Chapter 4

The Nash Demand Game Revisited: a Logit-Equilibrium Approach

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Abstract

We prove existence of a logit equilibrium in the Nash Demand Game. If both bargainers are sufficiently 'irrational,' (i.e., if their error parameters are large enough), this equilibrium is unique. If, moreover, they have the same utility function, it is also symmetric. We then compute examples of logit equilibria in the Nash Demand game for various specifications of bargainers' utilities and error parameters. We focus on error parameters that were shown to be empirically relevant in a different context. Although we consider error parameters for which we were not able to show uniqueness of a logit equilibrium theoretically, in no example do we find multiple logit equilibria. Our computations accord well with the intuition that in a boundedly rational world it does not pay off to bargain aggressively: the danger of losing the 'cake' is greater than the loss from leaving a part of the cake unclaimed. Our computations strongly support the hypothesis that as bargainers' error parameters tend to zero at the same rate, the logit-equilibrium density functions converge to the Nash bargaining solution.

JEL Classification: C70, C78, D81

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1 Introduction

This note is concerned with a boundedly rational analysis of the Nash Demand Game (NDG). The NDG was proposed by Nash (1953) as a non-cooperative model of bargaining between two players. It describes a situation in which both players announce simultaneously their take-it-or-leave-it demands. If these demands do not exceed the overall size of the available 'cake,' each player receives his claim; otherwise, both players receive nothing.

The NDG possesses a continuum of equilibria, all of them Pareto-efficient. Bargaining theory has focused primarily on the selection among these. As Binmore *et al.* (1998) pointed out in a different context, however, the selection among multiple Pareto-efficient bargaining outcomes presupposes the solution of the more fundamental problem of how players coordinate on a Pareto-efficient outcome in the first place. Each Pareto-efficient outcome, after all, involves precarious balancing on the brink: if either party were to demand but a fraction of the cake more, an agreement would not be reached. In other words, each equilibrium in the NDG has the property that the loss incurred by deviating from it is highly asymmetric with respect to the direction of the deviation: demanding unilaterally a bit more than one's equilibrium share is far more costly than demanding a bit less.

While the precarious nature of Nash equilibria in the NDG does not matter in the world of perfectly rational agents, it might matter in a world that is less accommodating. To examine this intuition, we allow the bargaining players to make mistakes. Specifically, we suppose that each player, rather than selecting his demand with certainty, follows a "probabilistic choice rule" (Luce, 1959), with the probability of each demanded share depending on the expected utility from this share. There are at least three interpretations of such probabilistic choice behavior, which we review in section 2.1.

In a strategic context, such as the NDG, a player's expected utility from a claim depends on his belief about the other player's claim(s). If each player's belief matches the actual choice probabilities of his opponent (determined, in turn, by the opponent's beliefs about the first player), the resulting choice probabilities reinforce each other. If this is true, the situation constitutes a "boundedly rational" equilibrium. This equilibrium was first introduced by McKelvey and Palfrey (1995), who coined it a quantal response equilibrium and established its existence in finite-strategy normal-form games.

A special case of a quantal response equilibrium in which the probability of choosing each strategy is an exponential function of the expected utility associated with this strategy is called a logit equilibrium. It arises under the assumption that noise (players' mistakes in calculations, etc.) is distributed according to the double exponential function. Besides having the intuitive theoretical property that players' choice probabilities are unaffected by adding a constant to their payoffs, logit equilibrium is convenient to work with because various degrees of 'irrationality' (proclivity to mistakes or randomizing) may be modeled through varying a single parameter (the so-called error parameter). McKelvey and Palfrey (1995) proved that as players' error parameters tend to zero (i.e., as players become more and more perfectly rational), logit equilibrium converges to Nash equilibrium. Logit equilibrium thus may be thought of as a boundedly rational generalization of Nash equilibrium.

Logit equilibrium has been championed by Goeree and Holt (see, e.g., Goeree and Holt, 1999; Anderson, Goeree and Holt, 1998, 2001, forthcoming 1, fortcoming 2; Capra, *et al.*, 1999) as a tool for explaining seemingly anomalous behavior of experimental subjects in economic games. Goeree and Holt found a strong correspondence between logit equilibrium predictions for minimum effort coordination games, all-pay auctions, voluntary contribution games, etc. and the laboratory data available on these games. Although we feel there are some problems with interpreting their findings (see section 2.4), we believe they collected enough evidence to justify further research on logit equilibrium (and quantal response equilibrium in general), whether on properties of this equilibrium, circumstances under which it may arise or applications of this equilibrium to particular games.

This note is an extension of the theoretical research of Goeree and Holt to the NDG. After we introduce and discuss the concepts of quantal response equilibrium and logit equilibrium (see section 2), we define logit equilibrium in the NDG and prove its existence. If players are sufficiently 'irrational' (i.e., if their error parameters are large), we prove the uniqueness of this equilibrium. If, moreover, both players have the same utility, we prove that (the unique) logit equilibrium must be symmetric. Finally, we compute a few examples of logit equilibria for various specifications of players' utilities and error parameters and discuss their properties. In no case were we able to find more than one logit equilibrium, although all considered error parameters lay outside the range of theoretically established uniqueness. This supports our belief (based on results of Goeree and Holt for games similar to the NDG) that there is indeed a unique logit equilibrium in the NDG regardless of players' error parameters, although we were not able to prove so in general. All examples reflect the asymmetric nature of Nash equilibria in the NDG mentioned above: players indeed 'fear' to bargain aggressively and so put more weight on small claims.

Finally, the examples point decisively toward the conclusion that as players' error parameters go to zero at the same rate (i.e., as players become more and more perfectly rational at the same rate), logit equilibrium density functions converge to the Nash bargaining solution. This finding corroborates the relevance of the Nash bargaining solution when noise is added to the basic setup of the NDG (cf. also Young, 1993; and Binmore and Dasgupta, 1987, chapter 4).

2 Logit Equilibrium: Reviewing the Concept

2.1 Probabilistic Choice

Consider an agent who has to make a choice between two actions A_1 and A_2. He believes that A_1 has expected utility $U^e(A_1)$ and A_2 expected utility $U^e(A_2)$. A perfectly rational agent, who is absolutely sure about the accuracy of his assessment of $U^e(A_1)$ and $U^e(A_2)$, would opt for an action with the higher expected utility. Agents, however, may not always be perfectly rational. To model the departure from perfect rationality, suppose that a small degree of noise affects the agent's comparison of expected utilities. Then, the agent chooses the action with lower expected utility with non-zero probability. Algebraically,

$$P(A_1) = \Pr\left(U^e(A_1) - U^e(A_2) > \mu\varepsilon\right), \tag{1}$$

where $P(A_1)$ is the probability of choosing action A_1, ε is a random variable with zero mean, and μ is a positive real number. The distribution of ε describes the qualitative nature of the noise, while μ captures—in a sense to be made precise later—its significance. We call μ the agent's *error parameter*.

There are at least three interpretations of equation (1). First, the agent's preferences are subject to small random shocks. The agent always behaves in a perfectly rational manner but, his preferences being volatile and unobservable, he appears to the external observer as making mistakes in judgement. An example of such volatile preferences might be varying risk aversion. If action A_1 leads to a higher, but less certain, expected payoff than action A_2, then the agent may choose A_1 with high probability when feeling elated and A_2 when feeling dejected.

Second, the agent's preferences are stable, but he is unable to calculate $U^e(A_1)$ and $U^e(A_2)$ precisely. An auctioneer who has to make calculations under severe time constraints or who sometimes mishears the price quoted may serve as an example here. This interpretation of (1) is perhaps the easiest to imagine. In what follows, we refer to this interpretation as mistakes in computation.

Third, the agent has no conscious utility function, but he is, nonetheless, influenced by preferences buried in his subconscious mind. This interpretation was suggested by Chen, Friedman and Thisse (1997).

As μ becomes smaller, the agent's decision making becomes less noisy and equation (1) approaches the perfectly rational ideal. As μ grows larger, the error term $\mu\varepsilon$ starts to dominate the difference between $U^e(A_1)$ and $U^e(A_2)$. Consequently, both actions A_1 and A_2 will be selected with almost equal probabilities. Hence, by varying μ one is able to capture different degrees of imperfect rationality.

Equation (1) formalizes psychologist Luce's (1959) notion of "probabilistic choice." Depending on the specification of ε, this equation generates a number of different models of boundedly rational behavior. If, for instance, ε is distributed uniformly on $[-1, 1]$, then (1) turns into

$$P(A_1) = \begin{cases} \frac{1}{2} + \frac{U^e(A_1) - U^e(A_2)}{2\mu} & \text{if } |U^e(A_1) - U^e(A_2)| < \mu; \\ 0 & \text{if } U^e(A_1) - U^e(A_2) \leq -\mu; \\ 1 & \text{if } U^e(A_1) - U^e(A_2) \geq \mu. \end{cases} \qquad (2)$$

Here the agent randomizes between both actions, his choice probabilities being proportional to the difference in corresponding expected utilities unless this difference is too large. In the latter case, he chooses with certainty the action yielding the higher expected utility. Equation (2) is known as a linear probability choice model, applied first in a game-theoretical context by Rosenthal (1989).

If ε is drawn from a double-exponential distribution,[1] then (1) generates choice probabilities of a "logit" form

$$P(A_i) = \frac{\exp\left(\frac{U^e(A_1)}{\mu}\right)}{\exp\left(\frac{U^e(A_1)}{\mu}\right) + \exp\left(\frac{U^e(A_2)}{\mu}\right)}, \; i = 1, 2. \qquad (3)$$

2.2 Quantal Response Equilibrium and Logit Equilibrium in Finite-Strategy Games

Let us now switch from the simple decision problem of a single agent to a strategic situation involving n agents, $n \geq 2$. From now on we shall call these agents players and their actions strategies. While we still assume that each player $i = 1, \ldots, n$ behaves according to the probabilistic choice rule (1), properly generalized for more strategies at hand, the expected utilities that determine

[1]Double-exponential distribution (also called extreme value distribution) is defined by the cumulative density function $F(\varepsilon) = \exp(-\exp(\varepsilon))$.

the particular form of this rule now depend on player i's beliefs about other players' strategies.

To see this, let us adopt the mistakes-in-computation interpretation. Each player holds beliefs about his opponents' strategies and attempts to choose the best response to these beliefs: he calculates the expected utility from each of his pure strategies and adopts the strategy with the highest utility. Since his calculations are noisy, however, this strategy is unlikely to be the actual best response to his beliefs. Think of this pure strategy as a realization of a mixed strategy—a 'noisy best response.'

Assume that every player is hard-wired to play a particular noisy best response. If a player's noisy best response differs from what other players believe about him, these players are likely to revise their beliefs. If, in contrast, each player's noisy best response matches exactly what other players believe about him, an equilibrium emerges. This equilibrium was introduced by McKelvey and Palfrey (1995), who coined it a *quantal response equilibrium* and proved its existence in n-player normal-form games with a finite number of strategies.

Let us review and slightly refine the formal definition of this equilibrium. Consider a finite game Γ with n players; $N = \{1, \ldots, n\}$. Each player $i \in N$ has a strategy set $A_i = \{a_{i1}, \ldots, a_{iJ_i}\}$; $A \equiv \prod_i A_i$. Each player $i \in N$ has the (von Neumann-Morgenstern) utility function $U_i : A \to \mathsf{R}$. For each $i \in N$, Δ_i denotes the set of probability measures on A_i, and $\Delta \equiv \prod_i \Delta_i$.[2] Finally, for each $i \in N$, $\beta_i \in \Delta_{-i}$ describes player i's beliefs about other players' strategies.

Player i's expected utility from the mixed strategy $p_i \in \Delta_i$, $p_i = (p_{i1}, \ldots, p_{iJ_i})$, subject to his beliefs β_i is

$$U_i^e(p_i \mid \beta_i) = \sum_j p_{ij} \sum_{a_{-i} \in A_{-i}} \beta_i(a_{-i}) U_i(a_{ij}, a_{-i}).$$

Player i's best response $\rho_i^*(\beta_i)$ to β_i is defined as

$$\rho^*(\beta_i) \equiv \{p \in \Delta_i \mid U_i^e(p \mid \beta_i) \geq U_i^e(\widetilde{p} \mid \beta_i) \text{ for all } \widetilde{p} \in \Delta_i\}.$$

NASH EQUILIBRIUM: A vector $\mathsf{p} = (p_1, \ldots, p_n) \in \Delta$ is a Nash equilibrium if

1. For all $i \in N$, $\beta_i = p_{-i}$ and
2. For all $i \in N$, $p_i \in \rho_i^*(\beta_i)$.

[2] In what follows, we identify the pure strategy a_{ij} with the mixed strategy that places probability one on a_{ij}, and hence understand a_{ij} as an element of Δ_i.

Suppose that each player $i \in N$, when deciding what pure strategy to use, attempts to calculate $U_i^e(a_{ij} \mid \beta_i)$ for all $a_{ij} \in A_i$. Unfortunately, he makes a computational mistake and instead of $U_i^e(a_{ij} \mid \beta_i)$ obtains

$$\widehat{U}_i^e(a_{ij}|\ \beta_i) = U_i^e(a_{ij}|\ \beta_i) + \varepsilon_{ij},$$

the second term capturing the mistake. Let $\varepsilon_i \equiv (\varepsilon_{i1}, \ldots, \varepsilon_{iJ_i})$ be distributed according to the joint density function f_i, such that for each $j \in J_i$ the marginal density function exists, and $E(\varepsilon_{ij}) = 0$. Suppose that player $i \in N$ plays the best response as determined by his 'noisy' calculations of $\widehat{U}_i^e(a_{ij}|\ \beta_i)$, i.e., he plays strategy a_{ij} if and only if

$$\widehat{U}_i^e(a_{ij} \mid \beta_i) + \varepsilon_{ij} \geq \widehat{U}_i^e(a_{ik} \mid \beta_i) + \varepsilon_{ik}, \tag{4}$$

for all $k \in J_i$. Define $R_{ij}(\beta_i)$ as a set of profiles ε_i for which the above relationship holds. Then, $\rho_i(\beta_i) = (\rho_{i1}(\beta_i), \ldots, \rho_{iJ_i}(\beta_i)) \in \Delta_i$ defined as

$$\rho_{ij}(\beta_i) = \int_{R_{ij}(\beta_i)} f(\varepsilon)\mathrm{d}\varepsilon$$

determines player i's *noisy best response* to his beliefs β_i, as described verbally above. Quantal response equilibrium is a profile of noisy best responses that are consistent with beliefs that induce them. Formally,

QUANTAL RESPONSE EQUILIBRIUM: A vector $\mathsf{p} = (p_1, \ldots, p_n) \in \Delta$ is a quantal response equilibrium (QRE) if

1. For all $i \in N$, $\beta_i = p_{-i}$, and
2. For all $i \in N$, $p_i = \rho_i(\beta_i)$.

Comparison of this definition with that of Nash equilibrium reveals a strong link between both concepts.

If for some $i \in N$, the random variables ε_{ij}, $j \in J_i$, are independent and drawn from the same double exponential distribution with cumulative density function

$$F(\varepsilon) = \exp\left(-\exp\left(\frac{\varepsilon}{\mu_i}\right)\right), \tag{5}$$

player i's noisy best response becomes the *logit best response* resembling equation (3):

$$\rho_{ij}(\beta_i) = \frac{\exp\left(\frac{U_i^e(a_{ij} \mid \beta_i)}{\mu_i}\right)}{\sum_{k=1}^{J_i} \exp\left(\frac{U_i^e(a_{ij} \mid \beta_i)}{\mu_i}\right)}.$$

We call μ_i player i's *error parameter.*

Logit equilibrium is a QRE in which all players play the logit best responses. Formally,

LOGIT EQUILIBRIUM: A vector $\mathsf{p} = (p_1, \ldots, p_n) \in \Delta$ is a logit equilibrium if

$$\rho_{ij}(p_{-i}) = \frac{\exp\left(\frac{U^{\mathrm{e}}(a_{\mathrm{ij}} \mid p_{-\mathrm{i}})}{\mu_{\mathrm{i}}}\right)}{\sum_{k=1}^{J_{\mathrm{i}}} \exp\left(\frac{U^{\mathrm{e}}(a_{\mathrm{ik}} \mid p_{-\mathrm{i}})}{\mu_{\mathrm{i}}}\right)}, \text{ for all } i \in N,\ j \in J_i. \tag{6}$$

McKelvey and Palfrey (1995) proved that if all $\mu_i \to 0$ at the same rate, the logit equilibrium converges to a Nash equilibrium of the game Γ. If, however, all $\mu_i \to +\infty$, players' logit equilibrium responses degenerate to uniform randomization. Logit equilibrium therefore may be thought of as a boundedly rational generalization of Nash equilibrium, with the size of players' error parameters determining the distance from the latter.

A drawback of QRE is that it is, just as Nash equilibrium, a static concept. Moreover, QRE is an equilibrium in probability measures over players' strategy sets and hence it is more complex than a pure strategy Nash equilibrium (about as complex as a mixed-strategy Nash equilibrium). An important question is, therefore, how players are going to coordinate on a QRE. This question was addressed by Chen, Friedman and Thisse (1997), who investigated a boundedly rational learning process similar to fictitious play. In this process, players are matched in successive periods, with each player playing a particular type of noisy best response to the empirical distribution of his opponents' strategies derived from past plays. The authors show that if players are not too close to 'perfectly rational men' (i.e., in our notation, if players' error parameters are not too small), this learning process converges to a QRE.[3] They do not, however, characterize quantitatively the size of players' error parameters required for the convergence, except in an example (see Chen, Friedman and Thisse, 1997, p. 49). Since their proof of convergence is based on a the same 'contraction' argument as our proof of uniqueness of logit equilibrium in the NDG (see section 3) and since our proof works only for relatively large error parameters, our conjecture is that the learning process proposed by these authors also converges only for relatively large error parameters. Dynamic justification of QRE in a finite-strategy game for plausibly small error parameters still needs to be found.

[3]One cannot expect the process to converge for all games if players are almost perfectly rational (i.e., if their noisy best responses are almost best responses), because then the process becomes very close to traditional fictitious play, which does not converge in general (see Shapley, 1964).

2.3 Logit Equilibrium in Games with a Continuum of Strategies

Lopez (1995) extended logit equilibrium to games with a continuum of strategies. Consider a symmetric n-player game Γ with a continuous strategy set $[a, b] \subset \mathsf{R}$ common to all players. A profile $\mathsf{F} = (F_1, \ldots, F_n)$ of cumulative density functions F_i defined on $[a, b]$ constitutes a logit equilibrium in this game, if it solves a system of equations that are continuous analogues of (6):

$$f_i(x) = F_i'(x) = \frac{\exp\left(\frac{U_i^e(x \mid F_{-i})}{\mu_i}\right)}{\int_a^b \exp\left(\frac{U_i^e(s \mid F_{-i})}{\mu_i}\right) \mathrm{d}s}, \text{ for all } i \in N,\ x \in [a, b]. \qquad (7)$$

Here f_i is a density function corresponding to F_i, F_{-i} a profile of F_j's for all $j \neq i$, and $U_i^e(x \mid F_{-i})$ player i's utility from playing x, given that his beliefs about other players coincide with F_{-i}.[4] As before, μ_i stands for player i's error parameter. Note that for all i, the denominator in (7) is a constant guaranteeing that f_i integrates to one on $[a, b]$.

In practice, logit equilibrium (7) is typically found by solving a system of differential equations obtained by differentiating (7) with respect to x. As can be easily verified, this system is given as

$$\mu_i F_i''(x) = F_i'(x) U_i^{e\prime}(x \mid F_{-i}) \text{ s.t. } F_i(a) = 0,\ F_i(b) = 1 \qquad (8)$$

for all $i \in N$, $x \in [a, b]$.[5] It is important to note, however, that systems (8) and (7) are not equivalent. Since system (7) may have solutions that are not twice differentiable, some logit equilibria may not be found by solving system (8). However, if system (7) can be shown to have only twice differentiable solutions, then *any* logit equilibrium F is necessarily a solution to (8) as well.

Note that if $\mu_i \to 0$, the i-th equation in system (8) approaches

$$f_i(x) U_i^{e\prime}(x \mid F_{-i}) = 0 \text{ s.t. } F_i(a) = 0,\ F_i(b) = 1.$$

This means that the logit-equilibrium density function f_i converges to the degenerate density function f_i^∞ such that $f_i^\infty(x) = 0$ whenever $U_i^{e\prime}(x \mid F_{-i}) \neq 0$.[6]

[4]Equation (7) does not give a rigorous definition of logit equilibrium for games with a continuum of strategies, but rather a characterization of such an equilibrium. For a rigorous definition, one would first need to show how each player i's logit response (f_i) can be derived from his mistakes in expected utility computations, just as we did in the previous section. We are aware of no such definition. This makes us wonder whether the continuum of random variables involved in such a derivation is not a more fundamental problem than it first seems.

[5]Here prime denotes the first and a double prime denotes the second derivative with respect to x.

[6]Note that if $U_i'(x) = 0$ on a set of measure zero only, then f_i^∞ is a linear combination of

In other words, f_i^∞can only place positive probability on strategy x for which $U_i^{e\prime}(x \mid F_{-i}) = 0$. If for any such x, $U_i^{e\prime\prime}(x \mid F_{-i}) < 0$, then f_i^∞ is the player i's best response to F_{-i}. Suppose that this is true for all $i \in N$. Then, as all $\mu_i \to 0$, logit equilibrium must converge to an interior Nash equilibrium of the game Γ, given that such equilibrium exists.

If $\mu_i \to +\infty$, the i-th equation in (8) approaches

$$F_i''(x) = 0 \text{ s.t. } F_i(a) = 0,\, F_i(b) = 1.$$

In this case, the logit-equilibrium density function f_i converges to the uniform density function.

If all μ_i's are of intermediate size, the logit-equilibrium density functions are between the f_i^∞ (which correspond to an internal Nash equilibrium) and uniform density functions. Hence, even in the continuous case, logit equilibrium may be understood as a boundedly rational generalization of Nash equilibrium.

Anderson, Goeree and Holt (1997) proposed a boundedly rational learning process that converges to a logit equilibrium in a wide class of games that feature a continuum of strategies (the so-called potential games). The process arises if the game is played repeatedly, and if in each period players attempt to adjust their strategy optimally in the direction of increasing payoff, but make normally distributed mistakes. Convergence is independent of players' error parameters. What diminishes the beauty of this result are the strong epistemic foundations of the process: in each period each player knows precisely the mixed strategies of all other players, although he observes mere realizations of these. Another important question is whether this process represents the limit of some plausible learning process in a finite-strategy game. In particular, a connection to the learning process studied by Chen, Friedman and Thisse (1997) (see the previous section) seems worth examining.

2.4 Logit Equilibrium: A Perspective on Intuitive Behavioral Anomalies?

As mentioned in the introduction, logit equilibrium has been recently defended by Goeree and Holt (e.g., Goeree and Holt, 1999 or Anderson, Goeree and Holt, forthcoming 1 and forthcoming 2) as a means of reconciling game theory with 'anomalous' experimental results (i.e., results that go contrary to the Nash equilibrium predictions). The authors calculated logit equilibrium in more than ten symmetric games with a continuum of strategies (minimum effort coordination

Dirac delta functions $\delta_{x_0}(\cdot)$, well-known from theoretical physics. Dirac delta function $\delta_{x_0}(\cdot)$ is a generalized function that is zero everywhere except for x_0, infinite at x_0, and integrates to one on $[a, b]$.

games, all-pay auctions, voluntary contribution games, etc.) and compared this equilibrium with available experimental data. Their findings were quite compelling. In all games the average logit-equilibrium strategy was almost the same as the average strategy played in the few last rounds of the experiment. In particular, both the average logit-equilibrium strategy and the average actually played strategy were highly sensitive to a certain underlying parameter of the game (e.g. cost of effort in the minimum effort coordination game), whilst the Nash equilibrium strategy displayed no such sensitivity. In the Traveler's Dilemma game the logit equilibrium density function traced quite accurately the experimentally observed distribution of strategies (see Anderson, Goerre and Holt, forthcoming 1).

Although Goeree and Holt interpreted these results with great zeal, we feel some caution is necessary. One problem is the aggregate nature of the data. Logit equilibrium, as explained above, arises when each player in a game plays the logit best response. This, however, was not what Goeree and Holt observed in their experiments. Even in the Traveler's Dilemma, the most convincing of their experiments, they observed merely that the distribution of strategies among experimental subjects had a form of the logit best response. This distribution of strategies can arise even if no experimental subject actually playes the logit best response. To justify the comparison between logit best response and the distribution of strategies among players, one needs an interpreation of logit equilibrium as a population steady state, an interpretation similar to that of Nash equilibrium as an evolutionarily stable set (Maynard Smith and Price, 1973). Without this interpretation, it is difficult to conclude what the celebrated agreement between theory and the data means. Providing such interpretation, however, is not the aim of the present paper.

Even if Goeree and Holt's observations are a consequence of each experimental subject playing the logit best response, not all difficulties with interpreting their findings are resolved. The main problem is that no interpretation of the probabilistic choice rule (and hence of the logit best response) is very plausible in such a simple strategic situation as a laboratory experiment. Since Goeree and Holt's experimental subjects were motivated by monetary gains, their preferences over outcomes were by definition very simple. And, since all experiments took only a relatively short time, the subjects' attitude to risk could not vary much. The "random preference" interpretation of the probabilistic choice rule is thus of little avail, as is the "subconscious utility" interpretation. Only the "mistakes in computation" interpretation seems viable, but to make this interpretation sound in a strategic context, one would need to relate it to the formation of subjects' beliefs about each other.

In spite of the difficulties with interpreting Goeree and Holt's findings (and with justifying logit equilibrium by means of a learning process), we think that logit equilibrium and the like concepts represent a promising direction for the game theory. The application of logit equilibrium to the NDG, a game so far analyzed almost exclusively by traditional game-theoretic tools, is our contribution to the research in this vein.

3 Logit Equilibrium in the Nash Demand Game

We now establish existence of logit equilibrium in the NDG (Nash, 1953), and prove its uniqueness and symmetry for large error parameters.

Recall that in the NDG two players bargain over a cake, announcing their demands simultaneously. If the sum of the demands does not exceed the total size of the cake, each player gets what he wants; otherwise, both players get nothing. The game has a continuum of Nash equilibria in pure strategies, namely all outcomes where players' demands sum up to the size of the cake.

Suppose that the cake has unit size. Suppose also that player $i = 1, 2$, has an error parameter μ_i, and a utility function U_i that is non-decreasing, bounded and non-negative on $[0, 1]$.[7] Denote by F_i, $i = 1, 2$, the cumulative density function that governs player i's demand and by f_i the corresponding density function. Then, the expected utility player i derives from demanding x is

$$\begin{aligned} U_i^e(x) &= U_i(x) \Pr(\text{player } j \neq i \text{ demands no more than } 1-x) = \\ &= U_i(x) F_{3-i}(1-x). \end{aligned}$$

If a pair of cumulative density functions $\mathsf{F} = (F_1, F_2)$ constitutes a logit equilibrium in the NDG, it must satisfy

$$f_i(x) = F_i'(x) = \frac{\exp\left(\frac{U_i(x)F_{3-i}(1-x)}{\mu_i}\right)}{\displaystyle\int_0^1 \exp\left(\frac{U_i(s)F_{3-i}(1-s)}{\mu_i}\right) \mathrm{d}s}, \text{ for } i = 1, 2. \tag{9}$$

The following theorems assure that for any reasonable pair of utilities U_1 and U_2, logit equilibrium exists and, if both μ_i are large enough, it is unique.[8]

[7]The assumption that U_i, $i = 1, 2$, is non-negative on $[0, 1]$ is adopted only because it simplifies the ensuing analysis.

[8]Theorem 1 draws heavily on Proposition 1 in Anderson, Goeree and Holt (forthcoming 2). This proposition establishes the existence of logit equilibrium in all games with a continuum of strategies $[\underline{x}, \overline{x}]$ in which each player's expected utility (payoff) from playing strategy x

THEOREM 1: The NDG has a logit equilibrium $\mathsf{F} = (F_1, F_2)$, both cumulative functions F_1, F_2 being differentiable. If U_1 and U_2 are continuous on $[0,1]$, then so are f_1 and f_2. If, moreover, U_1 and U_2 are differentiable m-times on $[0,1]$, $1 \le m \le +\infty$, then both F_1 and F_2 are differentiable $m+1$ times (infinitely many times if $m = +\infty$).

Proof: Let $\mathcal{M}$ be defined as a set of continuous cumulative density functions on $[0,1]$, i.e.

$$\mathcal{M} \equiv \{F \in C[0,1] \mid F \text{ non-decreasing, } F(0) = 0,\ F(1) = 1\}.$$

Fix a pair of $\mu_1, \mu_2 \in \mathsf{R}_+$ and define the integral operator $T : \mathcal{M}\times\mathcal{M} \to \mathcal{M}\times\mathcal{M}$ as

$$T(F_1, F_2) = (\widetilde{T}_1(F_2), \widetilde{T}_2(F_1)), \tag{10}$$

where

$$\widetilde{T}_i(F)(x) = \frac{\displaystyle\int_0^x \exp\left(\frac{U_i(s)F_{3-i}(1-s)}{\mu_i}\right) \mathrm{d}s}{\displaystyle\int_0^1 \exp\left(\frac{U_i(s)F_{3-i}(1-s)}{\mu_i}\right) \mathrm{d}s}, \tag{11}$$

for $i = 1, 2$. Then, system (9) may be rewritten as

$$(F_1, F_2) = T(F_1, F_2), \tag{12}$$

and establishing the existence of logit equilibrium in the NDG is equivalent to showing that T has a fixed point. By Schauder's second theorem (see, e.g., Smart, 1974, p. 25), T has a fixed point if (i) its range of values $\mathcal{H} = \{T(F_1, F_2) \mid (F_1, F_2) \in \mathcal{M}\times\mathcal{M}\}$ is compact, and (ii) T is a continuous mapping.[9] We shall next prove points (i) and (ii).

Point (i): Note first that $\mathcal{H}$ may be written as $\mathcal{K}_1 \times \mathcal{K}_2$, where $\mathcal{K}_i \equiv \{\widetilde{T}_i(F) \mid F \in \mathcal{M}\}$ is a range of values of operator $\widetilde{T}_i$. Since a Cartesian product of two compact sets is a compact set, it suffices to prove that both sets $\mathcal{K}_i$, $i = 1, 2$, are compact. We do this just for $\mathcal{K}_1$; compactness of $\mathcal{K}_2$ may be established analogously.

is a continuous function of the distribution (density, cumulative density) functions of other players. Since it was not clear to us whether the authors meant the continuity with respect to the distribution functions of other players evaluated at x, or with respect to the distribution functions as elements of $C[\underline{x}, \overline{x}]$, and since we suspect that in the latter case—which would cover the NDG—the proposition may generally fail to hold, we preferred to prove the existence of logit equilibrium in the NDG ourselves.

[9] Here compactness and continuity are understood with respect to the norm in $C[0,1]$, $\|F\|_{\mathsf{sup}} = \sup_{x\in[0,1]} F(x)$.

To show that $\mathcal{K}_1$ is compact, we take advantage of the following proposition by Arzela (taken from Lusternik and Sobolev, 1961, p. 61):

Proposition (Arzela): *A set $\mathcal{K} \subset C[0,1]$ is compact if and only if the elements of $\mathcal{K}$ are (a) uniformly bounded and (b) equicontinuous.*

Uniform boundedness of $\mathcal{K}_1$ is evident since $\mathcal{K}_1 \subset \mathcal{M}$ and elements of $\mathcal{M}$ are uniformly bounded by definition. Equicontinuity of $\mathcal{K}_1$ requires us to show that for any given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any $F \in \mathcal{M}$ it holds that $\left|\widetilde{T}_1(F)(x) - \widetilde{T}_1(F)(x')\right| < \varepsilon$, whenever $|x - x'| < \delta(\varepsilon)$, with x, $x' \in [a,b]$. Since

$$\left|\widetilde{T}_1(F)(x) - \widetilde{T}_1(F)(x')\right| = \frac{\left|\int_x^{x'} \exp\left(\frac{U_1(s)F_2(1-s)}{\mu_1}\right) \mathrm{d}s\right|}{\int_0^1 \exp\left(\frac{U_1(s)F_2(1-s)}{\mu_1}\right) \mathrm{d}s} \leq$$
$$\leq \frac{\exp\left(\frac{U_1(1)}{\mu_1}\right)|x - x'|}{\exp\left(\frac{U_1(0)}{\mu_1}\right)(1-0)},$$

this condition is met for $\delta(\varepsilon) = \exp\left(\frac{U_1(0)-U_1(1)}{\mu_1}\right)\varepsilon$.

Point (ii): T is continuous on $\mathcal{M} \times \mathcal{M}$ if and only if each $\widetilde{T}_i$, $i \in 1,2$, is continuous on $\mathcal{M}$. We prove continuity of $\widetilde{T}_1$; the continuity of $\widetilde{T}_2$ may be established analogously.

To prove continuity of $\widetilde{T}_1$ on $\mathcal{M}$, choose an arbitrary function $F \in \mathcal{M}$ and consider any sequence $\{F_k\}_{k=0}^{+\infty}$, $F_k \in \mathcal{M}$, such that $\lim_{k\to+\infty} \|F_k - F\|_{\sup} = 0$. Then, $\widetilde{T}_1$ is continuous at F if $\lim_{k\to+\infty} \left\|\widetilde{T}_1(F_k) - \widetilde{T}_1(F)\right\|_{\sup} = 0$. Given that F is arbitrary, this result guarantees that $\widetilde{T}_1$ is continuous on $\mathcal{M}$.

From (11) it follows that for any $x \in [0,1]$, [10]

$$\left|\widetilde{T}_1(F_k)(x) - \widetilde{T}_1(F)(x)\right| \leq$$
$$\leq \frac{\left|\int_0^x e^{\frac{F_k U_1}{\mu_1}} \int_0^1 e^{\frac{F U_1}{\mu_1}} - \int_0^x e^{\frac{F U_1}{\mu_1}} \int_0^1 e^{\frac{F_k U_1}{\mu_1}}\right|}{\int_0^1 e^{\frac{F U_1}{\mu_1}} \int_0^1 e^{\frac{F_k U_1}{\mu_1}}} \leq$$
$$\leq \frac{\left|\int_0^x \left(e^{\frac{F_k U_1}{\mu_1}} - e^{\frac{F U_1}{\mu_1}}\right) \int_0^1 e^{\frac{F U_1}{\mu_1}} - \int_0^x e^{\frac{F U_1}{\mu_1}} \int_0^1 \left(e^{\frac{F_k U_1}{\mu_1}} - e^{\frac{F U_1}{\mu_1}}\right)\right|}{\int_0^1 e^{\frac{F U_1}{\mu_1}} \int_0^1 e^{\frac{F_k U_1}{\mu_1}}}. \quad (13)$$

[10] The arguments of integrands were dropped here for the sake of brevity.

Noting that for any $A, B \in \mathsf{R}$, $|A-B| \le |A|+|B|$, and applying this inequality to the numerator of (13), we arrive at

$$\left|\widetilde{T}_1(F_k)(x) - \widetilde{T}_1(F)(x)\right| \le \frac{2\exp\left(\frac{U_1(1)}{\mu_1}\right)}{\left[\exp\left(\frac{U_1(0)}{\mu_1}\right)(1-0)\right]^2} B_k, \qquad (14)$$

where

$$B_k = \int_0^1 \left|\exp\left(\frac{U_1(s)F_k(1-s)}{\mu_1}\right) - \exp\left(\frac{U_1(s)F(1-s)}{\mu_1}\right)\right| \mathrm{d}s.$$

But, since for any $U \ge 0$, $\mu > 0$, and $Y, Y' \in [0,1]$, it holds that

$$\left|\exp\left(\frac{UY'}{\mu}\right) - \exp\left(\frac{UY}{\mu}\right)\right| \le \exp\left(\frac{U}{\mu}\right)\left[\exp\left(\frac{U}{\mu}\right) - 1\right]\left|Y' - Y\right|,$$

we have

$$0 \le B_k \le \exp\left(\frac{U_1(1)}{\mu_1}\right)\left[\exp\left(\frac{U_1(1)}{\mu_1}\right) - 1\right]\|F_k - F\|_{\mathsf{sup}}. \qquad (15)$$

This finishes the proof of continuity of $\widetilde{T}_1$.

The differentiability of (F_1, F_2) that solve (12) follows from the equivalence between systems (12) and (9): since for any $i = 1, 2$, the right-hand side of (9) is a well-defined function of F_{3-i}, the derivative F_i' on the left-hand side of (9) is well-defined as well. If functions U_i, $i = 1, 2$, are continuous, then $f_i = F_i'$, $i = 1, 2$, given by (9), are continuous transformations of continuous functions F_1 and F_2 and hence themselves continuous. Finally, if U_i, $i = 1, 2$, are once differentiable, then both f_1 and f_2 are differentiable transformations of F_1 and F_2 which are also differentiable. Hence, for $i = 1, 2$, f_i can be differentiated, and so F_i possesses the second order derivatives. Higher order differentiability of F_i, $i = 1, 2$, may be established recursively in like manner.

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THEOREM 2: If μ_1 and μ_2 are large enough, logit equilibrium in the NDG is unique. Given that $U_1(0) = U_2(0) = 0$, 'large enough' means $\mu_i > \theta U_i(1)$, where

$$\theta = \ln^{-1}\left(\frac{1}{6}\sqrt[3]{\left(62 + 6\sqrt{105}\right)} + \frac{2}{3\sqrt[3]{\left(62 + 6\sqrt{105}\right)}} + \frac{1}{3}\right) \doteq \qquad (16)$$
$$\doteq 3.84.$$

Proof: From Theorem 1 we know that system (9) has a solution which is a fixed point of operator T defined by (10) and (11). This fixed point is unique if operator T is a contraction mapping. By definition of T, this holds if and only if both operators $\widetilde{T}_1$ and $\widetilde{T}_2$, defined by (11), have this property. As for $\widetilde{T}_1$, formulae (14) and (15) imply that for any pair of functions F, $G \in \mathcal{M}$, it holds that

$$\left|\widetilde{T}_1(F)(x) - \widetilde{T}_1(G)(x)\right| \leq \frac{2\exp\left(\frac{2U_1(1)}{\mu_1}\right)\left[\exp\left(\frac{U_1(1)}{\mu_1}\right) - 1\right]}{\exp\left(\frac{2U_1(0)}{\mu_1}\right)} \|F - G\|_{\mathsf{sup}}\,, \quad (17)$$

and hence $\widetilde{T}_1$ is a contraction whenever

$$\frac{2\exp\left(\frac{2U_1(1)}{\mu_1}\right)\left[\exp\left(\frac{U_1(1)}{\mu_1}\right) - 1\right]}{\exp\left(\frac{2U_1(0)}{\mu_1}\right)} < 1. \quad (18)$$

Since the limit of the left-hand side of (18) for $\mu_1 \to +\infty$ equals zero, (18) is satisfied for any large enough μ_1. If $U_1(0) = 0$, then the left-hand side of (18) is an increasing function of $\frac{U_1(1)}{\mu_1}$ and so (18) holds whenever $\frac{U_1(1)}{\mu_1} < \frac{1}{\theta}$, where θ is a root of $2\exp\left(\frac{2}{\theta}\right)\left[\exp\left(\frac{1}{\theta}\right) - 1\right] = 1$. Condition (16) rephrases this statement explicitly.

Performing the analogous exercise for $\widetilde{T}_2$ finishes the proof.

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COROLLARY: Suppose that both players have the same utility function U, $U(0) = 0$, and the same error parameter μ such that $\mu > \theta U(1)$, where θ is given by (16). Then, the Nash Demand Game possesses a unique logit equilibrium. This equilibrium is symmetric across players.

Proof: Uniqueness follows directly from Theorem 2. To see why this unique logit equilibrium must be symmetric, note the proof of Theorem 2 implies that $\widetilde{T} = \widetilde{T}_1 = \widetilde{T}_2$ is a contraction. Hence $\widetilde{T}$ must have a fixed point $F = \widetilde{T}F$. But this implies that $(F, F) = (\widetilde{T}(F), \widetilde{T}(F)) = T(F, F)$, and hence, by (12), (F, F) is a logit equilibrium.

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Theorem 2 asserts that if players' mistakes in computation are large in a well-defined sense, then the logit equilibrium is unique. A natural question is whether Theorem 2 does not establish uniqueness of logit equilibrium in the NDG only for error parameters that are too large (relative to the payoff range of the NDG) to be realistic. Computations of error parameters by Capra, Goeree and Holt give us some guidance. In the Traveler's Dilemma game, for instance, where players' payoffs ranged from 80 to 200, these authors estimated the common error parameter as 8.3 (see Capra *et al.*, 1999). In a two-person minimum effort coordination game with payoffs ranging from 110 to 170 this estimate was 7.4 (see Goeree and Holt, 1999). These results suggest that in a game with a given payoff range an error parameter can be considered "realistic" if it is about ten times smaller than this range. For the NDG, then, a "realistic" μ is about 0.1. To relate this estimate to Theorem 2, suppose that both bargainers are interested solely in payoffs: $U_1(x) = U_2(x) = x$. Then, Theorem 2 establishes uniqueness of logit equilibrium for $\mu_i > 3.84$. Since 3.84 is far greater than 0.1, Theorem 2 (and the ensuing corollary) is rather weak. This weakness is a consequence of the upper bound on the right-hand side of (17). Since this bound is quite coarse, we suppose that operator T is a contraction on a greater range of error parameters than we were able to prove. We return to the issue of uniqueness of logit equilibrium below.

Suppose now that U_1 and U_2 are differentiable. By Theorem 1, this assures that logit-equilibrium cumulative density functions are twice differentiable. As explained in section 2.3, this enables us to compute logit equilibrium by solving a system of differential equations in a form of (8). Since

$$\frac{d}{dx}U_i^e(x) = U_i'(x)F_{3-i}(1-x) - U_i(x)F_{3-i}'(1-x),$$

this system is

$$\begin{aligned} &\mu_i F_i''(x) = F_i'(x)\left[U_i'(x)F_{3-i}(1-x) - U_i(x)F_{3-i}'(1-x)\right] \\ &\text{s.t. } F_i(0) = 0, F_i(1) = 1 \end{aligned} \tag{19}$$

for $i = 1, 2$. Here the prime denotes the derivative with respect to x evaluated on whatever argument appropriate.

System (19) is not a system of ordinary differential equations of the second order because $F_i''(x)$ depends on F_{3-i} and F_{3-i}' evaluated at $1-x$ rather than at x. Differential equations of this kind are called functional differential equations, whose theory is less developed than the theory of their ordinary kin. Even less developed is the theory of *systems* of functional differential equations.

Fortunately, one can convert (19) to a system of ordinary differential equations:

LEMMA 1: System (19) is equivalent to the system

$$\begin{aligned} \mu_1 F''(x) &= F'(x)\left[U_1'(x)(1-G(x)) - U_1(x)G'(x)\right] \\ \mu_2 G''(x) &= G'(x)\left[U_2(1-x)F'(x) - U_2'(1-x)F(x)\right] \end{aligned} \tag{20}$$

subject to $F(0) = G(0) = 0$ and $F(1) = G(1) = 1$. The equivalence is given by the functional transformation $(F_1, F_2) \leftrightarrow (F, G)$ with $F(x) =$

$F_1(x)$ and $G(x) = 1 - F_2(1 - x)$.[11]

The proof is a simple exercise in substituting and differentiating and is therefore omitted.

Function G in Lemma 1 defines a cumulative density function that describes (distribution of) the complement to player 2's demand, i.e., how much he offers to player 1. Had we framed the NDG so that player 1 announces a demand whereas player 2 announces an offer, we could have arrived at system (20) directly.

System (20) does not have a closed-form solution even in the simple case of $U_1(x) = U_2(x) = x$. By Theorem 1, though, we know that the solution to this system exists for all pairs of error parameters μ_1, μ_2. Unfortunately, it is not clear whether this solution is unique. The theory of systems of differential equations is of no help here, since it deals mostly with initial-value problems (see e.g., Bear, 1962), and (20) is a system of boundary-value problems. Even the excellent book on boundary-value problems by Bailey *et al.* (1968) is silent about systems of equations. We cannot therefore, at this point, prove uniqueness of logit equilibrium in the NDG for other error parameters than those specified by Theorem 2.[12]

We therefore tried to prove whether there is at least a unique *symmetric* solution to (20) if $U_1 = U_2$ and $\mu_1 = \mu_2$ (by symmetric we mean a solution F, G such that $F(x) = 1 - G(1 - x)$ for all $x \in [0, 1]$). Even this attempt was futile, however. The "lens" technique (see, e.g., Anderson, Goeree and Holt, forthcoming 2) which is commonly employed in this type of proof cannot be used for (20) because this system is *not* symmetric.

Finally, following Anderson, Goeree and Holt (forthcoming 2) we tried to prove that any density function describing logit equilibrium in the NDG is

[11]Note that the substitution of F for F_1 is only formal. We introduce it mainly to lighten the notation in what follows.

[12]The uniqueness of logit equilibrium for all pairs of error parameters would contrast sharply with the continuum of Nash equilibria in this game. That a continuum of Nash equilibria needs not be an obstacle to having a unique logit equilibrium was demonstrated by Anderson, Goeree and Holt (2001) for minimum-effort coordination games. In the minimum-effort coordination games, however, all Nash equilibria are symmetric, while in the NDG all but one Nash equilibria are asymmetric. Still, we believe that there is a unique logit equilibrium in the NDG, if not for general utility functions and all pairs of error parameters, then at least for linear utilities and a range of error parameters much larger than indicated by Theorem 2. Our confidence here draws on a similarity between system (20) for linear utilities and a system describing logit equilibrium in the imperfect price competition game (Capra et al., forthcoming), for which uniqueness can be established. Numerical examples presented in the next section also support our belief in uniqueness even in the case of small error parameters.

single-peaked. Once again, however, we were betrayed by the asymmetry of (20).

4 The Examples

Since we were not able to characterize logit equilibrium in the NDG in general, we now show examples of this equilibrium for various specifications of players' utilities and error parameters. First we show symmetric examples and then asymmetric. All examples were calculated numerically by Maple V.

Figures 1, 2 and 3 illustrate a *symmetric* solution to system (20) in the case when both players have the same utility $U(x) = x$ and the same error parameter μ. The figures display both the cumulative density function F and the corresponding density function f. The common error parameter μ diminishes from Figure 1 to Figure 2 to Figure 3, taking values 1, 0.1, and 0.05.

Note that in all three cases the equilibrium density function f is, as we tried to prove, single-peaked. The same will be true about other examples to follow. Figure 1 shows that if $\mu = 1$, the equilibrium density function f is almost uniform (recall that the uniform distribution is the 'limit' logit equilibrium density function for $\mu \to +\infty$). This is not too surprising since $\mu = 1$ is quite large relative to the payoff range of the NDG (see the discussion following Theorem 2). In any case this indicates that as $\mu \to +\infty$ the convergence of the logit-equilibrium densities to uniform densities is quite fast. This provides one more reason why Theorem 2 ought to be considered weak.

Surprisingly, not even for $\mu = 0.05$ were we able to find multiple solutions to (20). While this does not prove that for even lower error parameters multiple logit equilibria cannot exist, it does support our conjecture that the range of error parameters for which there exists a unique logit equilibrium is much greater than shown in Theorem 2.

Figures 2 and 3 demonstrate how f accumulates as μ declines: in Figure 3 $f(x)$ is almost zero for $x \leq 0.2$ and $x \geq 0.6$. This accumulation of probability mass reflects the definition of logit equilibrium: the smaller μ, the more sensitive players are to differences in expected utilities from different demands, and so the closer their strategy is to the best response. Interestingly, as μ declines, f becomes markedly asymmetric. The asymmetry of f is especially conspicuous in Figure 3, where f grows relatively slowly between 0.3 and 0.45 to drop sharply to zero afterwards. This asymmetry reflects the asymmetric nature of Nash equilibria in the NDG and is in accord with the conjecture we made in the introduction that, when facing a boundedly rational opponent, players would be cautious to make demands close to or larger than one half. The smaller the

μ, the less such caution is necessary and the more closely the f approximates a demand of a half of the cake.

Table 1 provides a quantitative comparison of the logit equilibrium density functions shown in Figures 1, 2 and 3.

	Exp. value of f	Peak of f	Quantiles of f (10%, 50%, 90%)	Exp. equlib. payoff
$\mu = 1$	0.499	0.481	0.112, 0.498, 0.887	0.170
$\mu = 0.1$	0.431	0.456	0.264, 0.436, 0.573	0.316
$\mu = 0.05$	0.451	0.486	0.383, 0.464, 0.499	0.436

Table 1: Characterization of symmetric logit equilibrium when both players have linear utilities and the same error parameter μ.

The table adds details to our conjecture that boundedly rational players would be cautious. If μ equals 1, both players make such large mistakes in computation that they are almost indifferent to what to demand. The average demand is 0.499, the median demand 0.498. Low rationality thus implies no caution. If μ declines to 0.1, though, players are reluctant to make considerably larger demands than one half: the average demand drops to 0.431, the median to 0.436. Medium rationality thus implies some caution. If μ declines even more to 0.05, however, players can be less cautious again since certainty about their opponent's demands rises. The demands move closer to one half from the left: the average demand rises to 0.451, the median demand 0.464. High rationality thus implies little need for caution.

Quantiles of f in column 4 of Table 1 show how the asymmetry of f grows as μ decreases. They also demonstrate that the smaller the μ, the lower the proportion of demands above one half. Column 5 of Table 1 shows that a player's expected payoff from bargaining grows as μ declines. This is natural, since for small μ players' demands are well coordinated and the cake is rarely lost; rare also are situations when much of the cake remains unclaimed (when μ is large, however, both these instances of miscoordination are frequent).

Figure 4 shows an asymmetric logit equilibrium in the case when both players have the same utility $U(x) = x$, but differ in their error parameters. Player 1 is twenty times 'more rational' than player 2: $\mu_1 = 0.05$, $\mu_2 = 1$. Again, only one such equilibrium was found. Table 2 is an analog of Table 1 in this case.[13]

[13] In this table, and others to follow, symbol $f(x)$ is used somewhat loosely as a shorthand for equilibrum "density function." It refers to player 1's equilibrium density function (which is the hitherto used meaning of $f(x)$) as well as to player 2's equibrium density function ($g(1-x)$). In contrast, Figure 4 shows function $g(x)$. A peak of the density function of player 2 in Table 2, column 3, therefore does not correspond to the peak of $g(x)$ in Figure 4.

	Exp. value of f	Peak of f	Quantiles of f (10%, 50%, 90%)	Exp. equlib. profit
player 1	0.490	0.489	0.305, 0.490, 0.676	0.240
player 2	0.485	0.396	0.110, 0.471, 0.884	0.153

Table 2: Characterization of asymmetric logit equilibrium when both players have linear utilities but differ in their error parameters: $\mu_1 = 0.05$, $\mu_2 = 1$.

Here the equilibrium density function of player 1 is much flatter than in the case when he faces an equally rational opponent (see column 3 of Table 1). Since player 2's demands are very volatile, the probability of a miscoordination is great and the differences in player 1's expected payoffs from different demands small. Player 1's choice probabilities are therefore less dispersed—his density function flatter—than in case when his opponent's error parameter is small.

Player 2's equilibrium density function has a spread similar to that of the density function in the first row of Table 1, but it places more weight on smaller demands. This might be interpreted so that it is worse for player 2 to be confronted with a more rational opponent, than with the selfsame opponent. The last column of Table 2 supports this interpretation: player 2's expected payoff against a more rational player 1 (0.153) is lower than his payoff against an equally irrational player 1 (0.170; see the first row of Table 1). It cannot be said, however, player 2 is exploited by more rational player 1. Player 1 himself is better off when facing an equally rational opponent (compare expected payoffs in the first row of Table 2 and the last row of Table 1). In fact, as far as loss in expected payoff is concerned, player 1 is hurt more by player 2's irrationality than player 2 by player 1's rationality.

Table 1 and Figures 1, 2 and 3 demonstrate that as μ declines, the logit equilibrium (which, as said, appears to be unique), approaches a fair division of the cake $(\frac{1}{2}, \frac{1}{2})$. If both players have identical utility functions, this division coincides with both the Nash bargaining solution (Nash, 1950) and the Raiffa-Kalai-Smorodinsky bargaining solution (henceforth RKS solution; see Raiffa, 1953, and Kalai and Smorodinsky, 1975). It is therefore not clear whether for small μ the logit equilibrium approximates the former or the latter bargaining solution.

To discriminate between these two hypotheses, we calculated logit equilibrium in two cases when players' utility functions differ. In both cases μ_1 and μ_2 were set as identical and equal to 0.05, the smallest error parameter Maple V could handle. In the first case players' utilities were chosen as follows: $U_1(x) = \log(1+21.933x)$ and $U_2(x) = x$. The rather weird choice of U_1 guarantees a nice Nash bargaining solution $(0.3, 0.7)$.[14] The RKS bargaining solution is $(0.226, 0.773)$.

As before, we found only one logit equilibrium. Figure 5 and Table 3 describe the density functions of this equilibrium in detail.

	Exp. value of f	Peak of f	Quantiles of f (5%, 50%, 95%)	Exp. equlib. profit
player 1	0.293	0.306	0.262, 0.289, 0.296	0.291
player 2	0.641	0.685	0.580, 0.667, 0.694	0.641

Table 3: Characterization of asymmetric logit equilibrium when both players have the same error parameters $\mu_1 = 0.05$ but differ in their utility functions: $U_1(x) = \log(1 + 21.933x)$, $U_2(x) = x$.

Both the figure and the table provide strong support for the hypothesis that for small μ logit equilibrium approximates the Nash bargaining solution. This is best seen at players' expected or median payoffs, which are much closer to their Nash-bargaining-solution shares than to their RKS shares. The RKS bargaining solution therefore cannot be considered a point approximated by the logit equilibrium.

The second example points to the same conclusion even more sharply. In this example, players' utilities were chosen as $U_1(x) = \sqrt{1 + 20x} - 1$, $U_2(x) = x$. This choice guarantees the Nash bargaining solution $(0.4, 0.6)$; the RKS solution is $(0.126, 0.874)$. Table 4 and Figure 6 show the logit equilibrium in this case. Again, neither player made demands close to his RKS share with high probability, but both players demanded on average almost their Nash-bargaining-solution shares.

	Exp. value of f	Peak of f	Quantiles of f (10%, 50%, 90%)	Exp. equlib. profit
player 1	0.389	0.400	0.370, 0.393, 0.403	0.387
player 2	0.546	0.591	0.481, 0.562, 0.591	0.544

Table 4: Characterization of asymmetric logit equilibrium when both players have the same error parameters $\mu_1 = 0.05$ but differ in their utility functions: $U_1(x) = \sqrt{1 + 20x} - 1$, $U_2(x) = x$.

Both examples are quite persuasive. The Nash bargaining solution seems to be an accumulation point of logit equilibrium when players' error parameters

[14] $U_1(x)$ was determined ex post after the Nash bargaining solution was fixed at $(0.3, 0.7)$ and $U_2(x)$ put equal to x.

tend to zero. Since there may be more than one logit equilibrium for small error parameters, however, there may be more than one such accumulation point. In spite of this difficulty, a rigorous hypothesis about the limit behavior of logit equilibrium can be framed with the aid of logit equilibrium correspondence, as done by McKelvey and Palfrey (1995) for finite-strategy games. The *logit equilibrium correspondence* $\Lambda : \mathsf{R}_+ \to 2^{\mathcal{M}\times\mathcal{M}}$ is given as

$$\Lambda(\mu) \equiv \{(F_1, F_2) \in \mathcal{M} \times \mathcal{M} \mid (F_1, F_2) = T_\mu(F_1, F_2)\},$$

where T_μ stands for operator T defined by (10), when $\mu_1 = \mu_2 = \mu$. Then, the hypothesis is that there is a unique branch in the graph of Λ that starts at a pair of uniform cumulative density functions at $\mu = +\infty$ (the so-called centroid) and converges to the Nash bargaining solution as μ tends to zero. We see a theoretical investigation of this hypothesis as a salient task for future research.

5 Conclusion

In this note we presented a boundedly rational analysis of the Nash Demand Game based on the concept of quantal response equilibrium (logit equilibrium). We reviewed the concept and discussed its evolutive support. We then showed that logit equilibrium exists in the Nash Demand Game and it is unique if neither bargainer is too close to the ideal of a "perfectly rational man." If, moreover, both bargainers have the same utility function, then the logit equilibrium in the Nash Demand Game is symmetric.

We also computed numerical examples of logit equilibria in the Nash Demand Game for various specifications of players' utilities and various degrees of their 'irrationality' (error parameters). These examples reflected our conjecture that boundedly rational players would be cautious in their demands, since overreaching oneself in bargaining may be more costly than leaving a part of the 'cake' unclaimed. More importantly, the examples supported the hypothesis that, as players' error parameters tend to zero at the same rate, logit equilibrium in the Nash Demand Game selects a unique bargaining solution, namely the Nash bargaining solution. This finding corroborates the robustness of the Nash bargaining solution to "trembles," as evidenced by Young (1993) and Binmore and Dasgupta (1987).

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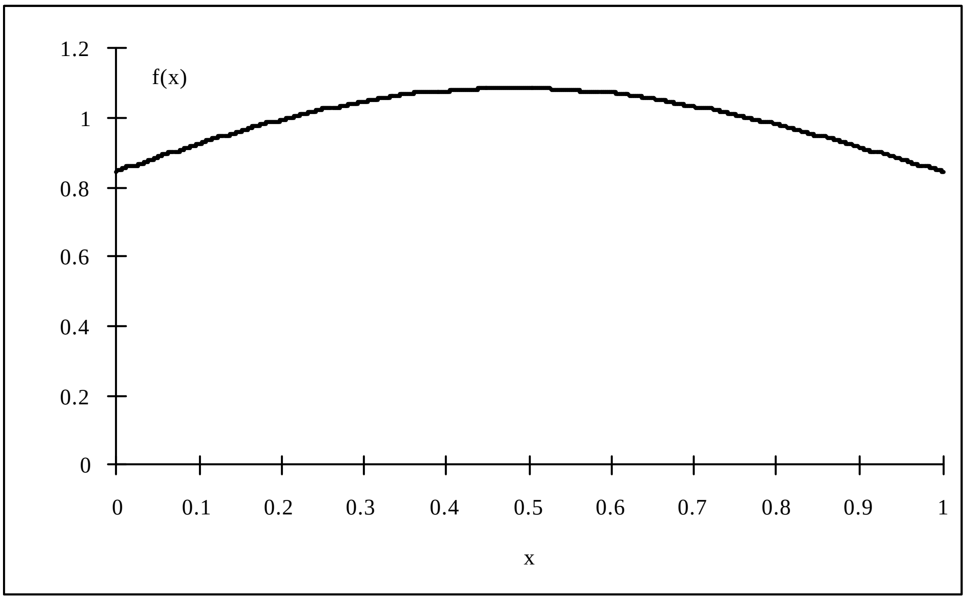
Figures



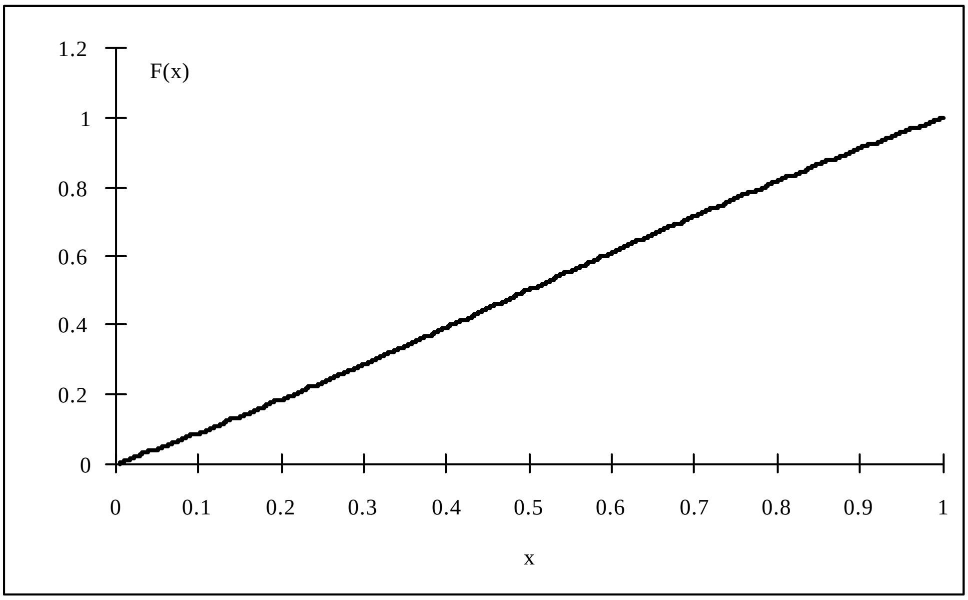


Figure 1: Logit equilibrium density function (top) and cumulative density function (bottom) when both players have linear utilities and ${}^{1}{}_{1} = {}^{1}{}_{2} = 1$.

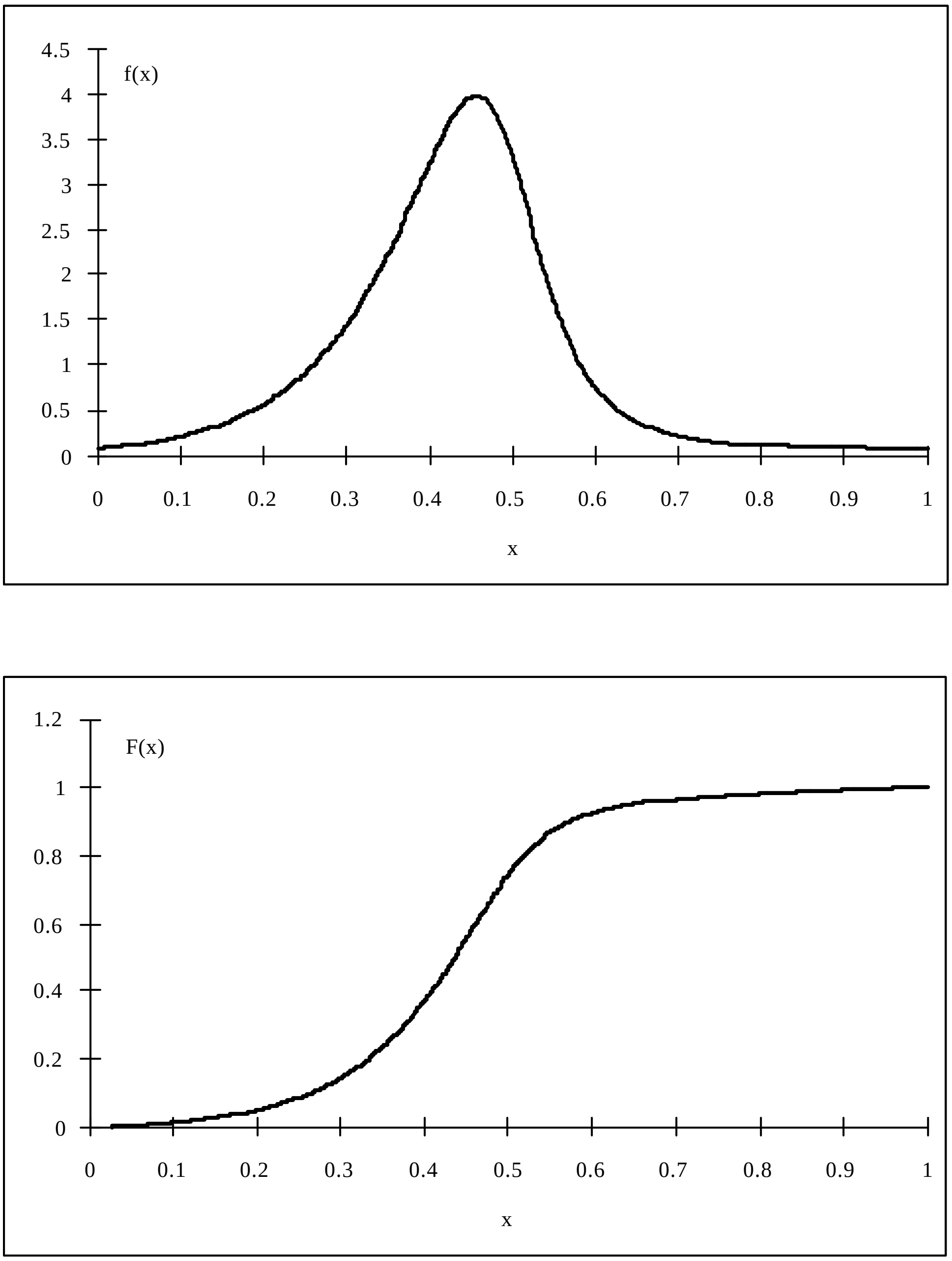


Figure 2: Logit equilibrium density function (top) and cumulative density function (bottom) when both players have linear utilities and $\mu_1 = \mu_2 = 0.1$.

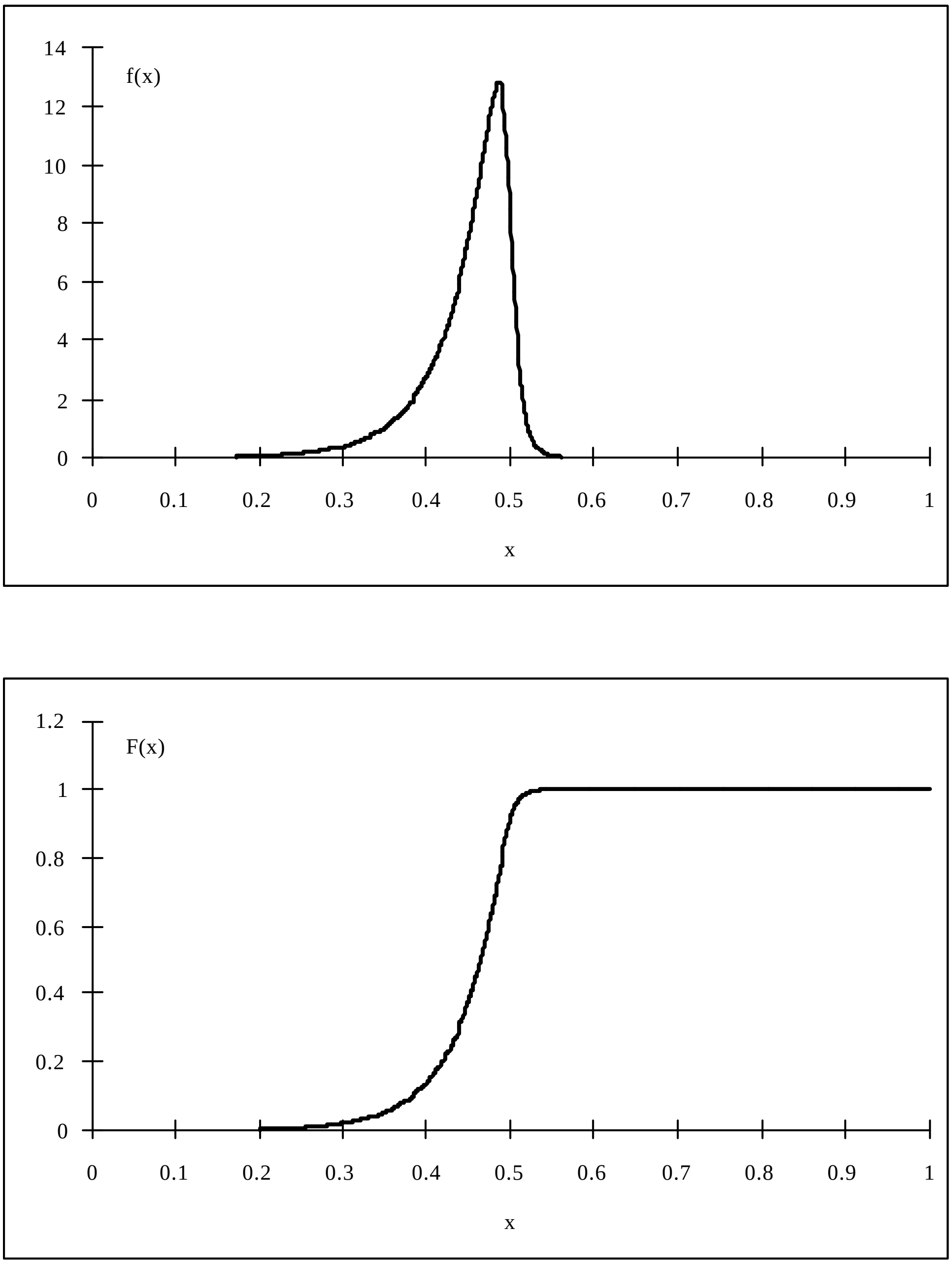


Figure 3: Logit equilibrium density function (top) and cumulative density function (bottom) when both players have linear utilities and $\mu_1 = \mu_2 = 0.05$.

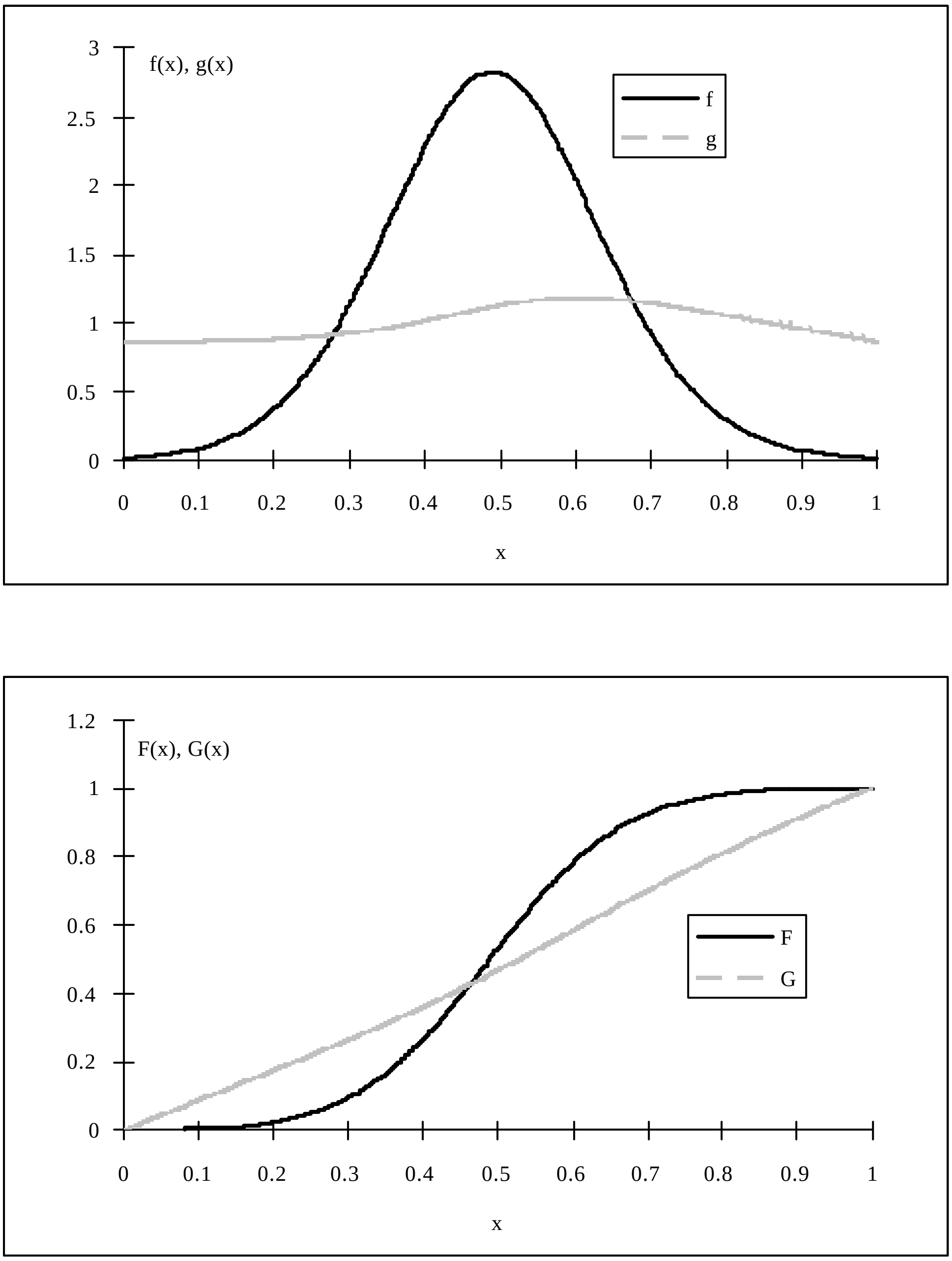
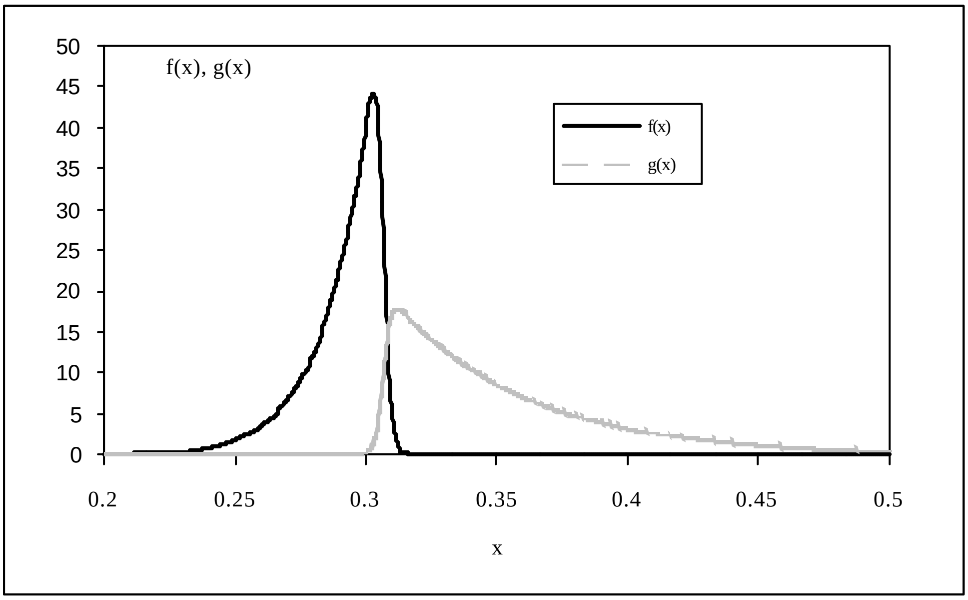


Figure 4: Logit equilibrium density functions (top) and cumulative density functions (bottom) when both players have linear utilities and $\mu_1 = 0.05$, $\mu_2 = 1$.



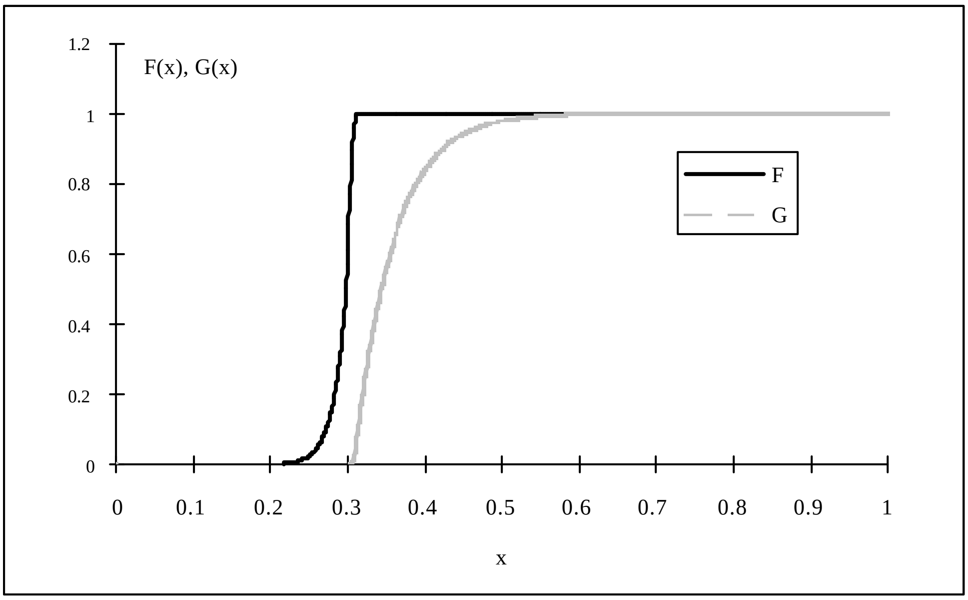


Figure 5: Logit equilibrium density functions (top) and cumulative density functions (bottom) when $U_1(x) = \log(1 + 21.933x)$, $U_2(x) = x$, and $\mu_1 = \mu_2 = 0.05$.

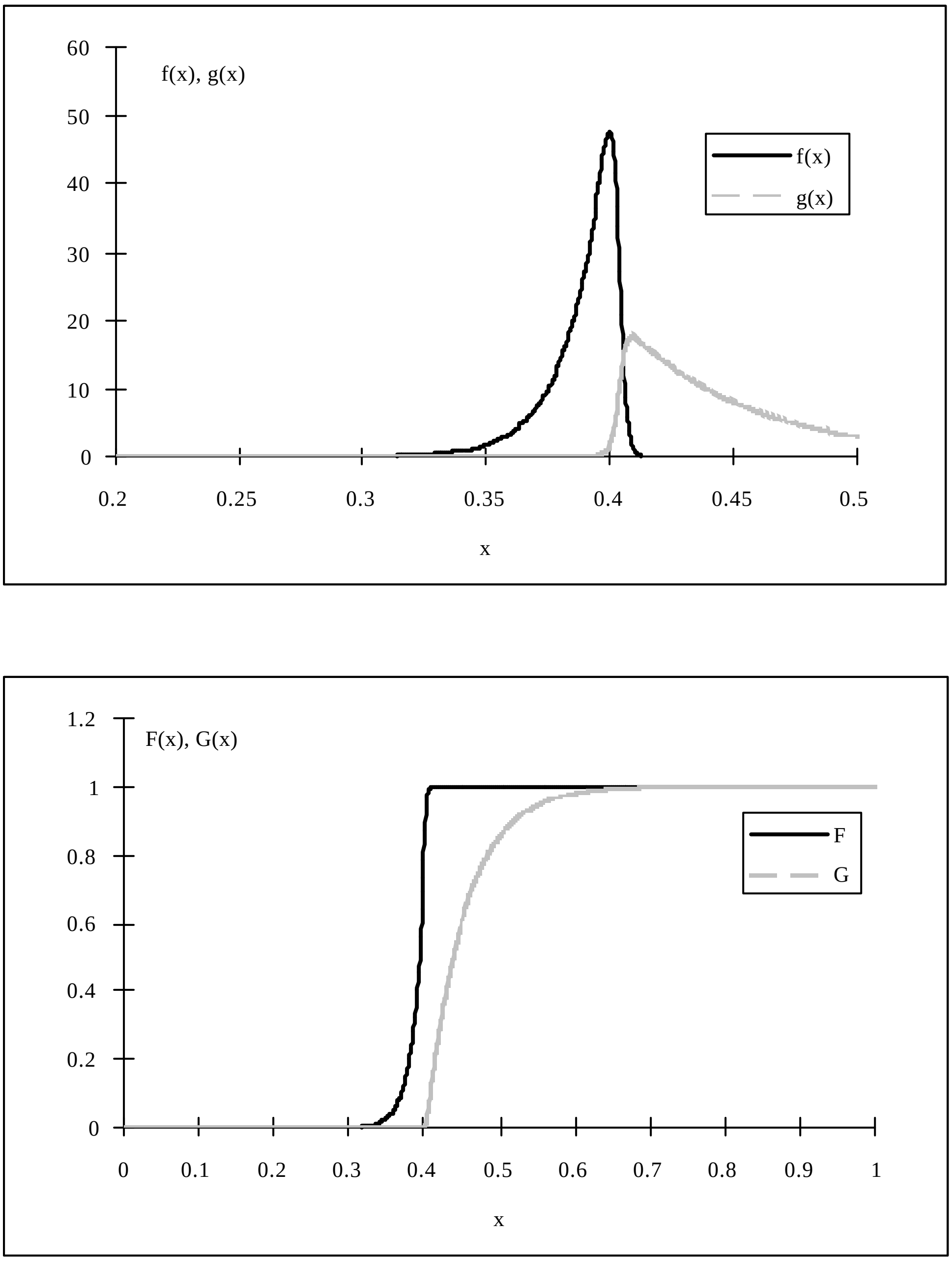


Figure 6: Logit equilibrium density functions (top) and cumulative density functions (bottom) when $U_1(x) = \sqrt{1 + 20x} - 1$, $U_2(x) = x$, and $\mu_1 = \mu_2 = 0.05$.