)\Delta t + \sum_{k=1}^{r} a_{ik}(t)\Delta W_t^k, \quad (5) \]

where \( \Delta W_t^k \) is an increase at time \( t \) of \( k \)th Brownian motion, \( a_{ik}(t) \) are instantaneous volatilities of \( i \)th LIBOR rate belonging to the \( k \)th factor (or Brownian motion), \( \mu_i(t) \) is drift of \( i \)th LIBOR rate.

Let us suppose that we have \( T \) observations \( y_t^i \) on the returns of \( k \) interest rate series with various maturities (i.e. one week, two weeks, one month, etc.). (G)O-GARCH models are based on so-called principal component analysis, each component being a simple linear combinations of the original returns series. The weights in these linear combinations are determined by the eigenvectors of the correlation matrix of the returns matrix. The principal components are ordered according to the size of eigenvalues (which are in fact variances of principal components) so that the first principal component, the one corresponding to the largest eigenvalue (i.e. the one with the largest variance) explains most of the variation. If the system is highly correlated (as it is assumed for interest rates with various maturities), only first few eigenvalues will be significantly different from zero. That means, that one can simplify the task by taking just few principal components into account to represent the original variables to a fairly high degree of accuracy.

The following text is based on Alexander (2002). Let us have the original returns in \( T \times k \) matrix \( Y \). We normalize these \( k \) series into series with zero mean and unit variance, so that we get matrix \( X \). Now, let matrix \( W \) be the matrix of eigenvectors of \( X'X/T \), and \( \Lambda \) be the associated diagonal matrix of eigenvalues, ordered according to decreasing magnitude of eigenvalue. The principal components of \( Y \) are given by the matrix \( P \):

\[ P = XW. \quad (6) \]

It can be shown, that the matrix \( P \) is orthogonal. Because of orthogonality of matrix \( W \), (6) can be rewritten as \( X = PW' \), which means:

\[ x_i = w_{i1}p_1 + \cdots + w_{ik}p_k \]

or

\[ y_i = \mu_i + \omega_{i1}p_1 + \cdots + \omega_{ir}p_r + \epsilon_r, \quad (7) \]
where $\omega_{ij} = w_{ij}\sigma_i$, $\mu_i, \sigma_i$ are the mean and standard deviation of $y_i$ and the error term $\epsilon_i$ means the approximation from using only $r$ out of the $k$ principal factors. When we take variances of (7), we get

$$V = ADA' + V_{\epsilon},$$

where $D = \text{diag}(V(p_1), \ldots, V(p_r))$ is a diagonal (because of orthogonality) covariance matrix of chosen $r$ principal factors, $A = (\omega_{ij})$ and $V_{\epsilon}$ is the covariance matrix of the errors. Ignoring the error term gives us the approximation that forms the basis for the model of covariance matrix $V$

$$V \approx ADA'.$$

Because the matrix $D$ is known, it is enough to model the matrix $D$, which can be achieved by running $r$ simple GARCH models on the first $r$ principal components from $P$. This is the basis of O-GARCH model.

The main limitation of this approach is that the principal components are only unconditionally uncorrelated so the assumption that off-diagonal elements of $D$ are zero may be unnecessarily strong. This assumption has been relaxed by van der Weide (2002), who develops a generalization of the model called Generalized O-GARCH. In this model the univariate GARCH specifications are applied to transformed variables $P^* = PU$, where $U$ is an orthonormal matrix which can be estimated using conditional information from the observed data.

Now, let us closely look at the specification (5) and (7). Because the series $p_1, \ldots, p_r$ are generated from the series with zero mean and unit variance, we can consider them as increases of $r$ Brownian motions, so that estimates of coefficients $\omega_{ij}$ are actually estimates of the conditional volatility belonging to $j$th Brownian motion. After obtaining these estimates (using (G)O-GARCH) the next step is needed.

As these estimates are just averages of instantaneous volatilities, we need to choose the appropriate functional form for the time profile of these volatilities. As suitable form Rebonato (2002) recommends to write instantaneous volatility $a_{ik}(t, T_i)$ (we added redundant parameter $T_i$ to better determine that we think the rate maturing at time $T_i$) as a product of maturity specific term and of time-to-maturity specific term:

$$a_{ik}(t, T_i) = g(T_i)h(T_i - t).$$

Rebonato (2002) also argues what is the best possible functional form for both $g, h$ functions.

The last step in calibration process is to perform numerical simulation of future term structure evolution. One possible simulation algorithm is.
described in Brace, Musiela and Schloegl (1998). These numerical simulation then can be used for the construction of processes of derivatives as well as replicating portfolios, needed for their comparisons.

5 Data description

For our research data for 4 countries are used: Slovak Republic, Czech Republic, Hungary and Poland. Various time spans have to be used as the quotation vary across the countries. The data-sets for Slovak Republic and Czech Republic are from the web pages of central banks, the data-sets for Hungary and Poland come from Reuters’ databases. In this work we use only data for the short end of term structure (the tenor with maturity from overnight to 1 year). There are a few reasons for such a restriction.

The markets in transition countries are often imperfect and non-developed and the interest-rate market is not an exception. Although countries mentioned in the previous paragraph have the most developed markets among the transition countries, they are still not at the level of developed countries. The interest-rate market is a very good example as banking institutions lend and borrow mostly money with the shortest maturities and the official quotation of interest rates exists only for maturities up to 1 year. The pricing of the instruments with longer maturities is based on swaps and rates calculated from swaps. These rates are quoted at Reuters (except Slovak Republic), but as was mentioned above, with longer maturities the market is even more imperfect. Mostly there exists only government bonds with higher maturities, municipal or corporate bonds exists only rarely.

So the reasons for restricting the data-sets to maturities up to one year are the market imperfections (no trading with longer maturities) and data availability. The calibration is thus performed for the model, where 12 forward rates are modelled, whose maturities differ by one month. We are concentrating on the interbank offered rates. We have few reasons for that. Firstly, the BGM model describes the evolution exactly of interbank offered rates. Secondly, these rates, although in general not risk-free (there is always a risk of bankruptcy of a bank, incorporated in the rates), are used by banks and other financial institutions as the lending and borrowing rates, so it is needed to know their parameters.

The following data are used:

- Slovak Republic - Bratislava InterBank Offered Rate (BRIBOR) - the analogue of LIBOR rates, time span is from 5th June 2000 to 28th November 2003 (earlier data are not usable as the rates were quoted only up to 6 months maturity)
- Czech Republic - Prague InterBank Offered Rate (PRIBOR) - the analogue of LIBOR rates, time span is from 2\textsuperscript{nd} January 1998 to 28\textsuperscript{th} November 2003

- Poland - Warsawa InterBank Offered Rate (WIBOR) - the analogue of LIBOR rates, time span is from 2\textsuperscript{nd} January 2001 to 28\textsuperscript{th} November 2003 (again, no quoting of longer rates beforehand)

- Hungary - Budapest InterBank Offered Rate (BUBOR) - the analogue of LIBOR rates, time span is from 2\textsuperscript{nd} May 2002 to 28\textsuperscript{th} November 2003 (previously only rates with maturities 1,3 and 6 months were quoted)

6 Estimation results

In this section we use the methodology stated above to analyze the development of interest rate covariances and correlations. This section is divided into the subsection corresponding to the countries analyzed. In each subsection we first characterize more closely the data, then state the results of estimation and at the end check whether the methodology captures the main characteristics of the data and its time evolution.

For all currencies the original interest rate time series were transformed in order to fit the BGM model specification. The time series used in estimations are constructed as yields of this original interest rate process.

6.1 Czech Republic

In this case we are using the 8 PRIBOR interest rate time series with the maturities from 1 week to 1 year. The data covers the time span from the beginning of January 1998 to the end of November 2003, what comprises 1492 daily observations. The time evolution can be found in figure 1. The basic characteristics of the data are in table 1.

In table 2 one can see the eigenvectors of the unconditional variance matrix, i.e. the weights of respective interest rates in principal factors. There are also the eigenvalues of the unconditional variance matrix (equal to the unconditional variances of principal factors) and the fraction of total variance explained by the concrete principal factor. We can see, that the first component directs the horizontal movements of the term structure, the second one directs the changes in slope and the third directs the changes in curvature of the term structure.
For the modelling of the Czech interest rate market we have chosen 3 principal factors, as they explain 96.4% of total variance. Autocorrelation is present only for the rates with the higher maturities, so we decided not to correct for autocorrelation in the modelling of principal factors. We calculated the time series of principal factors using the procedure described in the Methodology section. The results of O-GARCH procedures on this principal factors are in table 3.

Although principal factors have zero mean by definition, a constant term was used in estimation. The conditional variances were specified as GARCH(1,1) processes, so that:

$$p_t = c + \mu_t, \quad \mu_t \sim N(0, H_t),$$

(8)
Table 2: Principal components weights for CZK

<table>
<thead>
<tr>
<th>Component</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y, WEEK</td>
<td>0.322</td>
<td>-0.495</td>
<td>0.411</td>
<td>-0.249</td>
<td>0.012</td>
<td>0.133</td>
<td>0.631</td>
<td>0.050</td>
</tr>
<tr>
<td>Y, 2W</td>
<td>0.339</td>
<td>-0.477</td>
<td>0.316</td>
<td>-0.303</td>
<td>0.730</td>
<td>-0.169</td>
<td>-0.342</td>
<td>0.183</td>
</tr>
<tr>
<td>Y, M</td>
<td>0.365</td>
<td>-0.245</td>
<td>0.037</td>
<td>-0.056</td>
<td>0.133</td>
<td>0.631</td>
<td>0.050</td>
<td></td>
</tr>
<tr>
<td>Y, 2M</td>
<td>0.376</td>
<td>-0.058</td>
<td>-0.521</td>
<td>-0.021</td>
<td>0.206</td>
<td>0.729</td>
<td>-0.064</td>
<td>0.015</td>
</tr>
<tr>
<td>Y, 3M</td>
<td>0.376</td>
<td>0.357</td>
<td>0.357</td>
<td>0.357</td>
<td>0.357</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y, 6M</td>
<td>0.365</td>
<td>0.357</td>
<td>0.357</td>
<td>0.357</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y, 9M</td>
<td>0.376</td>
<td>0.357</td>
<td>0.357</td>
<td>0.357</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Y, Y</td>
<td>0.339</td>
<td>0.357</td>
<td>0.357</td>
<td>0.357</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eigenvalue</td>
<td>6.314</td>
<td>1.142</td>
<td>0.262</td>
<td>0.099</td>
<td>0.068</td>
<td>0.051</td>
<td>0.030</td>
<td>0.030</td>
</tr>
<tr>
<td>Total variance expl.</td>
<td>0.789</td>
<td>0.932</td>
<td>0.964</td>
<td>0.977</td>
<td>0.985</td>
<td>0.992</td>
<td>0.996</td>
<td>1.000</td>
</tr>
</tbody>
</table>

where the diagonal elements of $H_t$ are described by

$$h_{i,t} = \alpha_0 + \alpha_1 \mu_{i,t-1}^2 + \beta_1 h_{i,t-1}, \ i = 1, 2, 3.$$  \hspace{1cm} (9)

Table 3: Regression results for CZK

<table>
<thead>
<tr>
<th>Component</th>
<th>First Coefficient</th>
<th>First t-stat</th>
<th>Second Coefficient</th>
<th>Second t-stat</th>
<th>Third Coefficient</th>
<th>Third t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.148</td>
<td>1.176</td>
<td>0.026</td>
<td>1.361</td>
<td>0.037</td>
<td>0.327</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>0.020</td>
<td>0.754</td>
<td>0.027</td>
<td>0.793</td>
<td>0.052</td>
<td>1.083</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.011</td>
<td>1.911</td>
<td>0.034</td>
<td>1.853</td>
<td>0.112</td>
<td>1.829</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.987</td>
<td>369.062</td>
<td>0.963</td>
<td>37.377</td>
<td>0.883</td>
<td>14.098</td>
</tr>
</tbody>
</table>

As expected, the constant coefficients are statistically unimportant. The results show that there is basis for a strong suspicion of the presence of a unit root in all conditional variance processes. However, all $\beta$ coefficients are large, showing that there are fairly persistent volatilities of components and that the components in the Czech case are almost nonreactive to the inflow of new information. This result is consistent with the observed facts that the interest rates in the Czech Republic show a very low level of volatility and they remain relatively fixed for a number of days. This is also supported by the levels of weights for the first principal component, which vary at around 0.35, so only one third of the shock is translated to the movement in horizontal direction.

Using the estimated results we are able to generate the time evolution of conditional covariance and correlation matrices. As an illustration, in
figure 2 can be found the conditional correlation matrix of PRIBOR rates as seen on the market on 28th November 2003. Figure 3 shows the estimated conditional volatility of 1 month PRIBOR rate.

Both pictures are fairly consistent with the data. Although the overall volatility level is low, the periods of “increased” volatility level in figure 3 corresponds to the periods with higher market activity. Similarly, the correlation surface calculated is consistent with the market development seen around the end of November 2003.

From these facts we can assume that the calibration procedure is able to reveal the true market development and as such can be used in the pricing of IR sensitive derivatives. We are able to ascertain the development of conditional correlations and volatilities among the rates, which are factors influencing the prices of such derivatives.

6.2 Slovak Republic

For Slovakia the data had a similar structure as for the Czech Republic. The data used were 8 BRIBOR interest rates time series with the maturities from 1 week to 1 year. They cover the time span from the beginning of June 2000 to the end of November 2003, which consists of 871 daily observations. The time evolution can be found in figure 4. The basic characteristics of the data are in table 4.
Figure 3: Estimated conditional volatility of 1 month PRIBOR rate

Table 4: Characteristics of time series for SKK

<table>
<thead>
<tr>
<th>Series</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Sum</th>
<th>$Q_{10}$ stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y_WEEK</td>
<td>0.000221</td>
<td>0.0298</td>
<td>-0.161</td>
<td>0.1661</td>
<td>0.193</td>
<td>72.9</td>
</tr>
<tr>
<td>Y_2W</td>
<td>-0.000064</td>
<td>0.0179</td>
<td>-0.179</td>
<td>0.1101</td>
<td>-0.056</td>
<td>53.8</td>
</tr>
<tr>
<td>Y_M</td>
<td>-0.000300</td>
<td>0.0095</td>
<td>-0.187</td>
<td>0.0616</td>
<td>-0.261</td>
<td>34.6</td>
</tr>
<tr>
<td>Y_2M</td>
<td>-0.000352</td>
<td>0.0092</td>
<td>-0.193</td>
<td>0.0615</td>
<td>-0.307</td>
<td>63.3</td>
</tr>
<tr>
<td>Y_3M</td>
<td>-0.000365</td>
<td>0.0093</td>
<td>-0.179</td>
<td>0.0694</td>
<td>-0.317</td>
<td>65.5</td>
</tr>
<tr>
<td>Y_6M</td>
<td>-0.000443</td>
<td>0.0084</td>
<td>-0.141</td>
<td>0.0675</td>
<td>-0.386</td>
<td>189</td>
</tr>
<tr>
<td>Y_9M</td>
<td>-0.000490</td>
<td>0.0088</td>
<td>-0.158</td>
<td>0.0978</td>
<td>-0.427</td>
<td>180</td>
</tr>
<tr>
<td>Y_Y</td>
<td>-0.000503</td>
<td>0.0090</td>
<td>-0.159</td>
<td>0.0992</td>
<td>-0.438</td>
<td>165</td>
</tr>
</tbody>
</table>

Tables 5 and 6 show the eigenvectors of the unconditional covariance matrix together with eigenvalues and the results of GARCH procedures. Again, we choose 3 factors, which account for a total of 96% of variance. As there is strong autocorrelation presented in data (as we can see from the $Q$ statistics in table 4), we also used one lag in the mean equation for principal factors, so that the mean equation was

$$p_t = c + b p_{t-1} + \mu_t, \quad \mu_t \sim N(0, H_t),$$

(10)

with the equation for conditional variance the same as in the previous case.
The analysis of results is slightly different from the previous case. The reaction to the inflow of new information concerning the variance of the first component is very high, so there is a high probability that a shock in the first component will influence the volatilities of interest rates in influential amounts. However on the other side, the persistence of volatility is rather low and thus any shock will soon disappear. This is in accordance with the fact, that volatility of interest rates in Slovakia is higher than in the case of the Czech Republic, as we can observe in figure 4. Again, it is possible to observe the presence of unit root in the GARCH processes.

The next figure is the correlation surface as of 28th November 2003 (figure 5). The time evolution of volatility of 1 month BRIBOR rate is in figure 6.
Table 6: Regression results for SKK

<table>
<thead>
<tr>
<th>Component</th>
<th>First</th>
<th>Second</th>
<th>Third</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coefficient</td>
<td>t-stat</td>
<td>Coefficient</td>
</tr>
<tr>
<td>Constant</td>
<td>-0.019</td>
<td>-0.264</td>
<td>0.055</td>
</tr>
<tr>
<td>b</td>
<td>-0.387</td>
<td>-1.723</td>
<td>-0.065</td>
</tr>
<tr>
<td>α₀</td>
<td>0.205</td>
<td>1.957</td>
<td>0.042</td>
</tr>
<tr>
<td>α₁</td>
<td>0.501</td>
<td>4.498</td>
<td>0.038</td>
</tr>
<tr>
<td>β₁</td>
<td>0.495</td>
<td>4.487</td>
<td>0.956</td>
</tr>
</tbody>
</table>

Figure 5: Estimated correlation surface of SVK as of 28th Nov. 2003

The 10 observations were from 18th Nov. 2002 to 29th Nov. 2002 changed to 0.5, as their original values were in interval from 0.8 to 2.3, in order to increase the resolution of the figure. On 18th Nov. 2002 the Slovak central bank lowered the discount rate (at that time the official rate for the refinancing of banks) from 8% to 6.5%. Thus in the next few days the conditional volatility was affected for all rates, as can be seen from figure 4. Otherwise, the conditional volatility is low, mostly under 1%. Due to this sharp change in level it would be more suitable to use some type of switching regime GARCH. However, there are only 250 observations after the break, which are not enough to ensure stability of estimated coefficients. This use of switching regime would also be more reasonable from the pricing point of view. The estimated volatility levels are unnecessarily high and it took some time for the shock to disapper.
With the exception of this break in level, it seems reasonable to use our approach to pricing the IR sensitive derivatives. For example, the correlation surface showed can reveal the fact that in the last examined trading days the longer maturities (from 6 to 12 months) were stable, while the shorter ones were increasing. This is in accordance with figure 5, where there is a small negative correlation between short-term and long-term IR rates.

6.3 Hungary

In the case of Hungary we used 7 interest rate time series with the maturities from 1 week to 1 year (we did not have the series with 2 months’ maturity at our disposal). The time span covered by these rates is the beginning of May 2002 to the end of November 2003, what includes 406 daily observations. The time span is much shorter than in the previous cases. It is due to the fact that until May 2002 only rates with one, three and six months maturity were quoted in the market. The evolution of these rates and their basic characteristics are in figure 7 and table 7.

There is no autocorrelation presented, so we can use the same estimation equation as for the Czech crown. Although the first 2 principal components account for almost 97% of total variation in examined time series, we used again 3 components in order to allow for the changes of curvature of term structure.
The results of the estimation are quite interesting. Coefficients by innovation term \((\alpha_1)\) are very high for all 3 estimated principal components. This means that the persistence of shocks to volatility is even lower than in the case of the Slovak crown. The \(\alpha_0\) coefficient, meaning the constant in the volatility process, is unusually high, so that the volatility connected with the first coefficient should be higher than with the previous currencies. With the next figures it is possible track the consequences of these facts. The shocks in volatility are much more frequent than in the previous cases and they also do not have a long duration. Also the estimated correlations are in line with the evolution of the market (figures 8,9).
The previously stated facts may cause problems when using this calibration for pricing of IR derivatives, as the external shocks to volatility are too high and too frequent. When comparing the estimated periods of high volatility with figure 7, we see that we are able to capture the periods of high volatility in data however, these periods may cause the instability of prices dependent on these factors.

6.4 Poland

We used 6 interest rate time series with the maturities from 1 week to 1 year (we did not have series with the 2 weeks and 2 months maturities at our disposal). Time span covered is from the beginning of January 2001 to the end of November 2003, what comprises 739 daily observations. The evolution of these rates and their basic characteristics are in figure 10 and table 10. The $Q$ statistics are statistically significant for all rates except the one with three month maturity. Therefore we chose to model principal factors with the autocorrelation lag. We used 3 factors, as they explain 93% of total variation. The standard deviation of the changes in interest rates are higher than in the previous cases. Also the weights of principal components are higher. These facts signalize, that the volatilities for interest rates in the

<table>
<thead>
<tr>
<th>Component</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y_WEEK</td>
<td>0.337</td>
<td>-0.509</td>
<td>0.593</td>
<td>0.114</td>
<td>0.018</td>
<td>0.488</td>
<td>-0.155</td>
</tr>
<tr>
<td>Y_2W</td>
<td>0.367</td>
<td>-0.445</td>
<td>0.109</td>
<td>-0.015</td>
<td>0.029</td>
<td>-0.785</td>
<td>0.196</td>
</tr>
<tr>
<td>Y_M</td>
<td>0.389</td>
<td>-0.282</td>
<td>-0.570</td>
<td>-0.595</td>
<td>-0.144</td>
<td>0.245</td>
<td>-0.095</td>
</tr>
<tr>
<td>Y_3M</td>
<td>0.414</td>
<td>0.035</td>
<td>-0.435</td>
<td>0.596</td>
<td>0.518</td>
<td>0.113</td>
<td>-0.030</td>
</tr>
<tr>
<td>Y_6M</td>
<td>0.401</td>
<td>0.277</td>
<td>0.007</td>
<td>0.243</td>
<td>-0.613</td>
<td>0.144</td>
<td>0.554</td>
</tr>
<tr>
<td>Y_9M</td>
<td>0.380</td>
<td>0.393</td>
<td>0.114</td>
<td>0.046</td>
<td>-0.281</td>
<td>-0.227</td>
<td>-0.745</td>
</tr>
<tr>
<td>Y_Y</td>
<td>0.352</td>
<td>0.481</td>
<td>0.329</td>
<td>-0.464</td>
<td>0.505</td>
<td>0.029</td>
<td>0.255</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Component</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>5.567</td>
<td>1.219</td>
<td>0.142</td>
<td>0.040</td>
<td>0.012</td>
<td>0.011</td>
<td>0.008</td>
</tr>
<tr>
<td>Total variance expl.</td>
<td>0.795</td>
<td>0.969</td>
<td>0.990</td>
<td>0.996</td>
<td>0.997</td>
<td>0.999</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 8: Principal components weights for HUF

<table>
<thead>
<tr>
<th>Component</th>
<th>First Coefficient</th>
<th>First t-stat</th>
<th>Second Coefficient</th>
<th>Second t-stat</th>
<th>Third Coefficient</th>
<th>Third t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>-0.238</td>
<td>-1.990</td>
<td>0.105</td>
<td>1.147</td>
<td>0.023</td>
<td>0.507</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>1.040</td>
<td>1.255</td>
<td>0.074</td>
<td>0.956</td>
<td>0.010</td>
<td>0.554</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.585</td>
<td>7.438</td>
<td>0.537</td>
<td>5.420</td>
<td>0.626</td>
<td>0.956</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.386</td>
<td>4.738</td>
<td>0.451</td>
<td>4.512</td>
<td>0.351</td>
<td>0.539</td>
</tr>
</tbody>
</table>

Table 9: Regression results for HUF
Figure 8: Estimated correlation surface of HUF as of 28th Nov. 2003

<table>
<thead>
<tr>
<th>Series</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Sum</th>
<th>$Q_{10}$ stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y_WEEK</td>
<td>-0.00127</td>
<td>0.0298</td>
<td>-0.166</td>
<td>0.196</td>
<td>-0.938</td>
<td>27.0</td>
</tr>
<tr>
<td>Y_M</td>
<td>-0.00163</td>
<td>0.0113</td>
<td>-0.0811</td>
<td>0.0543</td>
<td>-1.207</td>
<td>40.3</td>
</tr>
<tr>
<td>Y_3M</td>
<td>-0.00160</td>
<td>0.00816</td>
<td>-0.0523</td>
<td>0.0540</td>
<td>-1.183</td>
<td>8.9</td>
</tr>
<tr>
<td>Y_6M</td>
<td>-0.00154</td>
<td>0.00755</td>
<td>-0.0455</td>
<td>0.0516</td>
<td>-1.140</td>
<td>32.0</td>
</tr>
<tr>
<td>Y_9M</td>
<td>-0.00151</td>
<td>0.00741</td>
<td>-0.0431</td>
<td>0.0544</td>
<td>-1.118</td>
<td>55.7</td>
</tr>
<tr>
<td>Y_Y</td>
<td>-0.00148</td>
<td>0.00755</td>
<td>-0.0326</td>
<td>0.0573</td>
<td>-1.097</td>
<td>66.1</td>
</tr>
</tbody>
</table>

Table 10: Characteristics of time series for PLZ

Poland have higher levels as those for the previous currencies.

From the results of regression in table 12 we can see that this is the only case, where there is no need for the GARCH regression with the use of unit root correction. The first component has a fairly persistent volatility, however the second coefficient shows a high responsiveness to the random shocks. This means that the trend in the changes in level is stable (with small influences from innovations), while the changes in the slope are more chaotic, but last only for a short time.
Figure 9: Estimated conditional volatility of 1 month BUBOR rate

Figure 10: Time evolution of WIBOR rates
Table 11: Principal components weights for PLZ

<table>
<thead>
<tr>
<th>Component</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>YWEEK</td>
<td>0.259</td>
<td>0.717</td>
<td>-0.590</td>
<td>0.254</td>
<td>-0.060</td>
<td>0.003</td>
</tr>
<tr>
<td>Y_M</td>
<td>0.387</td>
<td>0.453</td>
<td>0.418</td>
<td>-0.659</td>
<td>0.182</td>
<td>-0.001</td>
</tr>
<tr>
<td>Y_3M</td>
<td>0.445</td>
<td>0.060</td>
<td>0.539</td>
<td>0.507</td>
<td>-0.499</td>
<td>0.025</td>
</tr>
<tr>
<td>Y_6M</td>
<td>0.451</td>
<td>-0.201</td>
<td>0.029</td>
<td>0.360</td>
<td>0.782</td>
<td>0.112</td>
</tr>
<tr>
<td>Y_9M</td>
<td>0.442</td>
<td>-0.326</td>
<td>-0.264</td>
<td>-0.169</td>
<td>-0.142</td>
<td>-0.760</td>
</tr>
<tr>
<td>Y_Y</td>
<td>0.429</td>
<td>-0.357</td>
<td>-0.337</td>
<td>-0.288</td>
<td>-0.285</td>
<td>0.639</td>
</tr>
<tr>
<td>Eigenvalue</td>
<td>4.091</td>
<td>1.154</td>
<td>0.331</td>
<td>0.194</td>
<td>0.154</td>
<td>0.074</td>
</tr>
<tr>
<td>Total variance expl.</td>
<td>0.682</td>
<td>0.874</td>
<td>0.930</td>
<td>0.962</td>
<td>0.988</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table 12: Regression results for PLZ

<table>
<thead>
<tr>
<th>Component</th>
<th>First Coefficient</th>
<th>First t-stat</th>
<th>Second Coefficient</th>
<th>Second t-stat</th>
<th>Third Coefficient</th>
<th>Third t-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.063</td>
<td>.929</td>
<td>0.055</td>
<td>1.472</td>
<td>-0.048</td>
<td>-0.269</td>
</tr>
<tr>
<td>b</td>
<td>0.168</td>
<td>3.617</td>
<td>0.072</td>
<td>1.224</td>
<td>-0.125</td>
<td>-2.465</td>
</tr>
<tr>
<td>α0</td>
<td>0.203</td>
<td>1.636</td>
<td>0.365</td>
<td>2.598</td>
<td>0.015</td>
<td>0.767</td>
</tr>
<tr>
<td>α1</td>
<td>0.115</td>
<td>2.577</td>
<td>0.399</td>
<td>2.579</td>
<td>0.071</td>
<td>1.146</td>
</tr>
<tr>
<td>β1</td>
<td>0.831</td>
<td>13.233</td>
<td>0.363</td>
<td>2.477</td>
<td>0.878</td>
<td>7.223</td>
</tr>
</tbody>
</table>

In figure 12 we can see that in November 2003 the one month interest rate had a higher volatility than the long-term average. This may be the reason for a lower correlations of shorter interest rates (week and month) with the longer, as is it observable in figure 11. This can be understood as a confirmation that also in this case the calibration methodology is suitable and the conditional parameters are well in coincidence with the market development.
Figure 11: Estimated correlation surface of PLZ as of 28th Nov. 2003

7 Conclusions

In this research we wanted to calibrate the Brace-Gatarek-Musiela model of interest rates to fit transition markets. This model is the most sophisticated model of interest rates and it has very good pricing implications. Because the standard calibration technique (calibrate to fit prices of caps or swaptions) could not be used, a new technique has to be proposed.

The exact pricing of derivatives is very important for business and for policymakers too. Hedging based on derivatives can be successful only if correct prices of derivatives are available, otherwise there is a possibility of arbitrage. Also, the regulators of markets can set up the pricing rules and can avoid markets failure. The other important argument is that it may help to start up trades with derivatives in the emerging markets.

The estimation results and estimated evolution of conditional volatilities and correlations are in correspondence with the true market development and thus we conclude, that it is possible to calibrate interest rate models to the markets of transition countries as well as to the markets of developed countries.
A The models of interest rates

A.1 The Heath, Jarrow, Morton (1992) model

The earlier models of term structure were based on the explicit modelling of short rate evolution. This approaches have arisen from the need to price simple derivatives of term structure, as for example options or swaps, which depend on one underlying bond. The approach by Heath et al. (1992), to the modelling of term structure evolution is on the other hand based on the explicit specification of the dynamic of instant forward rates $f(t, T)$. This method is the generalization of the simple models, as shown in Baxter and Rennie (1996).

Let $W$ be $d$-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. With the dot symbol $\cdot$ we will denote the standard product of vectors. The HJM model is based on the following assumptions:

1. For the arbitrary fixed maturity $T \leq T^*$, the forward rate $f(t, T)$ is evolving in the following manner:

$$f(t, T) = \int_0^t \alpha(s, T) ds + \int_0^t \sigma(s, T) \cdot dW_s, \quad 0 \leq t \leq T, \quad (11)$$

or in the differences

$$d_t f(t, T) = \alpha(t, T) dt + \sigma(t, T) \cdot dW_t, \quad (12)$$

where the drift $\alpha$ and volatility $\sigma$ are stochastic processes with the values in $\mathbb{R}$, resp. $\mathbb{R}^d$. Formally $\alpha : C \times \Omega \rightarrow \mathbb{R}$, $\sigma : C \times \Omega \rightarrow \mathbb{R}^d$, kde
\[ C = \{(u, t); 0 \leq u \leq t \leq T^*\}. \]

**HJM.2** For the arbitrary maturity \( T \), processes \( \alpha(., T) \) and \( \sigma(., T) \) are such that
\[
\int_0^T |\alpha(u, T)|du + \int_0^T |\sigma(u, T)|^2du < \infty \quad \mathbb{P}\text{-almost everywhere}
\]

Our definition of forward rates \( f(t, T) \) allows us to write the equation for the instant interest rate \( r_t = f(t, t) \). Then, savings account satisfies following equation:
\[
B_t = \exp \left( \int_0^t f(u, u)du \right) \quad \forall t \in (0, T^*]. \quad (13)
\]

The following lemma describes the dynamics of prices of bonds \( P(t, T) \) under the actual (real) probability measure \( \mathbb{P} \).

**Lemma 1** The prices of bonds \( P(t, T) \) satisfy relationship
\[
dP(t, T) = P(t, T) \left( a(t, T)dt + b(t, T) \cdot dW_t \right), \quad (14)
\]
where \( a \) and \( b \) are defined as
\[ a(t, T) = f(t, t) - \alpha^*(t, T) + \frac{1}{2}|\sigma^*(t, T)|^2, \quad b(t, T) = -\sigma^*(t, T), \]
and for arbitrary \( t \in (0, T) \) is
\[
\alpha^*(t, T) = \int_t^T \alpha(t, u)du, \quad \sigma^*(t, T) = \int_t^T \sigma(t, u)du. \quad (15)
\]

Let us now consider \( T \) as a particular fixed maturity. If we define the discounted bond process as \( Z(t, T) = B_t^{-1}P(t, T) \), then it satisfy the following equation
\[
dZ(t, T) = Z(t, T) \left( b(t, T) \cdot dW_t + (a(t, T) - r_t) dt \right),
\]

Also let us define process \( \gamma_t \) as such change of drift of process \( Z(t, T) \) that it becomes a martingale. Then with the help of the Girsanov theorem there exists a measure \( \mathbb{P}^* \), equivalent with the real measure \( \mathbb{P} \) such that \( W_t^* = W_t + \int_0^t \gamma_s ds \) is \( \mathbb{P}^* \)-Brownian motion. These measure we will denote as risk-neutral. Then the process for the discounted bond can be written as
\[
dZ(t, T) = Z(t, T)b(t, T)dW_t^*.
\]

Then, the following theorem can be proved:
Theorem 2 For the arbitrary maturity $T \leq T^*$, under the assumption of non-existence of arbitrage, the dynamics of the price of bond $P(t, T)$ under the risk-neutral measure $\mathbb{P}^*$ is

$$dP(t, T) = P(t, T)\left(r_t dt - b(t, T)\right) \cdot dW^*_t$$

and the forward rate $f(t, T)$ satisfies

$$df(t, T) = \sigma(t, T)b(t, T)dt + \sigma(t, T) \cdot dW^*_t.$$  (17)

This theorem states, that under the risk-neutral measure the forward rates cannot have arbitrary drifts, but only drifts derived from the volatility process.

A.2 The Brace, Gatarek, Musiela (1997) model

The common feature of earlier models of interest rates (up to the HJM model) is the fact that (explicitly or implicitly) they include a specification of the stochastic behavior of non-observable financial quantities, as for example instantaneous forward rates. The calibration of these models to the set of market data thus needs some transformation of these data through the “black-box” of model to the dynamics of non-observable quantities.

This picture has radically changed with the introduction of the BGM (Brace et al. (1997)) model, which describes directly observable market quantities, as discrete LIBOR forward rates.

Let us fix a positive real number $\delta$. Following the definition, the forward $\delta$-LIBOR rate $L(t, T)$ is a discrete forward rate over the interval $(T, T + \delta)$ and is given by relationship

$$1 + \delta L(t, T) = \frac{P(t, T)}{P(t, T + \delta)} \forall t \in (0, T).$$

(18)

Now, we derive the dynamics of $L(t, T)$ under the risk-neutral measure. This derivation is based on the original Brace et al. (1997) article. The advantage of the BGM is that the $L(t, T)$ rates can be modelled as lognormal.

From (18) a (1) we get

$$L(t, T) = \exp\left(\frac{\int_T^{T+\delta} f(t, u)du}{\delta}\right) - 1.$$  (19)

In (17) we want to choose the volatility $\sigma(t, T)$ such that we would obtain the equation for $L(t, T)$ in the following form:

$$dL(t, T) = (\cdots)dt + L(t, T)\gamma(t, T) \cdot dW^*_t.$$
for some $\gamma(t, T)$. From (17) we get:

$$d \int_T^{T+\delta} f(t, u)du = \int_T^{T+\delta} df(t, u)du =$$

$$= \int_T^{T+\delta} \sigma(t, u)b(t, u)du + \int_T^{T+\delta} \sigma(t, u)dW_t^* =$$

$$= \int_T^{T+\delta} \frac{1}{2} \frac{\partial b^2(t, u)}{\partial u} du + [b(t, T) - b(t, T + \delta)] dW_t^* =$$

$$= \frac{1}{2} [b^2(t, T) - b^2(t, T + \delta)] dt + [b(t, T + \delta) - b(t, T)] dW_t^*.$$

Then

$$dL(t, T) = \frac{\exp \left( \int_T^{T+\delta} f(t, u)du \right) - 1}{\delta}$$

$$= \frac{1}{\delta} \exp \left( \int_T^{T+\delta} f(t, u)du \right) d \int_T^{T+\delta} f(t, u)du +$$

$$+ \frac{1}{2\delta} \exp \left( \int_T^{T+\delta} f(t, u)du \right) \left( d \int_T^{T+\delta} f(t, u)du \right)^2 =$$

$$\overset{(20)}{=} \frac{1}{\delta} [1 + \delta L(t, T)] \left[ \frac{1}{2} [b^2(t, T + \delta) - b^2(t, T)] \right] dt +$$

$$+ [b(t, T) - b(t, T + \delta)] dW_t^* + \frac{1}{2} [b(t, T) - b(t, T + \delta)]^2 dt$$

$$= \frac{1}{\delta} [1 + \delta L(t, T)] [b(t, T) - b(t, T + \delta)] [-b(t, T + \delta) dt + dW_t^*].$$

If we now define the process $\lambda(t, T)$ as following:

$$\lambda(t, T)L(t, T) = \frac{1}{\delta} [1 + \delta L(t, T)] [b(t, T) - b(t, T + \delta)],$$

we obtain

$$dL(t, T) = -\lambda(t, T)L(t, T)b(t, T + \delta)dt + \lambda(t, T)L(t, T)dW_t^*. \quad (23)$$

Equation (23) can be conveniently rewritten as

$$dL(t, T) = \lambda(t, T)L(t, T) [-b(t, T + \delta) dt + dW_t^*]. \quad (24)$$

If we combine the previous condition (23) with the Girsanov theorem, we obtain

$$dL(t, T) = \lambda(t, T)L(t, T)dW_t^{T+\delta}, \quad (25)$$
where for all $t \in (0, T + \delta)$

$$W_t^{T+\delta} = W_t^* - \int_0^{T+\delta} b(u, T + \delta) du.$$  

(26)

The process $W_t^{T+\delta}$ is Brownian motion under the measure $\mathbb{P}_{T+\delta} \sim \mathbb{P}^*$, defined with the help of Radon-Nikodym derivative as

$$\frac{d\mathbb{P}_{T+\delta}}{d\mathbb{P}^*} = \exp \left( \int_0^{T+\delta} b(u, T + \delta) \cdot dW_u^* - \frac{1}{2} \int_0^{T+\delta} |b(u, T + \delta)|^2 du \right).$$  

(27)

Let us denote the measure $\mathbb{P}_{\tau+\delta}$ as forward rate connected with the maturity $T + \delta$. Musiela and Rutkowski (1998) show, that if the price of some tradable asset (with no dividends and coupons), expressed in $P(t, T)$ units, is martingale under the $\mathbb{P}_\tau$ measure, so as numeraire under this measure is the price of a bond maturing in the time $T$.

In the following part we construct the model of forward LIBOR rates for the case of discrete time tenor, based on the following assumptions:

**(LR.1)**: For the arbitrary maturity $T \leq T^* - \delta$ we have given bounded, deterministic function $\lambda(\cdot, T) \in \mathbb{R}^d$, which represents the volatility of the forward rate $L(\cdot, T)$ process.

**(LR.2)** We assume the existence of a strictly decreasing and positive initial term structure $P(0, T), T \in (0, T^*)$, that means also of the initial curve $L(0, T)$ of forward rates.

$$L(0, T) = \delta^{-1} \left( \frac{P(0, T)}{P(0, T + \delta)} - 1 \right) \quad \forall T \in (0, T^* - \delta).$$

**Discrete tenor**

Let us assume that the time horizon $T^*$ is a multiple of $\delta$, let us say $T^* = M\delta$ for some natural $M$. In this subpart we will concentrate on the forward LIBOR rates with maturities in discrete time tenor $\{0, T_{(M-1)\delta}, T_{(M-2)\delta}, \ldots, T_\delta, T^*\}$, where $T_{m\delta} = T^* - m\delta$ for $m = 1, 2, \ldots, M - 1$. This procedure is based on backward induction, when one begins with the definition of LIBOR rate with the longest maturity possible $L(t, T_\delta)$. We will assume that we have specified lognormal volatilities $\lambda(t, T_{m\delta})$ for $m = 1, 2, \ldots, M - 1$. Let us postulate, that the rate $L(t, T_\delta)$ is under the probability measure $\mathbb{P}_{T^*}$ driven by the following stochastic differential equation:

$$dL(t, T_\delta) = L(t, T_\delta) \lambda(t, T_\delta) \cdot dW_t^{T^*},$$  

with initial condition
\[ dL(0, T_\delta) = \delta^{-1} \left( \frac{P(0, T_\delta)}{P(0, T^*)} - 1 \right). \]  

(29)

Because the initial term structure is strictly decreasing, it is clear that \( L(t, T_\delta) \) is positive and for fixed \( t \leq T^* - \delta \) the random variable \( L(t, T_\delta) \) has the lognormal distribution under \( \mathbb{P}_{T^*} \). This way the dynamics of LIBOR rates with the maturity in the last date of our tenor is defined.

In the next step we will define forward LIBOR rate for the date \( T_{2\delta}^* \) with the use of (22), where \( T = T_\delta \), so we have mean and volatility specified as

\[
\lambda(t, T_\delta) = \frac{1 + \delta L(t, T_\delta)}{\delta L(t, T_\delta)} [b(t, T_\delta) - b(t, T^*)],
\]

\[
\mu(t, T_\delta, T^*) = \frac{\delta L(t, T_\delta)}{1 + \delta L(t, T_\delta)} \lambda(t, T_\delta),
\]

(30)

where as \( \mu(t, T, T + \delta) \) we denoted \( b(t, T) - b(t, T + \delta) \). Let us define process \( W_t^{T_\delta} \), corresponding with the date \( T_\delta \) as

\[
W_t^{T_\delta} = W_t^{T^*} - \int_0^t \mu(u, T_\delta, T^*) du \quad \forall t \in (0, T_\delta).
\]

This process is connected with the date \( T_\delta \) (due to (26), it describes the relationship between Brownian motions under measures \( \mathbb{P}_{T + \delta} \) a \( \mathbb{P}^* \))

Because \( \mu(t, T_\delta, T^*) \) is bounded, from the Girsanov theorem it is clear the existence of this process and to it associated probability measure \( \mathbb{P}_{T_\delta} \sim \mathbb{P}^\ast \) under which \( W_{T_\delta} \) an Brownian motion. It is given by Radon-Nikodym derivative

\[
\frac{d\mathbb{P}_{T_\delta}}{d\mathbb{P}^\ast} = \exp \left( \int_0^{T_\delta} \mu(u, T + \delta) \cdot dW_u^{T^*} - \frac{1}{2} \int_0^{T_\delta} |\mu(u, T + \delta)|^2 du \right).
\]

From (27) we can see that it is the forward rate connected with maturity \( T_\delta \). Now we can specify the dynamics of LIBOR rate for the maturity \( T_{2\delta} \) under the measure \( \mathbb{P}_{T_\delta} \). Analogically as in (28) we define

\[
dl(t, T_{2\delta}) = L(t, T_{2\delta}) \lambda(t, T_{2\delta}) \cdot dW_t^{T_{2\delta}},
\]

(31)

with the initial condition

\[
L(0, T_{2\delta}) = \delta^{-1} \left( \frac{P(0, T_{2\delta})}{P(0, T_\delta)} - 1 \right).
\]

(32)
From (22) we get the value of the needed change of Brownian motion $W_t^{T_{2\delta}}$ in order to get to the values connected with the date $T_{2\delta}$

$$\mu(t, T_{2\delta}, T_{\delta}) = \frac{\delta L(t, T_{2\delta})}{1 + \delta L(t, T_{2\delta})} \lambda(t, T_{2\delta}) = b(t, T_{2\delta}) - b(t, T_{\delta})$$

If we have defined the process $\mu(t, T_{2\delta}, T_{\delta})$, we can define the pair $(W^{T_{2\delta}}, P_{T_{2\delta}})$, connected with the maturity $T_{2\delta}$, and so on. With backward induction to the first relevant date $T_{(M-1)\delta}$ we can construct the class of forward LIBOR rates $L(t, T_{m\delta})$, $m = 1, \cdots, M - 1$. With this procedure we assure lognormal distribution of each process $L(t, T_{m\delta})$ under corresponding forward probability measure $P_{T_{(m-1)\delta}}$. We have for all $m = 1, \cdots, M - 1$

$$dL(t, T_{m\delta}) = L(t, T_{m\delta})\lambda(t, T_{m\delta}) \cdot dW_t^{T_{(m-1)\delta}}, \quad (33)$$

where $dW_t^{T_{(m-1)\delta}}$ is a Brownian motion under $P_{T_{(m-1)\delta}}$.

This finishes the derivation of the lognormal model of forward LIBOR rates under discrete tenor. Before the end, we bring in the explicit relationship among the Brownian motions connected with the adjacent maturities:

$$W_t^{T_{(m-1)\delta}} = W_t^{T_{m\delta}} + \int_0^t \frac{\delta L(u, T_{m\delta})}{1 + \delta L(u, T_{m\delta})} \lambda(u, T_{m\delta}) du. \quad (34)$$
References


