CERGE
Center for Economic Research and Graduate Education
Charles University

# Estimation of Peer Effects Model with Selective Assignment of Pupils into Classes

**Lukáš Supik**

Master's Thesis

Prague, July 2021

**Thesis Committee:**

Name Surname (Name of institution, Chair)
Name Surname (Name or institution)
Name Surname (Name of institution)

**Referees:**

Supervisor: Doc. Ing. Daniel Münich, Ph.D. (CERGE)
Opponent:

# Contents

Acknowledgements ..... v
Abstract ..... vii
Abstrakt ..... ix
Introduction ..... 1
1 Literature Review ..... 4
1.1 Examples of Peer Effects in Different Environments ..... 4
1.1.1 Peer Effects in Smoking ..... 5
1.1.2 Peer Effects in Sexual Behaviour ..... 7
1.2 Peer Effects in Education ..... 7
1.2.1 Peer Effects in Education with Selective Assignment ..... 10
2 Estimation of the Linear-in-Means Peer Effects Model ..... 13
2.1 Definition of the Linear-in-Means Peer Effects Model ..... 13
2.2 Reduced Form of the Linear-in-Means Peer Effects Model ..... 16
2.3 Identification of the Linear-in-Means Peer Effects Models – General Conditions ..... 26
2.4 Identification of the Linear-in-Means Peer Effects Model – Specific Cases and the Reflection Problem ..... 30
2.5 The Problem of Correlated Effects ..... 36
2.6 The Description of Selection ..... 37
2.7 Identification in Linear-in-Means Peer Effects Models in a Selective Environment ..... 40
3 Empirical Analysis ..... 46
3.1 The Description of Czech Education and Data ..... 46
3.2 The Estimation of Peer Effects Model ..... 49
3.3 Placebo Test ..... 53
3.4 Simple Analysis of Social Effects ..... 55
4 Further Research ..... 58
Summary ..... 60
References ..... 61

# Acknowledgements

I am extremely grateful to my supervisor, Doc. Ing. Daniel Münich, Ph.D., who came with the idea to analyse peer effects in the Czech environment. However, in the beginning, we both were unaware of the complexity of peer effects research and I was steadily realising the necessity of cooperation with the specialist of peer effects research. Here, I would like to give my special thanks to Paolo Zacchia, Ph.D., for his irreplaceable advice. Moreover, Paolo is one of the fastest responders whom I know and he was always prepared to help me within his time possibilities. Regarding professional help, I also thanks Bohdana Kurylo, who gave me very effective feedback during our second-year courses and she significantly contributed to the final form of my thesis.

With regard to English writing, I thank all members of the Academic Skills Centre at CERGE-EI who helped me to impart my thoughts to the reader of my thesis, especially to Grayson Krueger, M.A., who edited a great part of my thesis. I also thank the further staff of CERGE-EI, who gave me support during my studies, especially women in the Study Affairs Office.

As regards my life path, I would like to thanks my parents who supported me in my life and my studies from the very beginning. Without them, I would not be where I am now. Last, but not least, I give my special thanks to my girlfriend Julia, who supported me during my studies. She helped me to stay in a good mood and keep my mental health despite tough times in my studies.

Prague, Czech Republic Lukáš Supik

July, 2021

# Abstract

People are by nature social beings. Most of us have a complex social network that connects us with other people in numerous aspects of our lives: neighbours, co-workers or peers in schools, and friends. Moreover, it is widely believed that people's behaviour is to some extent affected by others in their social networks, which is known as peer effects. Therefore, a precise understanding of the behaviour of an individual necessarily includes understanding her interactions with others within her social network.

The first part of this thesis, literature review, summarizes contemporary research on peer effects, shows which aspects of human behaviour may be affected by social interactions, and highlights the importance of peer effects research. In the second part, the estimation of the linear-in-means peer effects model, we provide a detailed description of the model, derivations of its alternative formulations, and show the identification conditions. The main contribution of the second part is that we provide a step-by-step analysis of the linear-in-means peer effects model and detailed proofs of theorems in one place.

The third part provides an empirical analysis of peer effects in education in the Czech Republic. In particular, we examine how the test scores of pupils are affected by their classmates. We observe that pupils' test scores are negatively affected by the test scores of their peers and positively affected by the abilities of their peers. The results are statistically significant; however, they are also excessively high compared with previous research. Therefore, we conduct bootstrap simulation and find that the estimators of standard errors are probably underestimated. Moreover, we conduct a placebo check randomly allocating pupils among classes and show that widely used peer effects estimators are slightly biased in both directions, which could explain high and significant peer effects estimators. Therefore, we conclude that corrected peer effects estimators are likely unsignificant in our setting, which is mainly caused by the small data sample. Finally, we briefly propose possible extensions of the linear-in-means peer effects model, which may give a more realistic description of peer effects in real world.

# Abstrakt

Lidé jsou ve své přirozenosti společenské bytosti. Většina z nás má složitou sociální síť, která nás propojuje s druhými v mnoha aspektech našich životů: sousedé, spolupracovníci nebo spolužáci a přátelé. Navíc je jasné, že chování jedince je do jisté míry ovlivněno druhými v jeho sociální síti, a toto ovlivňování obvykle označujeme jako peer efekty. Proto přesné poznání chování jedince nezbytně zahrnuje poznání interakcí s druhými v rámci sociální sítě.

První část naší práce, rešerše literatury, shrnuje současný výzkum peer efektů, ukazuje, které aspekty lidského chování mohou být ovlivněny sociálními interakcemi a zdůrazňuje tak význam výzkumu peer efektů. Ve druhé části, estimaci lineárního-v-průměrech modelu, přinášíme detailní popis modelu, odvození jeho alternativních formulací a podmínky identifikace. Hlavním přínosem druhé části je, že přinášíme analýzu modelu krok za krokem a detailní důkazy matematických vět na jednom místě.

Třetí část podává empirickou analýzu peer efektů ve vzdělávání v České republice. Konkrétně zkoumáme, jak je testové skóre žáků ovlivněno jejich spolužáky a pozorujeme, že je negativně ovlivněno testovým skórem a pozitivně ovlivněno schopnostmi jejich spolužáků. Výsledky jsou statisticky signifikantní, ale také jsou příliš velké ve srovnání s předchozím výzkumem. Proto provádíme bootstrap simulace and zjišťujeme, že odhad standardní odchylky je pravděpodobně podhodnocen. Dále také provádíme placebo kontrolu tak, že náhodně umisťujeme žáky po třídách a ukazujeme, že obvykle používané estimátory peer efektů jsou vychýlené v obou směrech, což by mohlo vysvětlit vysoké a statisticky signifikantní estimátory peer efektů. Proto se domníváme, že korigované estimátory peer efektů již pravděpodobně signifikantní nejsou, což je způsobeno zejména malým vzorkem dat. Nakonec ve stručnosti navrhujeme možná rozšíření lineárního-v-průměrech modelu peer efektů, což může přinést realističtější popis peer efektů.

# Declaration of Authorship

I hereby proclaim that I wrote my thesis on my own under the leadership of my supervisor and that the reference includes all resources I have used.

Prague, Czech Republic | Lukáš Supik

July, 2021

# Diploma Thesis Proposal

Center for Economic Research and Graduate Education – economics Institute
Joint workplace of Charles University and the Economics Institute of the Czech Academy of sciences

| | | | |
|---|---|---|---|
| Author: | Lukas Supik | Supervisor: | doc. Ing. Daniel Münich, Ph.D. |
| E-mail: | lukas.supik@gmail.com | E-mail: | daniel.munich@cerge-ei.cz |
| Phone: | +420 732 330 047 | | |


## Proposed Topic:

**Economics of sklil formation**

## Motivation

Human capital has significant socio-economic impacts and it is useful to examine factors that can positively influence it and bring future benefits. Psacharopoulos (1985) estimated returns to investments in education and emphasized the role of suitable public policies. However, the preparation of good public policies should be supported by research analysing data from specific countries, because school systems are diverse and the same policies may have different impacts in different countries. The aim of my thesis is to analyse Czech panel data and reveal factors that can play significant roles in educational outcomes of pupils.

## Contribution

There are several longitudinal studies that have observed educational outcomes of pupils, their socio-economic status, family background, the environment where they live and the schools they attend. Goodman and Washbrook (2011) conducted a large analysis of four longitudinal studies in England and correlated educational outcomes of pupils with these factors. His analysis suggest which factors could play key roles in the development of human capital and how it can be affected by public policies. Crawford et al. (2017) focussed on role of school quality and estimates how important could it be in the development of human capital. However, neither study provides the causal effects of the factors considered on the educational outcomes of pupils, although their results propose promising areas for further research and public policies.

To the best of my knowledge, there is no study examining longitudinal data about Czech pupils. However, from 2012 to 2018 Czech Longitudinal Study in Education (CLOSE) was conducted and it contains information about educational outcomes of pupils, their family background, environment where they live, schools which they attend and teachers. The aim of my thesis is to conduct the first and unique observational analysis of longitudinal patterns specific to individual study paths of individual pupils in the Czech Republic. My research should identify correlations between educational outcomes of pupils and other factors included in the data from CLOSE. Additionally, my results could provide useful support for policy makers in the Czech Republic and suggest promising further research of causal effects on educational outcomes of pupils, which could have international significance.

## Methodology

I will explore the content of data from CLOSE using provided questionnaires. Then I will track individual study paths of individual pupils and correlate educational outcomes of pupils with factors described in the data from CLOSE. I will want to propose public policies based on my results. Finally, I will compare data from CLOSE with data used in related research and discuss opportunities of further research.

## Outline

Introduction

Literature review

Description of data from CLOSE

How was data from CLOSE already analysed

How could be data from CLOSE analysed

Conclusion

## Bibliography

Crawford, C., Macmillan, L., Vignoles, A., 2017. When and why do initially high-achieving poor children fall behind? Oxf. Rev. Educ. 43, 88–108. https://doi.org/10.1080/03054985.2016.1240672

Goodman, A., Washbrook, E., 2011. Children's educational attainment and the aspirations, attitudes and behaviours of parents and children through childhood in the UK 18.

Psacharopoulos, G., 1985. Returns to Education: A Further International Update and Implications. J. Hum. Resour. 20, 583. https://doi.org/10.2307/145686

. . . . . . . . . . . . . . .                . . . . . . . . . . . . . . .
Author                Supervisor

# Introduction

Classical economic theories assume that an individual chooses her actions taking the state of the world as given, and thus that the actions of an individual do not affect the actions of other individuals. This assumption is relatively reasonable in environments where a large number of individuals interact together and their interactions are evenly distributed, for example a stock exchange where a transaction can be undertaken by individuals from different parts of the world. However, it is also possible that a large number of individuals is partitioned into smaller groups and individuals may affect each other within these groups. Peer effects describe almost any externality in which the action of an individual directly affects the actions of individuals in her group.

Peer effects are evident in different environments, e.g., investments decisions, the labour market, and health, among others. In the case of investing, individuals may be affected by the decisions of their neighbours. Specifically, the neighbour of individual $i$ may affect which stocks individual $i$ will buy. Alternatively, individual $i$ may not currently be purchasing stocks and having a neighbour who is investing may encourage individual $i$ to start investing in stocks too. As regards peer effects in the labour market, having a co-worker who is highly motivated and chooses high effort in work may increase another individual's effort, especially in a team task. With regard to peer effects on health, being surrounded by individuals who are vaccinated against an infectious illness may lower the probability that a non-vaccinated individual will be infected. To provide another example, having peers who are overweight may reduce the effort that an individual puts into reducing her own weight.

In addition to the above, peer effects are also found in the context of education, which is particularly relevant for our thesis. Educational outcomes are usually measured by math, reading, and language tests. The analysis of peer effects in education examines how the educational outcomes of a pupil $i$ are affected by characteristics of her peers and Manski (1993) classify three types of peer effects. First, endogenous peer effects describe how the outcome of individual $i$ is affected by the outcomes of her peers. Second, exogenous peer effects describe how the outcome of individual $i$ is affected by the background characteristics of her peers. Third, correlated effects describe the situation in which individuals are sorted into groups according to unobservable characteristics that directly affect the examined outcome. As a consequence, the outcome of an individual $i$ is correlated with the outcomes of their peers. As a consequence, correlated effects have to be taken into account for the unbiassed estimation of endogenous and exogenous peer effects.

Regarding the main goals of our work and potential contributions, we start with a literature review discussing several papers examining peer effects. Our examples are focused mainly on peer effects in education and on the behaviour of youth since we examine peer effects in education in our analysis. However, as there is nothing particularly special about peer effects in education, our methodology applies to all types of peer effects. In this section, we motivate our peer effects research on examples from previous studies.

The theoretical part of our work is devoted to the analysis of the linear-in-means peer effects model and we summarise previous research in detail. First, we motivate the definitions using intuition by providing a simplified example and extend it in several logically connected steps. Specifically, in the case of the network matrix, we start with the matrices of ones and generalise them into the matrix describing an arbitrary social network. Second, we provide step-by-step derivations of different functional forms of the peer effects model. In separated propositions, we summarise important properties of the objects used in our derivations. Third, we provide all proofs in one place since scientists refer sometimes to previous papers and do not repeat all necessary proofs. An example is the proof of proposition 2.1; since we were unable to find a rigorous proof in the original papers, we provide our own proof.

In the empirical part of our work, we provide a unique analysis of the Czech Longitudinal Study in Education (CLOSE). We estimate peer effects in test scores among pupils attending junior secondary grammar schools preparing pupils for academic track. We focus on this subset of pupils since we can observe their test scores at the beginning of their studies at junior secondary grammar schools and four years later. Therefore, we observe test scores not affected by their peers, which can be used as the explanatory variable summarising pupils' background characteristics. The test score after four years is analysed as the outcome variable that is affected by peer effects. Both test scores enable us to estimate our peer effects model and we find negative and statistically significant endogenous peer effects in Math, Reading and English. Moreover, we find positive and statistically significant exogenous peer effects in Math, Reading and English. However, our estimates are noisy and unrealistically high. Therefore, we compute bootstrap errors which suggest that the observed significant results may be caused by the underestimated standard errors of the estimators. Moreover, we conduct a placebo check suggesting our peer effects estimators are slightly biased and taking this bias into account makes our estimators insignificant. We hypothesise that the underestimation of standard errors of estimators and the bias of estimators can be explained by the small sample size and relatively low class-size variation.

Finally, we describe extensions of the linear-in-means peer effects model. We provide a summary of the recent research and discuss developed methods. The linear-in-means peer effects model can be naturally extended in two dimensions. First, previous research suggests that peer effects may be non-linear, and hence it is promising to extend the linear-in-means peer effects model to some non-linear models. Second, peer effects may include particular heterogeneities. This means that the effect of peers on an individual $i$ may depend on her characteristics. Specifically, high-achieving individuals may affect differently their high- or low-achieving peers. A different extension of the linear-in-means peer effects model considers endogenous network formation. The assumption is that individual pupils are not affected by all their peers, but they are mostly affected by their close friends. The assumption of endogenous network formation seems to be reasonable; nevertheless, the description of peer effects is much more involved in this model. Importantly, the described extensions may provide a realistic description of the real-world environments based on recent research.

# 1 Literature Review

## 1.1 Examples of Peer Effects in Different Environments

We start our literature review describing peer effects research in different environments. The following examples show how the topic is relevant since it covers various areas of humans' lives. Considering economic decisions, Duflo and Saez (2002) examine peer effects in retirement savings decisions. They use data from employees of a large university about the participation in Tax Deferred Account plan sponsored by the same university. They examine if an individual participation decision may be affected by the decisions of other employees in the same department. Their estimation suggests that peers from the same department may affect the participation decision of an individual. Banerjee et al. (2013) examine the diffusion of information about microfinance loans. A microfinance institution first conducted an informal meeting with the community leaders including teachers and shopkeepers. Subsequently, Banerjee et al. (2013) examine how a relationship with an informed individual increases the probability of taking out a loan. They find positive and significant peer effects, e.g., community leaders spread the information about loans and increased the probability that others would take a loan. The effect is significant for both community leaders who took out a loan and those who did not. Community leaders who took out loan had seven times higher power in influencing their peers. However; the number of non-takers was higher; hence they account for one third of the net peer effect.

Considering labour market decisions, Nicoletti et al. (2018) examine peer effects with a decision regarding mothers' labour supply. The peer group is presented by peers from individuals' families and neighbours. Using Norwegian administrative data, they find positive and statistically significant peer effects in the individuals' working supply. Specifically, having a peer who increases her labour supply for one hour increases an individual's labour supply by about half an hour. However, regarding labour market participation, Nicoletti et al. (2018) do not find statistically significant peer effects. This means that having a peer who increases her labour supply for one hour does not affect an individual's decision whether to work or not.

Regarding health outcomes, Miguel and Kremer (2004) examine a deworming policy intervention when pupils receive deworming drugs in schools. The researchers argue that a simple analysis of the treatment effect may underestimate the positive effects of policy interventions. They explain that the distribution of deworming drugs also reduces the spread of worms in locations that were not treated. They evaluate the treatment effects and consequently peer effects on absenteeism of pupils in schools. They find that untreated schools in the neighborhood of treated schools experience a significant reduction in absenteeism. Trogdon et al. (2008) examines peer effects in obesity among adolescents. They define peer groups according self-reported friendships. Therefore, they have to account for endogenous network formation and avoid plausible selection bias. The results suggest that having friends with high BMI increases an individual's BMI. Moreover, this peer effect is higher among girls and is also higher for individuals with the highest BMI.

### 1.1.1 Peer Effects in Smoking

Research on peer effects in smoking shows that smoking can be significantly affected by peer pressure. Having actively smoking peers increases the probability of individuals to start smoking. Moreover, the latest research reveals a certain heterogeneity. In particular, mentally unstable individuals may be more vulnerable to peer pressure (Hsieh and Kippersluis 2018).

Eisenberg et al. (2014) examine peer effects in risky behaviour among college roommates. College roommates are randomly allocated into rooms; hence the estimation of a standard peer effects model leads to unbiassed estimates. They find an insignificant effect of roommates' smoking on individual behaviour. However, the analysis of heterogeneities suggests that there are positive peer effects among men and negative among women. The authors cannot explain observed differences and stress that the statistical power of these results is rather small. In our opinion, the peer effects among roommates are specific since they measure peer effects on the subset of friends' network. Indeed, college students may have more friends who are not their roommates.

Nakajima (2007) examines peer effects in smoking behaviour among teenagers in the United States. He defines peer groups at the level of schools and assumes that all pupils interact within and not between schools. This assumption is reasonable since the author cannot observe the friendship network among teens. Parents can select a school for their children; hence the selection bias can be expected. Nakajima (2007) introduces and estimates a random utility model that can account for the selection bias. He finds that peer effects are positive and statistically significant, i.e., having peers who are smokers significantly increases individuals' probability of starting smoking. Examining heterogeneity Nakajima (2007) shows that peers effects occur within and between gender. A similar result applies to race.

Card and Giuliano (2013) examine risky behaviour among adolescents from the United States. They use Add Health data that contains information about up to five friends. The best friend is listed as the first, hence the researchers have detailed knowledge about friends' networks. However, the friends are likely similar in observable and unobservable characteristics. This tendency is usually called "homophily". As homophily is one of the causes of selection bias, Card and Giuliano (2013) construct bivariate ordered choice models for the behaviour of friends that include both social interaction effects and unobserved heterogeneity across pairs. They assume that risky behaviour brings individuals some utility based on their observed characteristics and behaviour of friends. Card and Giuliano (2013) estimate their model and measure peer effects in sexual behaviour, smoking cigarettes and marijuana, and truancy. The results are qualitatively comparable for different specifications of the model and suggest that there are significant peer effects in all cases of risky behaviour.

Hsieh and Kippersluis (2018) also use US Add Health data to examine peer effects in smoking initiation among adolescents. They observe friendship networks and extend the Selection Corrected Spatial Autoregressive (SC-SAR) model to account for the selection bias coming from endogenous friendship formation. Using SAR models, the researchers assume that friendship networks among adolescents can be observed. The central object of SAR models is a spatial weight matrix whose $i, j$ element is one if adolescents are friends. We will use the analogical network matrix in our model. However, as the SAR model likely leads to a selection bias, the authors introduce multivariate unobservable characteristics of individuals that can account for selection bias. The multivariate unobservable characteristics were first used in the SC-SAR model by Hsieh and Lee (2016). Hsieh and Kippersluis (2018) estimate their SC-SAR model using the Bayesian approach and show that peer effects play an important role in smoking initiation. Moreover, they examine the role of individuals' personality on the strength of peer effects and find that emotionally unstable individuals are more vulnerable to peer pressure. Consequently, the positive and statistically significant peer effects are driven mainly by emotionally unstable individuals.

### 1.1.2 Peer Effects in Sexual Behaviour

Ali and Dwyer (2011) estimate peer effects in sexual behaviour among adolescents in the United States and investigate whether having friends who already initiated intercourse or had more sexual partners affects individuals' sexual behaviour. They use data from a nationally representative sample of adolescents that contains friendship networks and attended school classes. Peer effects are estimated in both groups, friendship networks and school classes. Ali and Dwyer (2011) estimate the linear-in-means peer effects model and apply two strategies to avoid selection bias. First, they include school-level fixed effects to account for common shocks, teachers' qualities and selection bias. Second, they use parental characteristics as instrumental variables. Parental characteristics including sexual behaviour, number of sexual partners, and stability of marriage may directly affect an individual adolescent; however, parental characteristics are unlikely to affect peers' sexual behaviour. Hence, parental characteristics provide a valid and relevant instrument for the measurement of peer effects. The results show that a 10% increase in the fraction of friends who initiates sex increases individuals' probability of sex initiation by 5%. The analogical result applies to the number of sexual partners. However, the peer effects analysis among classmates provides insignificant estimates. The results suggest that peer effects may be more relevant in a friendship network than among classmates.

Card and Giuliano (2013) also examine peer effects estimation in sexual behaviour only. They find that individuals' sexual behaviour is significantly affected by their peers. Specifically, having a best friend who has intercourse increases the probability that an individual will also have intercourse by approximately 5%. To stress the overall significance of peer effects, Card and Giuliano (2013) estimate that 10% of inexperienced adolescents initiate their sexual behaviour based on the choice of their best friend.

## 1.2 Peer Effects in Education

We begin our review of peer effects in education by summarising Sacerdote's (2011) work entitled "*Peer Effects in Education: How Might They Work, How Big Are They and How Much Do We Know Thus Far?*". In this paper, Sacerdote (2011) provides a comprehensive introduction to peer effects in education, which contains a theoretical analysis, results and ideas that are highly relevant to our study of peer effects in education.

Sacerdote (2011) provides a general definition of peer effects, stating that it refers to almost any externality in which peers' backgrounds, current behaviour and outcomes affect the outcomes of an individual. Sacerdote (2011) uses almost any externality, since he excludes market-based effects. Specifically, if the inflow of rich people into a town increases the price of private schooling, this is not considered a peer effect in education. Second, Sacerdote (2011) excludes the effect of class size because it is expected that a reduction in class size will improve pupils' education through a more individual teaching approach, and thus we do not want to include this case into our peer effects analysis.

Elaborating on our definition, we provide some specific examples. The educational outcomes in primary and secondary education are usually the test scores in reading, math, and learning skills. The educational outcomes in post-secondary education are usually SAT (Scholastic Aptitude Test) scores and GPA (grade point average) in exams. Background characteristics are usually SES (Socio-Economic Status) of family, race, parents' education and behaviour, and the stability of family among others. Considering current behaviour, researchers usually focus on behavioural disorders such as disrupting during lessons, smoking, teenage pregnancy, bullying, and truancy, among others. We also introduce an example describing the potential mechanisms of peer effects, even though it is challenging to rigorously estimate them. Consider high-achieving primary-school students from high SES families with strong parental support for education. First, high-achieving students may motivate their peers and also teach them in their free time. Second, they can motivate their teachers to provide more enthusiastic teaching and cover more advanced topics. Third, the educated parents of high-achieving students may increase the aspirations for further education of their peers and update their beliefs about the returns to education. Fourth, high test scores of premiant students may motivate their peers to increase their effort in the preparation for exams. This example illustrates how peer effects can work through current behaviour, the effect on teachers' performance, background characteristics, and educational outcomes.

Regarding the empirical results of Sacerdote (2011), we start with the interpretation of estimated coefficients in the linear-in-means peer effect model. Test scores are usually normalised to have zero mean and unit variance. The peer effects estimator describes how an individual's test score would change if the average test score of her peers increased for one (which is a standard deviation of overall test scores). Sacerdote (2011) summarizes peer effects estimates from different studies and we will select the studies estimating peer effects in reading and math, since they are relevant for our own estimates.

Hoxby (2000) analyses peer effects among students in all Texas elementary schools in grades 3–6. Her estimation strategy relies on the exogenous variation in the gender composition of classes. Girls have about half of the standard deviation higher score than boys, and hence the variation in gender composition also affects the class average test score. The peer effects estimates are 0.3 to 0.5 in reading and 1.7 to 6.8 in math. Using students' fixed effects to control for positive selection, Betts and Zau (2004) examine students from San Diego Unified School District and arrive at estimates of peer effects of 1.4 in reading and 1.9 in math. They also find some evidence of nonlinearities suggesting that an average student is harmed more by low- than high-achieving students. Burke and Sass (2008) include both students' and teachers' fixed effects in their analysis of peer effects, estimating peer effects from 0.014 to 0.068 in reading and 0.04 in math. Lefgren (2004) estimates peer effects in the tracking[1] environment of Chicago Public Schools and arrives at estimates of peer effects of 0.027 in reading and 0.032 in math. Finally, Vigdor and Nechyba (2007) estimate peer effects in North Carolina. Their estimation strategy relies on random assignment of pupils into classes and they control for school and teacher fixed effects. Without teachers' fixed effects, the peer effects estimates are from 0.05 to 0.07 in reading and from 0.06 to 0.08 in math. However; after including students' fixed effects, the peer effects estimates are -0.10 in reading and -0.12 in math.

The results of the estimation of peer effects show that they are relatively small. Importantly, Vigdor and Nechyba (2007) suggest that peer effects may be driven by the selection of teachers into classes. Moreover, some of the summarized studies suggest that peer effects may be non-linear and heterogeneous.

[1] Tracking refers to sorting of students according to their achievements to high- and low-achieving classes.

### 1.2.1 Peer Effects in Education with Selective Assignment

In our work, we examine peer effects in education in a selective environment; hence, we summarize related papers in detail. Two chosen papers illustrate, how peer effects can be estimated in the real-world environment with a selection of pupils into classes according to some unobserved variables. Neidell and Waldfogel (2010) propose an efficient method that relies on particular assumptions regarding peers' groups. On the other hand, Boucher et al. (2014) provide a reliable method for the unbiased estimation of peer effects; however, the proposed method requires a relatively rich data set with small peer groups or high variance of the size of peer groups.

Neidell and Waldfogel (2010) examine how pupils who attended preschool education can affect their peers who did not attend. The authors note that these effects are usually called spillover effects; however, they are a special case of the broad definition of peer effects provided by Sacerdote (2011). Specifically, Neidell and Waldfogel (2010) investigate whether having peers who attended preschool education can affect individuals' math and reading test scores and non-cognitive outcomes. The analysed data contains information about preschool enrolments and several background variables including race, socioeconomic status, and parental education among others. Furthermore, the characteristics of teachers and classes are also included. The main outcome variables are math and reading test scores in the first grade and non-cognitive abilities. Non-cognitive abilities are difficult to measure and the authors use teachers' evaluations of each child in their class and also parental assessments of children's non-cognitive abilities. Finally, a key variable for the identification of peer effects is the fall-K score, which was measured at the beginning of preschool education, and hence it is unaffected by peer effects.

Neidell and Waldfogel (2010) argue that the inclusion of school fixed effects or fall-K score can lead to unbiassed estimators of peer effects. They start with the estimation without school fixed effects and fall-K score and gradually include other explanatory variables in four steps. Their peer effects estimators are 3.30, 1.21, 0.99, and 0.92 with standard deviations of approximately 0.3. The 3.30 coefficient means that moving from a class with no peers attending preschool education to a class with all peers attending preschool education would increase individuals' math score by 3.3 points (the standard deviation of math score is 10). The peer effects estimators steadily decrease while including more explanatory variables, which points to omitted variables bias. However, after the inclusion of the fall-K score as an explanatory variable, the peer effects estimators for other explanatory variables gradually included are 0.55, 0.61, 0.70, and 0.66 with standard deviations of approximately 0.2. We see that the decreasing pattern in peer effects estimators is no longer present. A qualitatively similar pattern is observed while including schools' fixed effects and not including fall-K scores and also while including both schools' fixed effects and fall-K scores. Neidell and Waldfogel (2010) argue that both schools' fixed effects and fall-K scores separately or together can reliably reduce selection bias of peer effects estimates, and consequently estimated peer effects represent a causal relationship.

The peer effects estimators in reading show a similar pattern as the estimators in math. After including schools' fixed effects or fall-K scores or both, the peer effects estimates stabilize and they are also the same in magnitude. Regarding peer effects in non-cognitive abilities, Neidell and Waldfogel (2010) do not find a significant result in any behavioural category. The estimation provided by Neidell and Waldfogel (2010) is surprisingly simple and efficient. Selection bias is a serious concern in peer effects estimation in a natural environment, since educational outcomes may be explained by unobservable variables - for example, inborn ability encoded in genetic code. Consequently, including more explanatory variables usually does not eliminate selection bias. Our intuition for the success of Neidell and Waldfogel (2010) is the following. Regarding the effect of fall-K scores on peer effects estimators, these scores are also affected by unobservable variables and provide information about unobserved variables especially relevant for further test scores (particularly since fall-K scores also examines math and reading abilities). Regarding schools' fixed effects, the allocation of pupils into classes within schools is probably as good as random, hence schools' fixed effects can solve the problem of selection of pupils into schools comparing randomly allocated students. We use this intuition and argumentation in our work too even though Neidell and Waldfogel (2010) estimate the peer effects in a different environment (spillover effects of preschool education vs. typical peer effects in test scores).

Boucher et al. (2014) examine peer effects in test scores among $4^{th}$- and $5^{th}$-grade grammar school students in the province of Québec (Canada). Hence, the context and data from their study are closely related to our research on peer effects. They implement the econometrical method proposed by Lee (2007) and we discuss it in detail in the section on Measurement of Peer Effects. For now, we provide a description of the method used and the identification in words. Boucher et al. (2014) analyse test scores from French, Science, Mathematics and History and examines how the test scores of peers may affect an individual's test scores. The data include most of the pupils in the province of Québec and the selection is likely to be present. Boucher et al. (2014) define peer groups at school levels, and thus they assume that all pupils can interact within one school. In contrast to Neidell and Waldfogel (2010), Boucher et al. (2014) include school fixed effects and define the peer group at the school level. More generally, Boucher et al. (2014) estimate the peer effects model with fixed effects of groups. The identification of this model is counterintuitive since the school fixed effect is the mean test score of pupils in a school and peer effects work through the average effect of peers' test scores. The trick is in the exact definition of peer groups: a particular individual is excluded from her peer group. This means that school fixed effects are exactly the mean scores of schools and peer effects work through the mean score of schools with one pupil excluded. The identification of this model is rigorously proved and the only potential problem comes from the variance of the estimators. The difference between the mean of school and the mean of school with one pupil excluded may be relatively small. Especially in the case of schools with a large number of pupils, the peer effects estimators may be noisy and Boucher et al. (2014) analyse this problem using Monte-Carlo simulations.

Moving to the empirical results, Boucher et al. (2014) estimate their model using the pseudo conditional maximum likelihood (CML) and instrumental variable (IV) approach. The CML approach provides less noisy estimates in comparison with the IV approach. However, the results are qualitatively identical. The peer effects estimators for French, Science, Mathematics and History are 0.33, -0.23, 0.82, and 0.65. Depending on the definition of standard errors, the estimates are significantly positive for Mathematics and History or Mathematics only. The peer effects estimator for Mathematics is highest, which is consistent with the overview provided by Sacerdote (2011). To conclude, Boucher et al. (2014) provide a useful econometric method for the estimation of peer effects in the linear-in-means model, which provides estimators comparable with the previous research, and hence it is likely to work in real-world settings. However, the method requires a relatively large sample size with a high variance of group sizes.

# 2 Estimation of the Linear-in-Means Peer Effects Model

## 2.1 Definition of the Linear-in-Means Peer Effects Model

The linear-in-means peer effects model is the simplest model describing peer effects. We estimate this model in our work and follow the notation of Bramoullé et al. (2009) since they significantly contributed to our understanding of the identification of the peer effects model. Scalars are denoted with lower case letters, vectors with bold lower-case letters, and matrices with bold capital letters. Let us denote $P_i$ the set of peers of an individual $i$ and $n_i$ the number of peers of an individual $i$. A crucial assumption for the identification is that an individual $i$ is excluded from her peer group, e.g., $i \notin P_i$. Let us denote $y_i$ the outcome of an individual $i$, which can be, for example, a math test score. Educational research usually assumes that outcome $y_i$ can be explained by background characteristics $\boldsymbol{x_i}$ of an individual $i$. To model peer effects, we will assume that outcome $y_i$ can also be explained by the average outcomes and background characteristics of an individual's peers. Formulating the peer effects model in mathematical notation gives.

$$y_i = \alpha + \beta \frac{1}{n_i} \sum_{j \in P_i} y_j + \boldsymbol{x_i \gamma} + \frac{1}{n_i} \sum_{j \in P_i} \boldsymbol{x_j}\, \boldsymbol{\delta} + \varepsilon_i \qquad (1)$$

where $\boldsymbol{x_i}$ is $1 \times k$ row vector of the background characteristics of an individual $i$. Parameters $\alpha$ and $\boldsymbol{\gamma}$ describe how background characteristics can explain educational outcomes. Parameter $\beta$ describes how the outcome of an individual $i$ is affected by the outcomes of her peers and it is usually called the *endogeneous peer effect*. Parameter $\boldsymbol{\gamma}$ describes how the outcome of an individual $i$ is affected by the background characteristics of her peers and is usually called *exogenous peer effect*. This terminology was established by Manski (1993) and is consistently used in the peer effects literature. Finally, $\varepsilon_i$ is an error term with the identification condition $\mathbb{E}[\varepsilon_i | \boldsymbol{x}_i] = 0$. Later we argue that the identification condition implies that pupils cannot be sorted into groups according to some unobservable variable. The violation of this implication will lead to a selection bias.

In empirical practice, we usually work with several background characteristics. However; for simplicity and better intuition in some proofs, we work with the simpler version of the peer effects model

$$y_i = \alpha + \beta \frac{1}{n_i} \sum_{j \in P_i} y_j + \gamma x_i + \delta \frac{1}{n_i} \sum_{j \in P_i} x_j + \varepsilon_i \tag{2}$$

assuming a unique background characteristic.

Now, we express our peer effects model in matrix notation, which will be useful for the analysis of the identification and also in the computational part. Let us denote $\boldsymbol{J}_{n \times m}$ the matrix of ones that is handy for expressing sums. Note for example that the product of $\boldsymbol{J}_{1 \times n}$ and $\boldsymbol{y}_{n \times 1}$ is simply the sum of the elements of $\boldsymbol{y}$.

$$\boldsymbol{J}_{1 \times n} \boldsymbol{y}_{n \times 1} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{j=1}^{n} y_j$$

Now, let us move to a slightly more complicated case. Assume one class with $n$ pupils and all pupils interact with each other; hence the number of peers is $n - 1$. If we want to express the average of peers' outcomes for each individual, we can sum up all outcomes in the class, subtract the outcome of an individual $i$ and divide by $n - 1$.

$$\frac{1}{n-1} \sum_{j \in P_i} y_j = \frac{1}{n-1} \left( \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = \frac{1}{n-1} (\boldsymbol{J}_{n \times n} \boldsymbol{y}_{n \times 1} - \boldsymbol{y}_{n \times 1})$$

We can further manipulate with the above equation to get the network matrix that calculates the average peer score for each individual $i$.

$$\frac{1}{n-1} \sum_{j \in P_i} y_j = \frac{1}{n-1} (\boldsymbol{J}_{n \times n} \boldsymbol{y}_{n \times 1} - \boldsymbol{y}_{n \times 1}) = \frac{1}{n-1} (\boldsymbol{J}_{n \times n} \boldsymbol{y}_{n \times 1} - \boldsymbol{I}_n \boldsymbol{y}_{n \times 1})$$

$$= \frac{1}{n-1} (\boldsymbol{J}_{n \times n} - \boldsymbol{I}_n) \boldsymbol{y}_{n \times 1}$$

where $\boldsymbol{I}_n$ is an $n \times n$ identity matrix. Now we can define a network matrix as

$$\boldsymbol{G} \coloneqq \frac{1}{n-1}(\boldsymbol{J}_{n\times n} - \boldsymbol{I}_n) = \frac{1}{n-1}\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}$$

We can note that a network matrix $\boldsymbol{G}$ has zeros at diagonal and $\frac{1}{n-1}$ otherwise. Using the element-wise definition, we can write

$$(\boldsymbol{G})_{ij} \coloneqq \begin{cases} \frac{1}{n-1} \text{ if } j \neq i \\ 0 \text{ if } j = i \end{cases}$$

Finally, let us move to the most general case in which an individual $i$ has a general set of peers $P_i$ with $n_i$ peers. Inspired by the previous special case, the $i^{th}$ row of a network matrix $\boldsymbol{G}$ should have zeros in $j^{th}$ columns for pupils $j$ outside of peers group $P_i$. In $j^{th}$ columns for pupils $j$ inside of peers group $P_i$, the elements should be $\frac{1}{n_i}$. For example, if the first of 5 individuals have three peers – second, fourth, and fifth, the corresponding row for the first individual is

$$\begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \frac{1}{3}(y_2 + y_4 + y_5)$$

One can realize that the general definition of a network matrix $\boldsymbol{G}$ is

$$(\boldsymbol{G})_{ij} \coloneqq \begin{cases} \frac{1}{n_i} \text{ if } j \in P_i \\ 0 \text{ if } j \notin P_i \end{cases}$$

Indeed, if we write the multiplication of $\boldsymbol{G}$ and $\boldsymbol{y}$ elements by elements, we get

$$(\boldsymbol{G}\boldsymbol{y})_i = \sum_j (\boldsymbol{G})_{ij} y_j = \frac{1}{n_i} \sum_{j \in P_i} y_j$$

Now we are prepared to formulate the linear-in-means peer effects model in matrix notation. Let us denote $n$ the number of all pupils in the sample and write

$$\boldsymbol{y} = \alpha \boldsymbol{J}_{n\times 1} + \beta \boldsymbol{G}\boldsymbol{y} + \boldsymbol{X}\boldsymbol{\gamma} + \boldsymbol{G}\boldsymbol{X}\boldsymbol{\delta} + \boldsymbol{\varepsilon} \tag{3}$$

Note that our definition of $\boldsymbol{G}$ works also properly for the $n \times k$ matrix $\boldsymbol{X}$ of peers' background characteristics. One way to see this directly is to express matrix $\boldsymbol{X}$ as a vector of $n$ rows $\boldsymbol{x}_i$.

The simplified version of our model with one background characteristic is

$$\boldsymbol{y} = \alpha \boldsymbol{J}_{n\times 1} + \beta \boldsymbol{G}\boldsymbol{y} + \gamma \boldsymbol{x} + \delta \boldsymbol{G}\boldsymbol{x} + \boldsymbol{\varepsilon} \tag{4}$$

## 2.2 Reduced Form of the Linear-in-Means Peer Effects Model

Before the analysis of the identification of the linear-in-means peer effects model, we derive its reduced form, which helps us with the intuition. The following parts are relatively technical; hence, we use the simplified version of our model with one background characteristic [equation (4)]. Note that Bramoullé et al. (2009) describe only the simplified version of the peer effects model. First, we separate educational outcomes $\boldsymbol{y}$ to the left-hand side and the rest of the variables to the righthand side

$$(\boldsymbol{I}_n - \beta \boldsymbol{G})\boldsymbol{y} = \alpha \boldsymbol{J}_{n\times 1} + (\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} + \boldsymbol{\varepsilon}$$

Multiplying by $(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}$ from the left-hand side gives

$$\boldsymbol{y} = \alpha (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1} \boldsymbol{J}_{n\times 1} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}(\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}\boldsymbol{\varepsilon} \tag{5}$$

This reduced form can be further simplified using special properties of matrices $\boldsymbol{G}$ and $(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}$. Bramoullé et al. (2009) provide usually brief derivations and the steps may not be clear while reading for the first time. They also sometimes refer to different papers with a similar issue considering proofs. Hence, we decided to provide some derivations and proofs in detail, which can be skipped while reading our work for the first time.

Now, we assume that no-one is isolated, e.g., $P_i$ is a non-empty set for all $i$. We impose this assumption on the beginning of our derivation, since all proofs are simpler. However, it should be possible to realise this assumption if needed and one needs to check what will be different.

Usually, the following statements are slightly modified, for example, the $G$ is not row normalised; however, some rows can be zeros. Consequently, $\boldsymbol{G}^k \boldsymbol{J}_{n\times1} \neq \boldsymbol{J}_{n\times1}$ and the elements of the resulting vector are zeros and ones. Isolated individuals cause some eigenvalues to be zeros; hence, proposition 1.3 still holds. Finally, proposition 2.3 should also be modified.

**Proposition 1. The Properties of a Network Matrix $\boldsymbol{G}$**

Assume that no-one is isolated, e.g., $P_i$ is a non-empty set for all $i$.

1.1. $\boldsymbol{G}$ is row normalised.

1.2. $\boldsymbol{G}^k \boldsymbol{J}_{n\times1} = \boldsymbol{J}_{n\times1}$

1.3. The eigenvalues $\lambda$ of $\boldsymbol{G}$ are less than one in an absolute value, e.g., $|\lambda| \leq 1$.

1.4. If $\boldsymbol{G}$ is diagonalizable, $\lim_{k\to\infty} \boldsymbol{G}^k = \widetilde{\boldsymbol{G}}$ where $\widetilde{\boldsymbol{G}}$ is a finite matrix.

**The Proof of Proposition 1.**

1.1. We can use the elementwise definition of $\boldsymbol{G}$ matrix

$$(\boldsymbol{G})_{ij} := \begin{cases} \frac{1}{n_i} \text{ if } j \in P_i \\ 0 \text{ if } j \notin P_i \end{cases}$$

and express the sum of its elements in $i^{th}$ row

$$\sum_{j=1}^{n} (\boldsymbol{G})_{ij} = \sum_{j\in P_i} \frac{1}{n_i} = 1$$

since we are summing over $n_i$ elements.

1.2. Let us start with $k = 1$. The $i^{th}$ element of the product of $\boldsymbol{G}$ and $\boldsymbol{J}_{n\times1}$ is

$$(\boldsymbol{G}\boldsymbol{J}_{n\times1})_i = \sum_{j=1}^{n} (\boldsymbol{G})_{ij} (\boldsymbol{J}_{n\times1})_{j1} = \sum_{j=1}^{n} (\boldsymbol{G})_{ij}$$

Proposition 1.1 implies that the above sum is equal to one. Consequently, each row of the analysed product is one and we can write

$$\boldsymbol{GJ}_{n\times 1} = \boldsymbol{J}_{n\times 1}$$

Finally, iteratively applying the above formula we get

$$\boldsymbol{G}^k\boldsymbol{J}_{n\times 1} = \boldsymbol{G}^{k-1}\boldsymbol{G}\boldsymbol{J}_{n\times 1} = \boldsymbol{G}^{k-1}\boldsymbol{J}_{n\times 1} = \boldsymbol{G}^{k-2}\boldsymbol{J}_{n\times 1} = \dots = \boldsymbol{GJ}_{n\times 1} = \boldsymbol{J}_{n\times 1}$$

1.3. This statement could be directly proved by recalling the *Gershgorin Circle Theorem*[2]. However, the proof is not too complex and we can modify it to our special case of a matrix $\boldsymbol{G}$. Let $\lambda$ be an eigenvalue and $\boldsymbol{v}$ be an eigenvector of a matrix $\boldsymbol{G}$. Then by the definition of eigenvalues and eigenvectors, we can write

$$\boldsymbol{Gv} = \lambda\boldsymbol{v}$$

Now we write the $i^{th}$ row of the above equality

$$\sum_{j=1}^{n} (\boldsymbol{G})_{ij} v_j = \lambda v_i$$

Now, we choose one particular row so that $v_i$ is the greatest component of a vector $\boldsymbol{v}$ in the absolute value. Moreover, the eigenvectors are determined up to normalization, so let us impose that $v_i = 1$. Thus, we can write

$$|v_j| \leq 1 \;\forall\, j$$

The analysed equation becomes

$$\sum_{j=1}^{n} (\boldsymbol{G})_{ij} v_j = \lambda$$

Let us take an absolute value of this equation

$$|\lambda| = \left|\sum_{j=1}^{n} (\boldsymbol{G})_{ij} v_j\right|$$

and apply the triangular inequality

[2] *Gershgorin Circle Theorem*: https://en.wikipedia.org/wiki/Gershgorin_circle_theorem

$$|\lambda| = \left|\sum_{j=1}^{n} (\boldsymbol{G})_{ij} v_j\right| \leq \sum_{j=1}^{n} (\boldsymbol{G})_{ij} |v_j|$$

Note that the elements $(\boldsymbol{G})_{ij}$ are positive, and hence their absolute value is the same number. The inequality $|v_j| \leq 1 \; \forall \, j$ implies that

$$|\lambda| \leq \sum_{j=1}^{n} (\boldsymbol{G})_{ij} = 1$$

where the last equality comes from Proposition 1.1.

1.4. We start our proof assuming that $\boldsymbol{G}$ is diagonalizable. Let us denote by $\boldsymbol{D}$ the diagonal matrix so that

$$\boldsymbol{G} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-\mathbf{1}}$$

Then for the $k^{th}$ power of matrix $\boldsymbol{G}$ holds

$$\boldsymbol{G}^k = \boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-\mathbf{1}}$$

The diagonal elements of a matrix $\boldsymbol{D}$ are the eigenvalues $\lambda$ of a matrix $\boldsymbol{G}$, and hence the examined limit is

$$\lim_{k\to\infty} \boldsymbol{G}^k = \boldsymbol{P}\left(\lim_{k\to\infty} \boldsymbol{D}^k\right)\boldsymbol{P}^{-1} = \boldsymbol{P}\begin{pmatrix} \lim\limits_{k\to\infty} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lim\limits_{k\to\infty} \lambda_2^k & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lim\limits_{k\to\infty} \lambda_n^k \end{pmatrix}\boldsymbol{P}^{-1}$$

However, proposition 1.3 implies that also $\lim\limits_{k\to\infty} \lambda_i^k$ is less than one in absolute value, and hence the matrix $\lim\limits_{k\to\infty} \boldsymbol{D}^k$ is finite. Noting that $\boldsymbol{P}$, $\lim\limits_{k\to\infty} \boldsymbol{D}^k$, and $\boldsymbol{P}^{-1}$ are finite, their product $\widetilde{\boldsymbol{G}}$ is also finite.

The properties of a matrix $G$ will be crucial for the analysis of the properties of matrix $\boldsymbol{I}_n - \beta\boldsymbol{G}$**.** Regarding proposition 1.4, one may ask if the matrix $\boldsymbol{G}$ is diagonalizable in general. We do not know; however, we will prove an important property of $\boldsymbol{I}_n - \beta\boldsymbol{G}$ even in the case in which $\boldsymbol{G}$ is not diagonalizable.

**Proposition 2. The Properties of a Matrix $\boldsymbol{I}_n - \beta\boldsymbol{G}$**

Assume that $|\beta| < 1$, then

2.1. the eigenvalues of a matrix $\boldsymbol{I}_n - \beta\boldsymbol{G}$ are positive, a matrix has full rank and is invertible.

2.2. the inverse of a matrix $\boldsymbol{I}_n - \beta\boldsymbol{G}$ can be expressed as an infinite sum

$$(\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1} = \sum_{k=0}^{\infty} \beta^k \boldsymbol{G}^k$$

2.3. $(\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}\boldsymbol{J}_{n\times 1} = \frac{1}{1-\beta}\boldsymbol{J}_{n\times 1}$

**The Proof of Proposition 2.**

2.1. Let $\lambda$ be an eigenvalue and $\boldsymbol{v}$ be an eigenvector of a matrix $\boldsymbol{G}$. First, we show that $\boldsymbol{v}$ is also an eigenvector of a matrix $\boldsymbol{I}_n - \beta\boldsymbol{G}$ with eigenvalue $1 - \beta\lambda$.

$$(\boldsymbol{I}_n - \beta\boldsymbol{G})\boldsymbol{v} = \boldsymbol{I}_n\boldsymbol{v} - \beta\boldsymbol{G}\boldsymbol{v} = \boldsymbol{v} - \beta\lambda\boldsymbol{v} = (1 - \beta\lambda)\boldsymbol{v}$$

where in the second equality, we are using $\boldsymbol{G}\boldsymbol{v} = \lambda\boldsymbol{v}$, which is the property of eigenvalues and eigenvectors of matrix $\boldsymbol{G}$**.** Using $|\lambda| \leq 1$ (proposition 1.3) and $|\beta| < 1$ implies $\lambda\beta < 1$. Now we can multiply this inequality by a negative one

$$-\lambda\beta > -1$$

and add one

$$1 - \lambda\beta > 0$$

Thus, we have proved that eigenvalues of a matrix $\boldsymbol{I}_n - \beta\boldsymbol{G}$ are positive and consequently, matrix $\boldsymbol{I}_n - \beta\boldsymbol{G}$ has full rank and is invertible.

2.2. The series expansion is analogical to the formula for the sum of a geometric sequence. The formula for the inverse of a matrix $\boldsymbol{A}$ can be verified by the relation

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}_n$$

Let us write the identity for an inverse matrix in our case. First, we should pay attention to the proper definition of an infinite series.

$$(\boldsymbol{I}_n - \beta\boldsymbol{G})\sum_{k=0}^{\infty}\beta^k\boldsymbol{G}^k = (\boldsymbol{I}_n - \beta\boldsymbol{G})\lim_{l\to\infty}\sum_{k=0}^{l}\beta^k\boldsymbol{G}^k = \lim_{l\to\infty}\left((\boldsymbol{I}_n - \beta\boldsymbol{G})\sum_{k=0}^{l}\beta^k\boldsymbol{G}^k\right)$$

Now we multiply the parenthesis and the series.

$$\lim_{l\to\infty}\left((\boldsymbol{I}_n - \beta\boldsymbol{G})\sum_{k=0}^{l}\beta^k\boldsymbol{G}^k\right) = \lim_{l\to\infty}\left(\sum_{k=0}^{l}\beta^k\boldsymbol{G}^k - \sum_{k=0}^{l}\beta^{k+1}\boldsymbol{G}^{k+1}\right) =$$
$$= \boldsymbol{I}_n - \lim_{l\to\infty}(\beta^{l+1}\boldsymbol{G}^{l+1})$$

We can see that our limit shares the properties of the telescoping series, that were used in the second equality.

Now, we split our proof into two cases. First, if $\boldsymbol{G}$ is diagonalizable, then proposition 1.4 implies that we can split the limit and obtain

$$\lim_{l\to\infty}(\beta^{l+1}\boldsymbol{G}^{k+1}) = \lim_{l\to\infty}(\beta^{l+1})\lim_{l\to\infty}(\boldsymbol{G}^{l+1}) = \lim_{l\to\infty}(\beta^{l+1})\,\widetilde{\boldsymbol{G}} = 0$$

since $\widetilde{\boldsymbol{G}}$ is finite and $|\beta| < 1$.

Second, if $\boldsymbol{G}$ is not diagonalizable, then $\widetilde{\boldsymbol{G}}$ is not finite (as we show below) and we cannot conduct so simple a proof to show that $\lim_{l\to\infty}(\beta^{l+1}\boldsymbol{G}^{l+1}) = 0$. Nevertheless, if $\boldsymbol{G}$ is not diagonalizable, there still exists the Jordan normal form $\boldsymbol{N}$ of a matrix $\boldsymbol{G}$ and we can use it. The following holds

$$\boldsymbol{G} = \boldsymbol{P}\begin{pmatrix}\boldsymbol{N}_1 & 0 & \cdots & 0\\ 0 & \boldsymbol{N}_2 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \boldsymbol{N}_m\end{pmatrix}\boldsymbol{P}^{-1}$$

where $\boldsymbol{N}_i$ are Jordan blocks and $m$ is their number ($m < n$). The Jordan block has eigenvalues on its diagonal, ones on the superdiagonal, and zeros otherwise.

$$\boldsymbol{N}_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

Now we need to express the powers of Jordan blocks. If one is familiar with the powers of Jordan normal form, she can move to the last rows of the following proof.

Now we can decompose the Jordan block to the sum of two matrices and use the binomial theorem.

$$\boldsymbol{N}_i^k = \left( \lambda_i \boldsymbol{I} + \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \right)^k$$

$$\boldsymbol{N}_i^k = \sum_{l=0}^{k} \binom{k}{l} \lambda_i^{k-l} \boldsymbol{I}^{k-l} \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^l$$

$$\boldsymbol{N}_i^k = \sum_{l=0}^{k} \binom{k}{l} \lambda_i^{k-l} \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^l$$

Now the expression seems to be even more complex. However, the powers of the matrix with superdiagonal of ones and zeros otherwise exhibit a special pattern. Each power shifts the superdiagonal one step to the right (one can see it multiplying these matrices directly). As a result, we can write

$$\boldsymbol{N}_i^k = \lambda_i^k \boldsymbol{I} + \binom{k}{1} \lambda_i^{k-1} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} + \binom{k}{2} \lambda_i^{k-2} \begin{pmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & \ddots & \\ & & 0 & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix} + \cdots$$

which can be summarised in one matrix

$$
\boldsymbol{N}_i^k = \begin{pmatrix} \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \binom{k}{2}\lambda_i^{k-2} & \cdots & \\ & \lambda_i^k & \binom{k}{1}\lambda_i^{k-1} & \ddots & \\ & & \lambda_i^k & \ddots & \binom{k}{2}\lambda_i^{k-2} \\ & & & \ddots & \binom{k}{1}\lambda_i^{k-1} \\ & & & & \lambda_i^k \end{pmatrix}
$$

Now, we can see that the limit $\lim_{k\to\infty} \boldsymbol{G}^k$ can be possibly infinity. Indeed, $\lambda_i$ can be possibly 1, then for example $\binom{k}{1} = k$ and it approaches infinity as $k$ goes to infinity. However, let us show that the desired limit is zero, e.g., $\lim_{l\to\infty}(\beta^k \boldsymbol{G}^k) = 0$.

$$
\lim_{k\to\infty}(\beta^k \boldsymbol{G}^k) = \lim_{k\to\infty}\left(\beta^k \boldsymbol{P}\begin{pmatrix} \boldsymbol{N}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{N}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \boldsymbol{N}_m \end{pmatrix}^k \boldsymbol{P}^{-1}\right) =
$$

$$
= \boldsymbol{P}\lim_{k\to\infty}\begin{pmatrix} \beta^k \boldsymbol{N}_1^k & 0 & \cdots & 0 \\ 0 & \beta^k \boldsymbol{N}_2^k & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \beta^k \boldsymbol{N}_m^k \end{pmatrix}\boldsymbol{P}^{-1}
$$

Note that $\beta^k$ is scalar and can be moved into the Jordan normal form. Moreover, matrices $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$ are finite, and hence they can be placed outside of the limit. Now we can examine the limit $\lim_{k\to\infty}(\beta^k \boldsymbol{N}_i^k)$ of an arbitrary Jordan block. Let us examine an arbitrary element above the diagonal of matrix $\lim_{k\to\infty}(\beta^k \boldsymbol{N}_i^k)$.

$$
\lim_{k\to\infty}\left(\beta^k \binom{k}{j} |\lambda_i^{k-j}|\right) \leq \lim_{k\to\infty}\left(\beta^k \binom{k}{j}\right)
$$

since $|\lambda| \leq 1$ (proposition 1.3).

$$
\lim_{k\to\infty}\left(\beta^k \binom{k}{j}\right) = \lim_{k\to\infty}\left(\beta^k \frac{k!}{j!\,(k-j)!}\right) = \lim_{k\to\infty}\left(\beta^k \frac{k(k-1)\ldots(k-j+1)}{j!}\right)
$$

Note that in the nominator, we have $j$ polynomials, hence multiplying them gives the polynomial of $k$ with the highest power $k^j$. Finally, using, e.g., L'Hospital's rule, gives that

$$\lim_{k\to\infty}\left(\beta^k\frac{k(k-1)\dots(k-j+1)}{j!}\right)=\lim_{k\to\infty}\left(\beta^k k^j\right)=0$$

since $|\beta|<1$. One may ask if this complicated proof was necessary; hence, we will provide an example of a network matrix that is not diagonalizable.

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$$

One may verify that this matrix is not diagonalizable executing the following code in Wolfram Alpha.

*diagonalize {{0,1/2,1/2}, {1/2,0,1/2}, {1,0,0}}*

which gives the corresponding Jordan normal form

$$\begin{pmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.3. The previous proof was quite technical; however, as a reward, its result will be efficiently used in this proof. Let us express the $(\boldsymbol{I}_n-\beta\boldsymbol{G})^{-1}$ as the infinite series.

$$(\boldsymbol{I}_n-\beta\boldsymbol{G})^{-1}\boldsymbol{J}_{n\times1}=\sum_{k=0}^{\infty}\beta^k\boldsymbol{G}^k\boldsymbol{J}_{n\times1}$$

Recall that $\boldsymbol{G}^k\boldsymbol{J}_{n\times1}=\boldsymbol{J}_{n\times1}$ (proposition 1.2).

$$(\boldsymbol{I}_n-\beta\boldsymbol{G})^{-1}\boldsymbol{J}_{n\times1}=\sum_{k=0}^{\infty}\beta^k\boldsymbol{J}_{n\times1}$$

Finally, note that $\sum_{k=0}^{\infty} \beta^k = \frac{1}{1-\beta}$ since $|\beta| < 1$. (The formula for the sum of geometrical sequence.)

$$(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1} \boldsymbol{J}_{n\times 1} = \frac{1}{1-\beta} \boldsymbol{J}_{n\times 1}$$

Propositions 2 enable us to simplify the reduced form of the peer effects model [equation (5)].

$$\boldsymbol{y} = \alpha(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1} \boldsymbol{J}_{n\times 1} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}(\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1} \boldsymbol{\varepsilon}$$

Specifically, let us apply proposition 2.3 in the first term in the right-hand side.

$$\boldsymbol{y} = \frac{\alpha}{1-\beta} \boldsymbol{J}_{n\times 1} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}(\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1} \boldsymbol{\varepsilon}$$

and express $(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}$ in the second term in the right-hand side as an infinite sum.

$$(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}(\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} = \sum_{k=0}^{\infty} \beta^k \boldsymbol{G}^k (\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} = \sum_{k=0}^{\infty} (\gamma \beta^k \boldsymbol{G}^k + \delta \beta^k \boldsymbol{G}^{k+1})\boldsymbol{x}$$

Now, let us separate the explanatory part of individuals' background characteristics.

$$(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}(\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} = \gamma \boldsymbol{x} + \sum_{k=0}^{\infty} (\gamma \beta^{k+1} \boldsymbol{G}^{k+1} + \delta \beta^k \boldsymbol{G}^{k+1})\boldsymbol{x}$$

Note that we had to split the sum and shift the summing indices one step forward in the first sum. Now, we can simplify the obtained sum.

$$(\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}(\gamma \boldsymbol{I}_n + \delta \boldsymbol{G})\boldsymbol{x} = \gamma \boldsymbol{x} + (\gamma\beta + \delta) \sum_{k=0}^{\infty} \beta^k \boldsymbol{G}^{k+1} \boldsymbol{x}$$

Taking all modifications together, we can obtain the simplified reduced form peer effects model.

$$\boldsymbol{y} = \frac{\alpha}{1-\beta} \boldsymbol{J}_{n\times 1} + \gamma \boldsymbol{x} + (\gamma\beta + \delta) \sum_{k=0}^{\infty} \beta^k \boldsymbol{G}^{k+1} \boldsymbol{x} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1} \boldsymbol{\varepsilon} \qquad (6)$$

## 2.3 Identification of the Linear-in-Means Peer Effects Models – General Conditions

For the discussion of the identification of the linear-in-means peer effects model, we maintain the simplification that we have only one background characteristic and $\gamma$ is a scalar. We discuss the matrix form of the model

$$\boldsymbol{y} = \alpha \boldsymbol{J}_{n\times 1} + \beta \boldsymbol{G}\boldsymbol{y} + \gamma \boldsymbol{x} + \delta \boldsymbol{G}\boldsymbol{x} + \boldsymbol{\varepsilon} \tag{4}$$

and its reduced form.

$$\boldsymbol{y} = \frac{\alpha}{1-\beta}\boldsymbol{J}_{n\times 1} + \gamma \boldsymbol{x} + (\gamma\beta + \delta)\sum_{k=0}^{\infty} \beta^k \boldsymbol{G}^{k+1}\boldsymbol{x} + (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1}\boldsymbol{\varepsilon} \tag{6}$$

First, we discuss the nature of a matrix $\boldsymbol{G}$. We can assume that $\boldsymbol{G}$ is a parameter of the model and it is not affected by random variables in the model. This assumption may hold if we consider classes as peer groups and the number of pupils in classes is given by some institution. Another possibility is that $\boldsymbol{G}$ is stochastic and naturally enters the identification conditions of our model.

Second, in this section, we assume that the conditional mean of errors given regressors is zero

$$\mathbb{E}[\varepsilon_i | \boldsymbol{x}_i] = 0$$

which is enough for the case in which $\boldsymbol{G}$ is fixed. If $\boldsymbol{G}$ is stochastic, then the conditional mean of errors should be zero given the realization of $\boldsymbol{G}$.

$$\mathbb{E}[\varepsilon_i | \boldsymbol{x}_i, \boldsymbol{G}_i] = 0$$

Third, equation (6) reveals that peer effects encoded in vectors $\boldsymbol{G}^{k+1}\boldsymbol{x}$ cannot be identified if $\gamma\beta + \delta = 0$. Indeed, if $\gamma\beta + \delta = 0$, then peer effects vanish from equation (6) and the corresponding information is lost. Fourth, considering identification in simple OLS, intuitively, several vectors $\boldsymbol{G}^{k+1}\boldsymbol{x}$ in the infinite sum should be linearly independent. Since we have constant $\alpha$ and three parameters $\beta, \gamma$, and $\delta$ related to regressors $\boldsymbol{x}$, we can expect that the necessary condition is that $\boldsymbol{I}_n$, $\boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly independent [the condition for $\boldsymbol{I}_n$ comes intuitively from the term $\gamma \boldsymbol{x}$ in equation (6)]. However, we have to prove that this condition is also sufficient for the identification in our peer effects model.

**Proposition 3. Identification of the Linear-in-Means Peer Effects Model**

Assume that $\gamma\beta + \delta \neq 0$ and $\mathbb{E}[\varepsilon_i | \boldsymbol{x}_i, \boldsymbol{G}_i] = 0$. If the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly independent, then the linear-in-means peer effects model is identified, e. g., parameters $\alpha, \beta, \gamma$, and $\delta$ are identified.

**The Proof of Proposition 3.**

Consider two sets of parameters $\{\alpha, \beta, \gamma, \delta\}$ and $\{\alpha', \beta', \gamma', \delta'\}$ leading to the same educational outcomes $\boldsymbol{y}$ in equation (5), which has the form

$$\boldsymbol{y} = \alpha(\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}\boldsymbol{J}_{n\times1} + (\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}(\gamma\boldsymbol{I}_n + \delta\boldsymbol{G})\boldsymbol{x} + (\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}\boldsymbol{\varepsilon}$$

This assumption implies the following two conditions

$$\alpha(\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}\boldsymbol{J}_{n\times1} = \alpha'(\boldsymbol{I}_n - \beta'\boldsymbol{G})^{-1}\boldsymbol{J}_{n\times1}$$

$$(\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}(\gamma\boldsymbol{I}_n + \delta\boldsymbol{G})\boldsymbol{x} = (\boldsymbol{I}_n - \beta'\boldsymbol{G})^{-1}(\gamma'\boldsymbol{I}_n + \delta'\boldsymbol{G})\boldsymbol{x}$$

Let us start with the second condition. Since $\boldsymbol{x}$ may be an arbitrary vector, the matrices in front of $\boldsymbol{x}$ have to be identical. (In the case of the first condition one can find the different matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$ that satisfy $\boldsymbol{A}\boldsymbol{J}_{n\times1} = \boldsymbol{A}'\boldsymbol{J}_{n\times1}$. Proposition 1.2 implies that any network matrix satisfies this condition.)

$$(\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}(\gamma\boldsymbol{I}_n + \delta\boldsymbol{G}) = (\boldsymbol{I}_n - \beta'\boldsymbol{G})^{-1}(\gamma'\boldsymbol{I}_n + \delta'\boldsymbol{G})$$

Now, let us multiply the above equation by $(\boldsymbol{I}_n - \beta'\boldsymbol{G})(\boldsymbol{I}_n - \beta\boldsymbol{G})$ from the left-hand side

$$(\boldsymbol{I}_n - \beta'\boldsymbol{G})(\gamma\boldsymbol{I}_n + \delta\boldsymbol{G}) = (\boldsymbol{I}_n - \beta'\boldsymbol{G})(\boldsymbol{I}_n - \beta\boldsymbol{G})(\boldsymbol{I}_n - \beta'\boldsymbol{G})^{-1}(\gamma'\boldsymbol{I}_n + \delta'\boldsymbol{G})$$

Since $\boldsymbol{I}_n$ and $\boldsymbol{G}$ are commutative, one may easily show that the product $(\boldsymbol{I}_n - \beta'\boldsymbol{G})(\boldsymbol{I}_n - \beta\boldsymbol{G})$ is also commutative. Therefore, we can cancel analogically matrices on the right-hand side

$$(\boldsymbol{I}_n - \beta'\boldsymbol{G})(\gamma\boldsymbol{I}_n + \delta\boldsymbol{G}) = (\boldsymbol{I}_n - \beta\boldsymbol{G})(\gamma'\boldsymbol{I}_n + \delta'\boldsymbol{G})$$

Now we can multiply parenthesis and obtain

$$\gamma \boldsymbol{I}_n + \delta \boldsymbol{G} - \beta' \gamma \boldsymbol{G} - \beta' \delta \boldsymbol{G}^2 = \gamma' \boldsymbol{I}_n + \delta' \boldsymbol{G} - \beta \gamma' \boldsymbol{G} - \beta \delta' \boldsymbol{G}^2$$

$$(\gamma - \gamma') \boldsymbol{I}_n + (\delta - \delta' + \beta \gamma' - \beta' \gamma) \boldsymbol{G} + (\beta \delta' - \beta' \delta) \boldsymbol{G}^2 = 0$$

If the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly independent, then the above equation implies the following three conditions

$$\gamma - \gamma' = 0$$

$$\delta - \delta' + \beta \gamma' - \beta' \gamma = 0$$

$$\beta \delta' - \beta' \delta = 0$$

First, we can see that $\gamma' = \gamma$. The remaining system of equations can be solved expressing $\delta'$ from the last condition

$$\delta' = \frac{\delta \beta'}{\beta}$$

and plugging into the second.

$$\delta - \frac{\delta \beta'}{\beta} + \beta \gamma - \beta' \gamma = 0$$

$$\delta + \beta \gamma = \beta' \left( \frac{\delta}{\beta} + \gamma \right)$$

$$\beta' = \frac{\delta + \beta \gamma}{\frac{\delta}{\beta} + \gamma}$$

$$\beta' = \beta \frac{\delta + \beta \gamma}{\delta + \beta \gamma} = \beta$$

Now we can express also $\delta'$

$$\delta' = \frac{\delta}{\beta} \beta = \delta$$

Finally, recalling our very first condition

$$\alpha (\boldsymbol{I}_n - \beta \boldsymbol{G})^{-1} \boldsymbol{J}_{n \times 1} = \alpha' (\boldsymbol{I}_n - \beta' \boldsymbol{G})^{-1} \boldsymbol{J}_{n \times 1}$$

we can use $\beta = \beta'$ and obtain that

$$\alpha' = \alpha$$

since $\alpha$ and $\alpha'$ are the factors of two identical vectors. Now we can see that all parameters in our model are uniquely identified. However, we should take care of some "degenerated" cases. In our derivation, we can see two conditions, $\beta \neq 0$ and $\delta + \beta\gamma \neq 0$. If we were to start expressing $\beta'$ instead of $\delta'$, we would get the condition $\delta \neq 0$. The condition $\delta + \beta\gamma \neq 0$ was initially assumed; hence, let us move to the conditions $\beta \neq 0$ and $\delta \neq 0$.

Let us examine the initial condition once again noting that $\gamma' = \gamma$. The conditions are

$$\delta - \delta' + \beta\gamma - \beta'\gamma = 0$$

$$\beta\delta' - \beta'\delta = 0$$

and let us analyse them in all 3 cases.

If $\beta = 0$ and $\delta \neq 0$, then the condition $\beta\delta' - \beta'\delta = 0$ implies that $\beta' = 0$ and consequently the condition $\delta - \delta' + \beta\gamma - \beta'\gamma = 0$ implies that $\delta = \delta'$. Hence, the model is identified.

If $\beta \neq 0$ and $\delta = 0$, then the condition $\beta\delta' - \beta'\delta = 0$ implies that $\delta' = 0$ and consequently the condition $\delta - \delta' + \beta\gamma - \beta'\gamma = 0$ implies that $\beta = \beta'$, since the condition $\delta + \beta\gamma \neq 0$ implies that $\gamma \neq 0$ in this case. Hence, the model is identified.

If $\beta = 0$ and $\delta = 0$, then the condition $\delta + \beta\gamma \neq 0$ is violated and we do not need to consider this case.

Now, let us think more about the condition on matrix $\boldsymbol{G}$. It is intuitive that the vector $\boldsymbol{Gx}$ of peers' background characteristics play an important role in the identification of peer effects. However, the vector $\boldsymbol{G}^2\boldsymbol{x}$ also has an interesting interpretation. Indeed, let us write.

$$\boldsymbol{G}^2\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{Gx})$$

Hence, we are calculating the effects of peers' peers. Alternatively, we can say that we are measuring how the peers of our peers can influence us through the effect on our peers. The condition that $\boldsymbol{Gx}$ and $\boldsymbol{G}^2\boldsymbol{x}$ have to be linearly independent, for example, means that the peers of our peers who are not directly our peers can enable the estimation.

On the other hand, however, there is one mathematical trick that enables the estimation in the case of school classes, where the peers of our peers are also our peers. The identification is then possible even though it is relatively weak. We address this in the next section.

## 2.4 Identification of the Linear-in-Means Peer Effects Model – Specific Cases and the Reflection Problem

In this section, we want to show that the identification of the peer effects model is possible using group size variation proposed by Lee (2007). Specifically, if we have at least two types of classes with different sizes, then the linear-in-means peer effects model can be identified. However, it contradicts the previous results of Manski (1993) and Moffitt (2001). Hence, we also show why the previous studies did not find the identification. Moreover, the examples of non-identification underline how the identification conditions are important. Hence, let us start with the identification proposed by Lee (2007).

**Proposition 4. The Identification of Peer Effects Proposed by Lee (2007).**

Let us assume that pupils interact within their classes only and there are at least two types of classes with sizes $m_1$ and $m_2$. If $\gamma\beta + \delta \neq 0$ then peer effects are identified.

**The Proof of Proposition 4**.

Since we assume that $\gamma\beta + \delta \neq 0$, we can recall proposition 3 and we need to show that the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly independent. Without loss of generality, we consider two classes only. Let us recall the definition of matrix $\boldsymbol{G}$

$$(\boldsymbol{G})_{ij} := \begin{cases} \dfrac{1}{n_i} \text{ if } j \in P_i \\ 0 \text{ if } j \notin P_i \end{cases}$$

where $n_i$ is the number of the peers of individual $i$. Note that individual $i$ is excluded from her peers' group, and hence $n_i$ is the class size of an individual $i$ minus one. (Note that there is particular inconvenience in the notation since for an individual 2 holds that $n_2 = m_1 - 1$).

One can write the matrix $\boldsymbol{G}$ for a simple particular case and then it is clear that the matrix $\boldsymbol{G}$ is block diagonal

$$\boldsymbol{G} = \begin{pmatrix} \boldsymbol{G}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_2 \end{pmatrix}$$

with blocks $\boldsymbol{G}_i$ that have zeros on diagonals and $\frac{1}{m_i-1}$ otherwise.

$$\boldsymbol{G}_i = \begin{pmatrix} 0 & \frac{1}{m_i-1} & \cdots & \frac{1}{m_i-1} \\ \frac{1}{m_i-1} & 0 & & \vdots \\ \vdots & & \ddots & \frac{1}{m_i-1} \\ \frac{1}{m_i-1} & \cdots & \frac{1}{m_i-1} & 0 \end{pmatrix}$$

It will be useful to write matrix $\boldsymbol{G}_i$ as

$$\boldsymbol{G}_i = \frac{1}{m_i-1}\left(\boldsymbol{J}_{m_i \times m_i} - \boldsymbol{I}_{m_i}\right)$$

Since we want to compare $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ we need to express $\boldsymbol{G}^2$, which is

$$\boldsymbol{G}^2 = \begin{pmatrix} \boldsymbol{G}_1^2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_2^2 \end{pmatrix}$$

and hence, we need to calculate $\boldsymbol{G}_i^2$

$$\boldsymbol{G}_i^2 = \left(\frac{1}{m_i-1}\right)^2 \left(\boldsymbol{J}_{m_i \times m_i} - \boldsymbol{I}_{m_i}\right)^2 = \left(\frac{1}{m_i-1}\right)^2 \left(\boldsymbol{J}_{m_i \times m_i}^2 - 2\boldsymbol{J}_{m_i \times m_i} + \boldsymbol{I}_{m_i}\right)$$

since $\boldsymbol{I}_{m_i}$ commutates with any matrix. One can think that the product of the row and column of ones is the dimension of both entities, and hence

$$\boldsymbol{J}_{m_i \times m_i}^2 = m_i \boldsymbol{J}_{m_i \times m_i}$$

consequently

$$\boldsymbol{G}_i^2 = \left(\frac{1}{m_i-1}\right)^2 \left((m_i-2)\boldsymbol{J}_{m_i \times m_i} + \boldsymbol{I}_{m_i}\right) = \frac{m_i-2}{(m_i-1)^2}\boldsymbol{J}_{m_i \times m_i} + \frac{1}{(m_i-1)^2}\boldsymbol{I}_{m_i}$$

Now we can see that $\boldsymbol{G}_i^2$ is linearly dependent on $\boldsymbol{G}_i$ and $\boldsymbol{I}_{m_i}$ since all are the linear combinations of the matrices $\boldsymbol{J}_{m_i \times m_i}$ and $\boldsymbol{I}_{m_i}$. Despite that, let us examine the whole matrix $\boldsymbol{G}$. It is useful to express the matrix $\boldsymbol{G}$ using the matrices $\boldsymbol{J}_{m_i \times m_i}$ and $\boldsymbol{I}_{m_i}$.

$$\boldsymbol{G} = \begin{pmatrix} \frac{1}{m_1 - 1}\left(\boldsymbol{J}_{m_1 \times m_1} - \boldsymbol{I}_{m_1}\right) & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{m_2 - 1}\left(\boldsymbol{J}_{m_2 \times m_2} - \boldsymbol{I}_{m_2}\right) \end{pmatrix}$$

$$\boldsymbol{G} = \begin{pmatrix} \frac{1}{m_1 - 1}\boldsymbol{J}_{m_1 \times m_1} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{m_2 - 1}\boldsymbol{J}_{m_2 \times m_2} \end{pmatrix} - \begin{pmatrix} \frac{1}{m_1 - 1}\boldsymbol{I}_{m_1} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{m_2 - 1}\boldsymbol{I}_{m_2} \end{pmatrix}$$

and similarly for the matrix $\boldsymbol{G}^2$.

$$\boldsymbol{G}^2 = \begin{pmatrix} \frac{m_1 - 2}{(m_1 - 1)^2}\boldsymbol{J}_{m_1 \times m_1} + \frac{1}{(m_1 - 1)^2}\boldsymbol{I}_{m_1} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{m_2 - 2}{(m_2 - 1)^2}\boldsymbol{J}_{m_2 \times m_2} + \frac{1}{(m_2 - 1)^2}\boldsymbol{I}_{m_2} \end{pmatrix}$$

$$\boldsymbol{G}^2 = \begin{pmatrix} \frac{m_1 - 2}{(m_1 - 1)^2}\boldsymbol{J}_{m_1 \times m_1} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{m_2 - 2}{(m_2 - 1)^2}\boldsymbol{J}_{m_2 \times m_2} \end{pmatrix} + \begin{pmatrix} \frac{1}{(m_1 - 1)^2}\boldsymbol{I}_{m_1} & \boldsymbol{0} \\ \boldsymbol{0} & \frac{1}{(m_2 - 1)^2}\boldsymbol{I}_{m_2} \end{pmatrix}$$

Finally, the examination of the linear dependence of the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ can be formulated as the question. Is there some positive $\lambda_i$ satisfying the following condition?

$$\lambda_1 \boldsymbol{I}_n + \lambda_2 \boldsymbol{G} + \lambda_3 \boldsymbol{G}^2 = 0$$

If there is some positive $\lambda_i$ satisfying the above condition, then the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly dependent. Otherwise, the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly independent.

One can imagine that plugging in for all three matrices would lead to a confusing expression. Hence, we will rewrite it for both blocks separately.

$$\lambda_1 \boldsymbol{I}_{m_i} + \lambda_2 \left(\frac{1}{m_i - 1}\boldsymbol{J}_{m_i \times m_i} - \frac{1}{m_i - 1}\boldsymbol{I}_{m_i}\right) + \lambda_3 \left(\frac{m_i - 2}{(m_i - 1)^2}\boldsymbol{J}_{m_i \times m_i} + \frac{1}{(m_i - 1)^2}\boldsymbol{I}_{m_i}\right) = 0$$

which can be expressed as

$$\boldsymbol{I}_{m_i}\left(\lambda_1 - \frac{1}{m_i - 1}\lambda_2 + \frac{1}{(m_i - 1)^2}\lambda_3\right) + \boldsymbol{J}_{m_i \times m_i}\left(\frac{1}{m_i - 1}\lambda_2 + \frac{m_i - 2}{(m_i - 1)^2}\lambda_3\right) = 0$$

Since $\boldsymbol{I}_{m_i}$ and $\boldsymbol{J}_{m_i \times m_i}$ are linearly independent the above condition implies the following

$$\lambda_1 - \frac{1}{m_i - 1}\lambda_2 + \frac{1}{(m_i - 1)^2}\lambda_3 = 0$$

$$\frac{1}{m_i - 1}\lambda_2 + \frac{m_i - 2}{(m_i - 1)^2}\lambda_3 = 0$$

Note that since $i \in \{1,2\}$, we have the four conditions for the three unknown parameters $\lambda_1, \lambda_2$, and $\lambda_3$. First, we will slightly modify the conditions multiplying both by $(m_i - 1)^2$

$$(m_i - 1)^2\lambda_1 - (m_i - 1)\lambda_2 + \lambda_3 = 0$$

$$(m_i - 1)\lambda_2 + (m_i - 2)\lambda_3 = 0$$

adding the second condition to the first

$$(m_i - 1)^2\lambda_1 + (m_i - 1)\lambda_3 = 0$$

$$(m_i - 1)\lambda_2 + (m_i - 2)\lambda_3 = 0$$

and dividing the first condition by $(m_i - 1)$.

$$(m_i - 1)\lambda_1 + \lambda_3 = 0$$

$$(m_i - 1)\lambda_2 + (m_i - 2)\lambda_3 = 0$$

Now, let us choose three of four conditions and write them explicitly in matrix notation (we are choosing the first condition twice)

$$\boldsymbol{A\lambda} = \begin{pmatrix} m_1 - 1 & 0 & 1 \\ m_2 - 1 & 0 & 1 \\ 0 & m_1 - 1 & m_1 - 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = 0$$

Obviously, if $m_1 = m_2$, then the matrix $\boldsymbol{A}$ is singular and the vector $\boldsymbol{\lambda}$ may not be zero. However, if $m_1 \neq m_2$, then the matrix $\boldsymbol{A}$ is not singular, the vector $\boldsymbol{\lambda}$ has to be zero and the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly independent. Indeed, one can show that $\boldsymbol{A}$ is singular calculating its determinant.

$$|\boldsymbol{A}| = \begin{vmatrix} m_1 - 1 & 0 & 1 \\ m_2 - 1 & 0 & 1 \\ 0 & m_1 - 1 & m_1 - 2 \end{vmatrix} = (m_1 - 1)(m_2 - 1) - (m_1 - 1)^2$$

Obviously, if $m_1 \neq m_2$ the $|\boldsymbol{A}| \neq 0$.

The result of proposition 4 can be also intuitively interpreted; however, let us firstly discuss the non-identification of the peer effects model analysed by Manski (1993) and Moffitt (2001). We provide an intuition for all cases together.

**Proposition 5. The Reflection Problem and the Non-Identification of Peer Effects in the Models Proposed by Manski (1993) and Moffitt (2001).**

Manski (1993): Let us assume that pupils interacted within their classes only and an individual $i$ is included in its set of peers, e.g., $i \in P_i$. Then peer effects are not identified (the reflection problem).

Moffitt (2001): Let us assume that pupils interacted within their classes only and an individual $i$ is excluded from the set of her peers, e.g., $i \notin P_i$. Moreover, assume that all classes have the same size $m$. Then peer effects are not identified.

**The proof of Proposition 5.**

Let us start with the model proposed by Manski (1993). Allowing class size to vary, assume that we have classes each with the size $c_j$. Additionally, let $p$ be the number of classes. Since individuals cannot interact between the classes, the network matrix $\boldsymbol{G}$ has a block-diagonal form.

$$\boldsymbol{G} = \begin{pmatrix} \mathbf{G}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{G}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{G}_\mathrm{p} \end{pmatrix}$$

The blocks $\mathbf{G}_\mathrm{j}$ are in this special case matrices of ones divided by $c_j$.

$$\mathbf{G}_\mathrm{j} = \frac{1}{c_j}\boldsymbol{J}_{c_j \times c_j}$$

Consequently, the second power of $\mathbf{G}_\mathrm{j}$ is

$$\mathbf{G}_\mathrm{j}^2 = \frac{1}{c_j^2}\boldsymbol{J}_{c_j\times c_j}^2 = \frac{1}{c_j^2}c_j\boldsymbol{J}_{c_j\times c_j} = \mathbf{G}_\mathrm{j}$$

We can see that the same holds for the whole matrix $\boldsymbol{G}$.

$$\boldsymbol{G}^2 = \boldsymbol{G}$$

This means that the assumption in proposition 3 is violated. One may go through its proof and see that if the matrices $\boldsymbol{I}_n$, $\boldsymbol{G}$, and $\boldsymbol{G}^2$ are linearly dependent, we lose the condition $\beta\delta' - \beta'\delta = 0$ and we have only two equations for three parameters.

$$\gamma - \gamma' = 0$$

$$\delta - \delta' + \beta\gamma' - \beta'\gamma = 0$$

Hence, the parameters of the peer effects model are not identified. Bramoullé et al. (2009) provides rigorous proof examining all special cases; however, we do not find it necessary in our work.

Let us move to the model proposed by Moffitt (2001). Without loss of generality, we can analyse just a sample with one class of size $m$. The network matrix $\boldsymbol{G}$ is

$$\boldsymbol{G} = \frac{1}{m-1}(\boldsymbol{J}_{m\times m} - \boldsymbol{I}_m)$$

as we derived in section 2.1. Consequently,

$$\boldsymbol{G}^2 = \frac{1}{(m-1)^2}(\boldsymbol{J}_{m\times m} - \boldsymbol{I}_m)^2 = \frac{1}{(m-1)^2}(\boldsymbol{J}_{m\times m}^2 - 2\boldsymbol{J}_{m\times m} + \boldsymbol{I}_m)$$

Since $\boldsymbol{J}_{m\times m}^2 = m\boldsymbol{J}_{m\times m}$, we can write

$$\boldsymbol{G}^2 = \frac{1}{(m-1)^2}\big((m-2)\boldsymbol{J}_{m\times m} + \boldsymbol{I}_m\big)$$

We can see that $\boldsymbol{G}^2$ is the linear combination of $\boldsymbol{J}_{m\times m}$ and $\boldsymbol{I}_m$, and hence $\boldsymbol{G}^2$ is linearly dependent on $\boldsymbol{G}$ and $\boldsymbol{I}_m$. Using the same argument as in the non-identification of the model proposed by Manski (1993), the peer effects model proposed by Moffitt (2001) is also unidentified.

Let us compare proposition 4 and proposition 5 in words. The non-identification in the model proposed by Manski (1993) and the identification in the model proposed by Lee (2007) seem to be a mathematical trick. Let us connect this result with the intuition that peers of peers who are not directly peers of an examined individual may provide an instrument for the identification (As we discussed in the end of section 2.3). If we include an individual in the peer group (Manski 1993), the peers of our peers are also our peers. However, if we exclude an individual $i$ from her peer group, then her peers have a different peer group. The difference is that the peer group of peers includes an individual $i$ and excludes one different individual.

On the other hand, however, one may imagine that the differences in means with excluded or included individual is rather small. Especially in case of relatively large peer groups the differences are also relatively small. This property of the model can be analysed using Monte-Carlo simulations and Boucher et al. (2014) confirms this intuition.

Furthermore, let us discuss the reflection problem introduced by Manski (1993), which is often cited in the peer effects literature. The reflection problem states that $\beta$ and $\delta$ are not separately identified. Using terminology proposed by Manski (1993) *endogenous and exogenous peer effects* are not separately identified. We can see that Bramoullé et al. (2009) provides instructions on how to avoid this type of reflection problem. Simply exclude an individual $i$ from her peer group. Importantly, Manski (1993) addresses another type of the reflection problem allowing for *correlated effects*. We describe it in detail in the following sections.

## 2.5 The Problem of Correlated Effects

Manski (1993) also addresses another type of reflection problem allowing for *correlated effects*. He assumes that peers may be sorted into a group according to some unobserved variable that directly affects educational outcomes. Consider the following example from the Czech educational system. For simplicity, we can model the situation assuming two types of schools: prestigious and non-prestigious. The average pupils' characteristics may differ between these types of schools and we can control for this. However, pupils may also be sorted according to some unobserved variable, for example, inborn ability. If pupils with higher inborn ability attend prestigious schools, educational outcomes of individual $i$ will be correlated with the average outcome of her peers even if there are no *endogenous and exogenous effects*. Consequently, the peer effects estimators will be biased.

## 2.6 The Description of Selection

If we want to check whether there is selection in our data, one may define the average correlation $\rho$ of background characteristics among pupils within classes. We can choose one relevant background characteristic (for example socioeconomic status) and examine it. We intuitively motivate our definition of average correlation $\rho$ and it is inspired by Feld and Zölitz (2017). Denote by $\mathcal{N}_i$ the set of classmates of an individual $i$ including also individual $i$. We examine the covariation of classmates

$$\text{Cov}[x_i, x_j] \quad \forall\ j \in \mathcal{N}_i$$

and we average it between all classes. Our analysis is specific since we are examining group interactions, and hence it is reasonable to describe the data that we want to analyse. Consider a class with three pupils whose characteristics are $\{-1,0,1\}$. The examined pairs $[x_i, x_j]$ can be represented as the cartesian second power of the set $\{-1,0,1\}$. The graphical representation of pairs $[x_i, x_j]$ is provided in figure 1 a). Note that diagonal pairs $[x_i, x_i]$ are also included since with them the correlation of peers in one class is zero.

Now we move to the case with 4 pupils assigned into 2 classes. Pupils' test scores are $\{-2,-1,1,2\}$ and we calculate $\text{Cov}[x_i, x_j]\ \forall\ j \in \mathcal{N}_i$ for different assignments. $\text{Cov}[x_i, x_j]$ is defined by

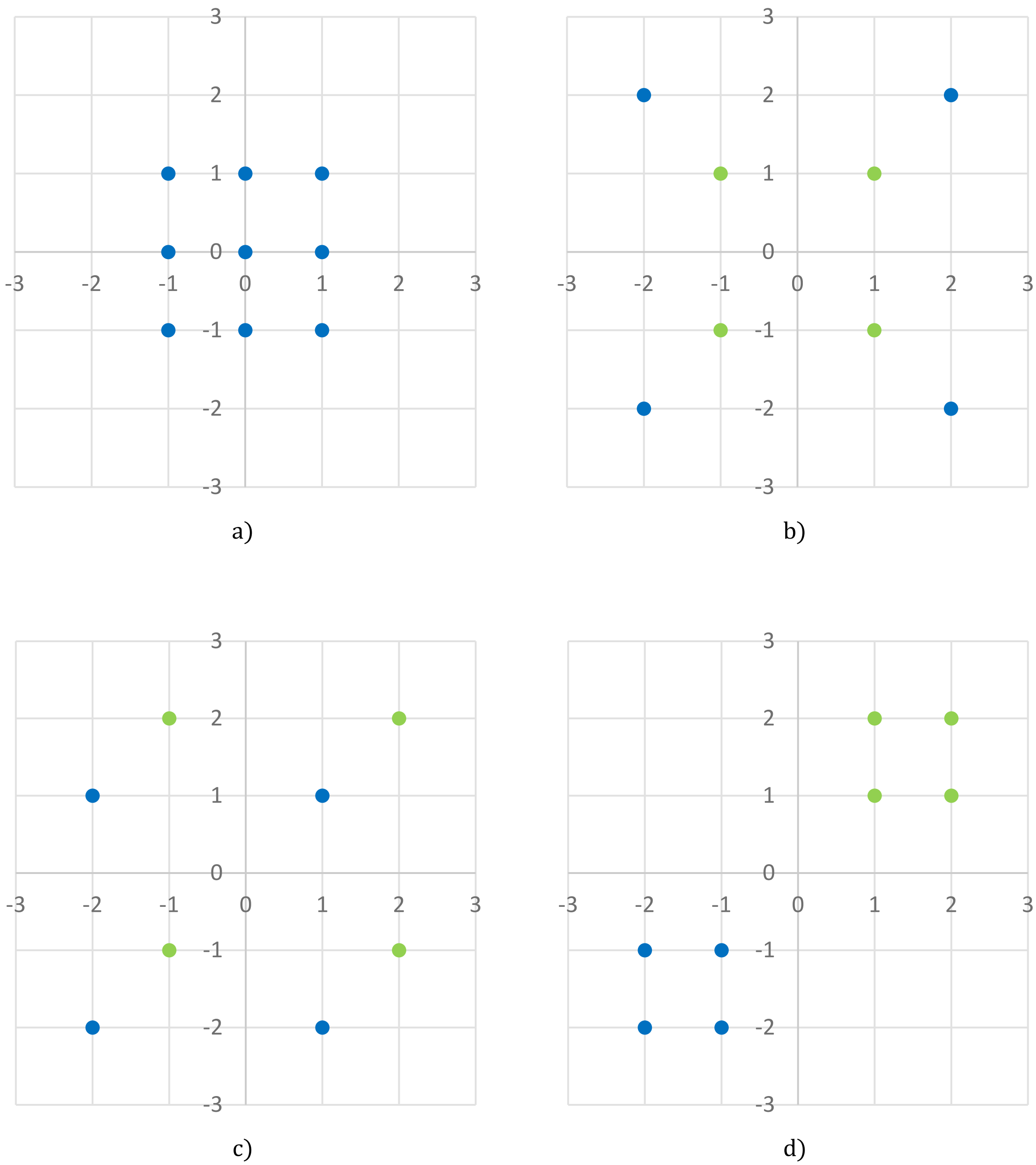
$$\text{Cov}[x_i, x_j] = \text{E}[x_i x_j] - \text{E}[x_i]\text{E}[x_j]$$

Since $x_j$ comes from the second power of $x_i$, both variables have identical expected values and we can write

$$\text{Cov}[x_i, x_j] = \text{E}[x_i x_j] - \text{E}[x_i]^2$$

**Figure 1: Cartesian's products of pupils' test scores.**

The figure graphically represents pairs$[y_i, y_j]$ used for the computation of average peers' correlation. Part a) represents one class with test scores $\{-1, 0, 1\}$. Parts b), c) and d) represent test scores of pupils allocated into two classes distinguished by blue and green colour.



Hence the corresponding estimator of average correlation is

$$\hat{\rho} = \widehat{\mathrm{Cov}}[x_i, x_j] = \frac{1}{N_p} \sum_{i,j=1}^{N_p} x_i x_j \; - \left( \frac{1}{N} \sum_{i=1}^{N} x_i \right)^2 \;\; \forall \; j \in \mathcal{N}_i \tag{7}$$

where $N_p$ is the number of pairs $[x_i, x_j]$ and $N$ is the number of pupils. If we write the numbers of pupils in each class into a vector, then $N$ is the sum of its elements and $N_p$ is the sum of its elements squared. To provide more intuition for our definition, we are enclosing the following examples.

Let us consider three different assignments of pupils into classes with background characteristics $\{-2, -1, 1, 2\}$. First, the classes are $\{-2, 2\}$ and $\{-1, 1\}$ and the pairs $[x_i, x_j]$ are graphically represented in figure 1 b). Second, the classes are $\{-2, 1\}$ and $\{-1, 2\}$ and the pairs $[x_i, x_j]$ are graphically represented in figure 1 c). Third, the classes are $\{-2, -1\}$ and $\{1, 2\}$ and the pairs $[x_i, x_j]$ are graphically represented in figure 1 d). The first case represents the most symmetrical assignment because the averages of both classes are zeros. The second case represents a slightly asymmetrical assignment since the averages of classes are -0.5 and 0.5. The third case is fully asymmetrical and the averages are -1.5 and 1.5.

Now we calculate the average correlation in the three cases above according to equation (7). Case 1 with classes $\{-2, 2\}$ and $\{-1, 1\}$ equation (7) gives

$$\frac{1}{N} \sum_{i,j=1}^{N} x_i x_j \;\; \forall \; j \in \mathcal{N}_i = \frac{1}{8} \left\{ x_1 \sum_{x_j \in \mathcal{N}_1} x_j + x_2 \sum_{x_j \in \mathcal{N}_2} x_j + x_3 \sum_{x_j \in \mathcal{N}_3} x_j + x_4 \sum_{x_j \in \mathcal{N}_4} x_j \right\} =$$

$$= \frac{1}{8} \{(-2) \cdot (-2 + 2) + (-1) \cdot (-1 + 1) + 1 \cdot (-1 + 1) + 2 \cdot (-2 + 2)\} = 0$$

In cases 2 and 3, the same calculation gives 0.25 and 2.25. Finally, we can define the average correlation $\rho$ as

$$\rho = \frac{\mathrm{Cov}[x_i, x_j]}{\sqrt{\mathrm{Var}[x_i]}\sqrt{\mathrm{Var}[x_j]}} = \frac{\mathrm{Cov}[x_i, x_j]}{\mathrm{Var}[x_i]}$$

where second equality follows from the fact that $x_i$ and $x_j$ have the same values. For the sample with zero mean, we can express the average correlation as

$$\hat{\rho} = \frac{\frac{1}{N_p}\sum_{i,j=1}^{N_p} x_i x_j \; - \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^2}{\frac{1}{N}\sum_{i}^{N} x_i^2} \tag{8}$$

An average variance in our sample is 2.5, hence the correlations in cases 1, 2, and 3 are 0, 0.1, and 0.9. We see that our definition of average correlation among peers is in the interval [0, 1]. The value $\hat{\rho} = 0$ describes the fully symmetrical assignment and the value $\hat{\rho} = 1$ describes fully selective assignment. Regarding class compositions, fully symmetrical assignment corresponds to the sample of classes with the same means and fully selective assignment corresponds to the case in which all pupils in each class have the same scores.

One may ask why we are calculating the average peers' correlation of characteristics $x$, if we are able to account for this selection in our peer effects model. The reason is that if there is selection according to an observed variable, there may also be an unobserved variable that affects educational outcomes and may cause the selection bias in peer effects estimation.

Finally, we would like to stress that the average peers' correlation $\hat{\rho}$ is equal to 0 in the case of fully symmetrical assignment. This means that under random sampling, there is some correlation $\hat{\rho}$ thanks to the random fluctuations. One can randomly assign pupils in the analysed sample into artificial classes and check what is the average correlation $\hat{\rho}$ under the random sampling.

## 2.7 Identification in Linear-in-Means Peer Effects Models in a Selective Environment

If there is a selection according to some unobserved variable that affects educational outcomes, then each class has an average value of an unobserved variable and average contribution to educational outcomes. Therefore, adding fixed effects for each class or generally peer group can solve the problem of selection. Hence, the corresponding extension of linear-in-means peer effects model is

$$y_{il} = \alpha_l + \beta \frac{1}{n_i} \sum_{j \in P_i} y_{jl} + \gamma x_{il} + \delta \frac{1}{n_i} \sum_{j \in P_i} x_{jl} + \varepsilon_i$$

where $l$ denotes a particular group. Now, we rewrite our peer effects model in matrix notation for one particular group $l$.

$$\boldsymbol{y}_l = \alpha_l \boldsymbol{J}_{m_l \times 1} + \beta \boldsymbol{G}_l \boldsymbol{y}_l + \gamma \boldsymbol{x}_l + \delta \boldsymbol{G}_l \boldsymbol{x}_l + \boldsymbol{\varepsilon}_l$$

where $m_l$ is the number of pupils in group $l$. The elimination of group fixed effect $\alpha_l$ can be conducted in different manners. We follow the technique used by Bramoullé et al. (2009) and apply matrix $\boldsymbol{G}_l$ to the whole equation.

$$\boldsymbol{G}_l \boldsymbol{y}_l = \alpha_l \boldsymbol{G}_l \boldsymbol{J}_{m_l \times 1} + \beta {\boldsymbol{G}_l}^2 \boldsymbol{y}_l + \gamma \boldsymbol{G}_l \boldsymbol{x}_l + \delta {\boldsymbol{G}_l}^2 \boldsymbol{x}_l + \boldsymbol{G}_l \boldsymbol{\varepsilon}_l$$

Note that $\boldsymbol{G}_l \boldsymbol{J}_{m_l \times 1} = \boldsymbol{J}_{m_l \times 1}$ since $\boldsymbol{G}_l$ is row normalised (proposition 1.2). Now we can subtract both equations and obtain

$$\boldsymbol{y}_l - \boldsymbol{G}_l \boldsymbol{y}_l = \beta \boldsymbol{G}_l \boldsymbol{y}_l - \beta {\boldsymbol{G}_l}^2 \boldsymbol{y}_l + \gamma \boldsymbol{x}_l - \gamma \boldsymbol{G}_l \boldsymbol{x}_l + \delta \boldsymbol{G}_l \boldsymbol{x}_l - \delta {\boldsymbol{G}_l}^2 \boldsymbol{x}_l + \boldsymbol{\varepsilon}_l - \boldsymbol{G}_l \boldsymbol{\varepsilon}_l$$

Now we can factorise the matrix $\boldsymbol{I}_{m_l} - \boldsymbol{G}_l$

$$(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{y}_l = \beta(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{G}_l \boldsymbol{y}_l + \gamma(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{x}_l + \delta(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{G}_l \boldsymbol{x}_l + (\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{\varepsilon}_l$$

We can also separate $\boldsymbol{y}_l$ to the right-hand side and fuse the terms including $\boldsymbol{x}_l$

$$(\boldsymbol{I}_{m_l} - \beta \boldsymbol{G}_l)(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{y}_l = (\gamma \boldsymbol{I}_{m_l} + \delta \boldsymbol{G}_l)(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{x}_l + (\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{\varepsilon}_l$$

Finally, we can multiply the above equation by $(\boldsymbol{I}_{m_l} - \beta \boldsymbol{G}_l)^{-1}$

$$\begin{aligned}(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{y}_l = (\boldsymbol{I}_{m_l} - \beta \boldsymbol{G}_l)^{-1}(\gamma \boldsymbol{I}_{m_l} + \delta \boldsymbol{G}_l)(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{x}_l \\ + (\boldsymbol{I}_{m_l} - \beta \boldsymbol{G}_l)^{-1}(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l)\boldsymbol{\varepsilon}_l\end{aligned} \tag{9}$$

Equation (9) is analogical to equation (5) derived for the non-selective environment. Let us provide conditions under which peer effects are identified. Intuitively, we can expect a more demanding condition in comparison with the model for a non-selective environment (proposition

3), since we need to eliminate group fixed effects. The conditions are summarised in the following proposition.

**Proposition 6. The Identification of Linear-in-Means Peer Effects Model in a Selective Environment**

Assume that $\gamma\beta + \delta \neq 0$ and $\mathbb{E}[\varepsilon_i | \boldsymbol{x}_i, \boldsymbol{G}_i] = 0$. If the matrices $\boldsymbol{I}_n, \boldsymbol{G}$, $\boldsymbol{G}^2$ and $\boldsymbol{G}^3$ are linearly independent, then the linear-in-means peer effects model is identified, e. g., parameters $\alpha, \beta, \gamma,$ and $\delta$ are identified.

**The Proof of Proposition 6**

The proof of proposition 6 is analogical to the proof of proposition 3. Consider two sets of parameters $\{\beta, \gamma, \delta\}$ and $\{\beta', \gamma', \delta'\}$ leading to the same educational outcomes $\boldsymbol{y}$ in equation (9). This implies the following conditions

$$\left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)^{-1}\left(\gamma\boldsymbol{I}_{m_l} + \delta\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\boldsymbol{x}_l = \left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)^{-1}\left(\gamma'\boldsymbol{I}_{m_l} + \delta'\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\boldsymbol{x}_l$$

Since $\boldsymbol{x}$ may be an arbitrary vector, the matrices in front of $\boldsymbol{x}$ have to be identical.

$$\left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)^{-1}\left(\gamma\boldsymbol{I}_{m_l} + \delta\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right) = \left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)^{-1}\left(\gamma'\boldsymbol{I}_{m_l} + \delta'\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)$$

Now, we can multiply the equation by $\left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)$ to eliminate matrix inverses.

$$\begin{aligned}&\left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)\left(\gamma\boldsymbol{I}_{m_l} + \delta\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\\ &\qquad = \left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)^{-1}\left(\gamma'\boldsymbol{I}_{m_l} + \delta'\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\end{aligned}$$

One may easily think that matrices $\left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)$ and $\left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)$ are commutative, and hence we can eliminate the second matrix inverse.

$$\left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)\left(\gamma\boldsymbol{I}_{m_l} + \delta\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right) = \left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)\left(\gamma'\boldsymbol{I}_{m_l} + \delta'\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)$$

Now, we can multiply parenthesis and obtain

$$\left(\gamma\boldsymbol{I}_{m_l} + \delta\boldsymbol{G}_l - \beta'\gamma\boldsymbol{G}_l - \beta'\delta\boldsymbol{G}_l^2\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right) = \left(\gamma'\boldsymbol{I}_{m_l} + \delta'\boldsymbol{G}_l - \beta\gamma'\boldsymbol{G}_l - \beta\delta'\boldsymbol{G}_l^2\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)$$

$$\gamma \boldsymbol{I}_{m_l} + \delta \boldsymbol{G}_l - \beta' \gamma \boldsymbol{G}_l - \beta' \delta \boldsymbol{G}_l^2 - \gamma \boldsymbol{G}_l - \delta \boldsymbol{G}_l^2 + \beta' \gamma \boldsymbol{G}_l^2 + \beta' \delta \boldsymbol{G}_l^3$$
$$= \gamma' \boldsymbol{I}_{m_l} + \delta' \boldsymbol{G}_l - \beta \gamma' \boldsymbol{G}_l - \beta \delta' \boldsymbol{G}_l^2 - \gamma' \boldsymbol{G}_l - \delta' \boldsymbol{G}_l^2 + \beta \gamma' \boldsymbol{G}_l^2 + \beta \delta' \boldsymbol{G}_l^3$$

Let us separate terms with the identity matrix $\boldsymbol{I}_{m_l}$ and different powers of matrix $\boldsymbol{G}_l$.

$$(\gamma - \gamma')\boldsymbol{I}_{m_l} + (\delta - \delta' + \beta\gamma' - \beta'\gamma + \gamma' - \gamma)\boldsymbol{G}_l + (\beta\delta' - \beta'\delta + \delta' - \delta + \beta'\gamma - \beta\gamma')\boldsymbol{G}_l^2 + (\beta'\delta - \beta\delta')\boldsymbol{G}_l^3 = 0$$

If $\boldsymbol{I}_{m_l}$, $\boldsymbol{G}_l$, $\boldsymbol{G}_l^2$, and $\boldsymbol{G}_l^3$ are linearly independent, then all parentheses have to equal zeros which gives

$$\gamma = \gamma'$$
$$\delta - \delta' + \beta\gamma' - \beta'\gamma = 0$$
$$\beta\delta' - \beta'\delta + \delta' - \delta + \beta'\gamma - \beta\gamma' = 0$$
$$\beta'\delta - \beta\delta' = 0$$

The last equation can be added to the third, which gives

$$\gamma = \gamma'$$
$$\delta - \delta' + \beta\gamma' - \beta'\gamma = 0$$
$$\delta' - \delta + \beta'\gamma - \beta\gamma' = 0$$
$$\beta'\delta - \beta\delta' = 0$$

The second and third equations are the same, and hence three equations are remaining

$$\gamma = \gamma'$$
$$\delta - \delta' + \beta\gamma' - \beta'\gamma = 0$$
$$\beta'\delta - \beta\delta' = 0$$

One may note that this is the same system of equations as in the proof of proposition 3. The same arguments imply that if $\delta + \beta\gamma \neq 0$, then parameters $\{\beta, \gamma, \delta\}$ are uniquely determined.

Importantly, we would like to stress that in the above derivation, one could make a serious mistake. Consider our conditions before we multiplied all parentheses

$$\left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)\left(\gamma\boldsymbol{I}_{m_l} + \delta\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right) = \left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)\left(\gamma'\boldsymbol{I}_{m_l} + \delta'\boldsymbol{G}_l\right)\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)$$

If we multiplied it by $\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)^{-1}$, we would obtain

$$\left(\boldsymbol{I}_{m_l} - \beta'\boldsymbol{G}_l\right)\left(\gamma\boldsymbol{I}_{m_l} + \delta\boldsymbol{G}_l\right) = \left(\boldsymbol{I}_{m_l} - \beta\boldsymbol{G}_l\right)\left(\gamma'\boldsymbol{I}_{m_l} + \delta'\boldsymbol{G}_l\right)$$

This is an identical condition as in the proof of proposition 3 which would lead to the identification condition that $\boldsymbol{I}_{m_l}$, $\boldsymbol{G}_l$, and $\boldsymbol{G}_l^2$ have to be linearly independent. However, multiplying the whole condition by $\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)^{-1}$ can lead to the loss of some information about matrix $\boldsymbol{G}_l$. We can test this hypothesis assuming that $\boldsymbol{G}_l^3$ linearly depends on $\boldsymbol{I}_{m_l}$, $\boldsymbol{G}_l$, and $\boldsymbol{G}_l^2$.

$$\boldsymbol{G}_l^3 = \lambda_0\boldsymbol{I}_{m_l} + \lambda_1\boldsymbol{G}_l + \lambda_2\boldsymbol{G}_l^2$$

Therefore, our identification condition is

$$(\gamma - \gamma')\boldsymbol{I}_{m_l} + (\delta - \delta' + \beta\gamma' - \beta'\gamma + \gamma' - \gamma)\boldsymbol{G}_l + (\beta\delta' - \beta'\delta + \delta' - \delta + \beta'\gamma - \beta\gamma')\boldsymbol{G}_l^2 + (\beta'\delta - \beta\delta')\left(\lambda_0\boldsymbol{I}_{m_l} + \lambda_1\boldsymbol{G}_l + \lambda_2\boldsymbol{G}_l^2\right) = 0$$

which can be rewritten as

$$\left(\gamma - \gamma' + \lambda_0(\beta'\delta - \beta\delta')\right)\boldsymbol{I}_{m_l} + (\delta - \delta' + \beta\gamma' - \beta'\gamma + \gamma' - \gamma + \lambda_1(\beta'\delta - \beta\delta')\,)\boldsymbol{G}_l + \left(\beta\delta' - \beta'\delta + \delta' - \delta + \beta'\gamma - \beta\gamma' + \lambda_2(\beta'\delta - \beta\delta')\right)\boldsymbol{G}_l^2 = 0$$

If $\boldsymbol{I}_{m_l}$, $\boldsymbol{G}_l$, and $\boldsymbol{G}_l^2$ are linearly independent, the following conditions have to be satisfied

$$\gamma - \gamma' + \lambda_0(\beta'\delta - \beta\delta') = 0$$

$$\delta - \delta' + \beta\gamma' - \beta'\gamma + \gamma' - \gamma + \lambda_1(\beta'\delta - \beta\delta') = 0$$

$$\beta\delta' - \beta'\delta + \delta' - \delta + \beta'\gamma - \beta\gamma' + \lambda_2(\beta'\delta - \beta\delta') = 0$$

Now we want to use a special property of matrix $\boldsymbol{G}_l$, which implies particular properties of parameters $\lambda_0$, $\lambda_1$, and $\lambda_2$. Let us multiply our definition of $\boldsymbol{G}_l$ by $\boldsymbol{J}_{m_l\times 1}$.

$$\boldsymbol{G}_l^3\boldsymbol{J}_{m_l\times 1} = \lambda_0\boldsymbol{J}_{m_l\times 1} + \lambda_1\boldsymbol{G}_l\boldsymbol{J}_{m_l\times 1} + \lambda_2\boldsymbol{G}_l^2\boldsymbol{J}_{m_l\times 1}$$

However, $\boldsymbol{G}_l^k\boldsymbol{J}_{m_l\times 1} = \boldsymbol{J}_{m_l\times 1}$ since the matrix $\boldsymbol{G}_l$ is row normalised (proposition 1.2). Therefore, we can write that

$$\boldsymbol{J}_{m_l \times 1} = (\lambda_0 + \lambda_1 + \lambda_2)\boldsymbol{J}_{m_l \times 1}$$

$$1 = \lambda_0 + \lambda_1 + \lambda_2$$

Now, we can sum up the first two equations

$$\delta - \delta' + \beta\gamma' - \beta'\gamma + (\lambda_0 + \lambda_1)(\beta'\delta - \beta\delta') = 0$$

and plug in the condition for lambdas

$$\delta - \delta' + \beta\gamma' - \beta'\gamma + (1 - \lambda_2)(\beta'\delta - \beta\delta') = 0$$

The multiplication of parenthesis leads to

$$\delta - \delta' + \beta\gamma' - \beta'\gamma + \beta'\delta - \beta\delta' - \lambda_2(\beta'\delta - \beta\delta') = 0$$

One may note that this is exactly the third condition. Summarising this result, we have shown that if $\boldsymbol{G}_l^3$ linearly depends on $\boldsymbol{I}_{m_l}$, $\boldsymbol{G}_l$, and $\boldsymbol{G}_l^2$, then three identification conditions can be reduced into two. Obviously, three conditions cannot uniquely determine three parameters $\{\beta', \gamma', \delta'\}$.

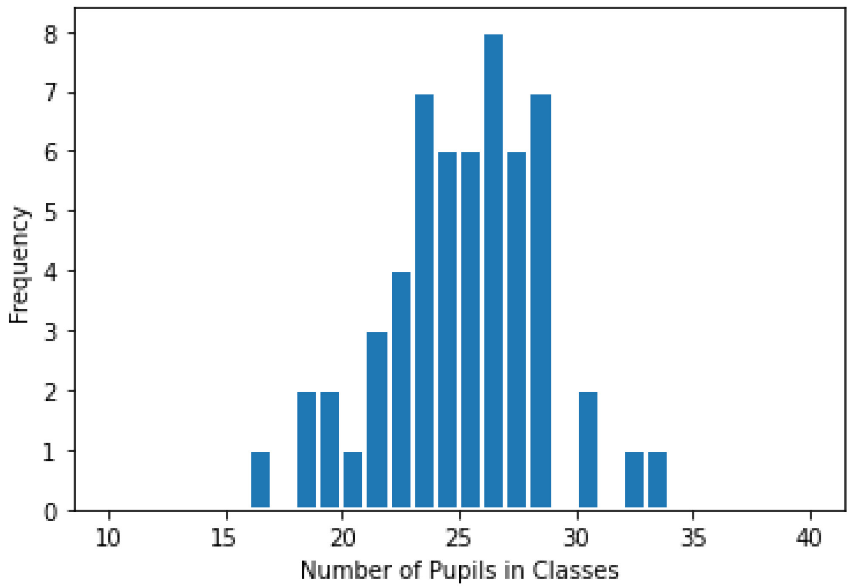
# 3 Empirical Analysis

## 3.1 The Description of Czech Education and Data

Children in the Czech Republic start their primary schooling when they are 6 years old and finish it usually when they are 15 years old. Primary schooling is also part of compulsory schooling hence all pupils have to spend 9 years in education. A majority of pupils spends their primary schooling in one primary school; however, there are two main exceptions. First, some schools provide education from 1st to 5th grade and pupils have to move to another school to continue their compulsory schooling.  Second, there are junior secondary grammar schools preparing pupils for the academic track. Junior secondary grammar schools accept pupils who finished 5th grade and offer them education for further 8 years. Junior secondary grammar schools usually select high-achieving pupils from different primary schools and thus pupils create new groups meeting new classmates, which is important for our analysis.

To examine peer effects among Czech pupils, we analyse data from Czech Longitudinal Study in Education (CLOSE). The data contains pupils' test scores in Math, Reading, English and Learning Skills from 4th, 6th, and 9th grades. The observational study started with a sample of pupils from primary schools in 4th grade as collected within he international joint PIRLS and TIMSS surveys in 2011. In 6th grade, the pupils transferring to junior secondary grammar schools were not followed, and therefore there was added the sample of pupils entering junior secondary grammar schools and we can observe their test scores also in 9th grade. The data contain also information about pupils' families, relationships in classes, relationships with teachers, among others. Typical empirical analysis of pupils' test scores usually uses the background characteristics of pupils as explanatory variables. However; in our case, it would involve using many variables with relatively weak predictive power, which would seriously complicate our analysis of peer effects. Moreover, it is widely believed that pupils' test scores are directly affected by pupils' innate ability which is not directly observable. Therefore, we decide to focus on the pupils from junior secondary grammar schools and use their 6th-grade score as an explanatory variable for the 9th-grade score. The advantage is that the 6th-grade score is not affected by peer effects from new peers and summarizes information about pupils' background characteristics and inborn abilities.

**Figure 2: Distribution of Class Sizes**

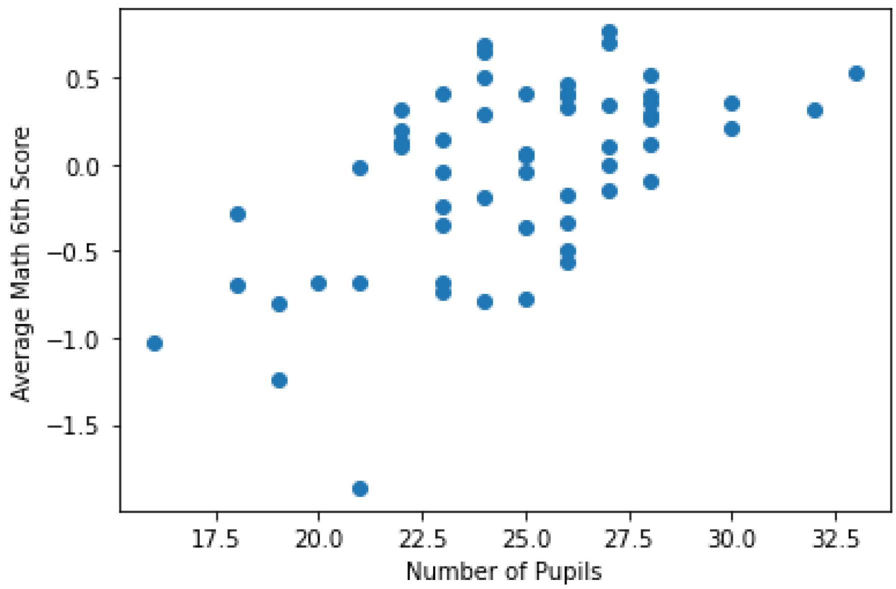



Another advantage is that pupils' $6^{th}$-grade score from a particular subject is a strong predictor of pupils' $9^{th}$-grade score in the same subject.

Regarding the data characteristics, we observe 1407 pupils in 57 classes. The average class size is 25 and the distribution of class sizes is shown by figure 2. All test scores are normalised to have zero mean and unit variance. Socioeconomic status (SES) is analogically normalised for the whole sample of data from CLOSE, therefore it is not normalised in our subsample of junior secondary grammar schools. The SES of pupils is calculated by the authors of CLOSE. For the estimation of peer effects, we have to check if there is a selection of pupils into classes according to their background characteristics ($6^{th}$-grade scores). We could calculate average peers' correlation according to equation (8) then permutate pupils test scores within the whole sample and compare average peers' correlation of original and permutated data. If permutated average peers' correlation were significantly lower than original, then we would have a selection present. However, we argue that we have a selection in our sample since we observe an even more special selection. Figure 3 shows that the class average in $6^{th}$-grade Math score is increasing with class size. We also conduct an OLS regression, which confirms that class averages in $6^{th}$-grade Math scores are increasing with class sizes and the regression coefficient is statistically significant.

**Figure 3: Correlation between Class Average Score and Size.**

The figure shows that class average Math score in 6th grade is positively correlated with class size. Therefore, we can observe selection of pupils into classes based on their achievements in Math.




The observed pattern is expectable in the Czech environment since class sizes should be lower than 30 (this rule is not fully enforced). It means that some schools are prestigious, can select high-achieving pupils and fulfil the capacity of 30 pupils in classes. On the other hand, some schools are less prestigious, select relatively low-achieving pupils and also have smaller class sizes. Summarising these observations, an environment of junior secondary grammar schools is selective and we need to estimate peer effects model with class fixed effects.

Considering the choice of explanatory variables, we decide to use three different specifications. We are measuring the explanatory power of explanatory variables by conducting OLS regression without peer effects.

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

Outcomes $y_i$ are in this case Math scores from 9th grade. As an explanatory variable, we use Math test scores in 6th grade, SES, and the average test scores in 6th grade from Math, Reading, English and Learning Skills. The estimated values for coefficients $\beta$ are reported in **table 1**, including also the coefficient of determination $R^2$. We can see that Math test scores in 6th grade

**Table 1: Linear regression of Math test scores in 9th grade using different explanatory variables.**

We conduct three different regressions with only one explanatory variable, since we also use only one explanatory variable in our peer effects model.

| Explanatory Variable | Math Test Score in 6th Grade | SES | The Average of Test Scores in 6th Grade |
|---|---|---|---|
| Regression Coefficient | 0.574<br>(0.027) | 0.214<br>(0.034) | 0.597<br>(0.024) |
| $R^2$ | 33% | 3% | 34% |

and average test scores in 6th grade are strongly predicting Math test scores in 9th grade. Moreover, regression coefficients are practically identical in magnitude and predictive power, which makes both measures a perfect instrument for robustness check in peer effects analysis. However, SES is a weaker predictor of the Math score in 9th grade, which complicates further analysis of peer effects.

## 3.2 The Estimation of Peer Effects Model

Since the environment of junior secondary grammar schools is selective, we estimate peer effects model with class fixed effects and follow the approach proposed by Bramoullé (2009). First, we generalise the model derived in section 2.7.

$$\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\boldsymbol{y}_l = \beta\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\boldsymbol{G}_l\boldsymbol{y}_l + \gamma\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\boldsymbol{x}_l + \delta\left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\boldsymbol{G}_l\boldsymbol{x}_l + \left(\boldsymbol{I}_{m_l} - \boldsymbol{G}_l\right)\boldsymbol{\varepsilon}_l$$

where $\boldsymbol{G}_l$ is the network matrix for class $l$. Now, we create network matrix $\boldsymbol{G}$, which has matrices $\boldsymbol{G}_l$ on its diagonal. Similarly, we can stack vectors $\boldsymbol{y}_l$ and $\boldsymbol{x}_l$ into vectors $\boldsymbol{y}$ and $\boldsymbol{x}$ describing pupils test scores for the whole sample. The general form of our model is

$$(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{y} = \beta(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{y} + \gamma(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{x} + \delta(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{x} + (\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{\varepsilon} \tag{10}$$

In the first step, we calculate 2SLS estimators of peer effects. We construct the matrix of instruments

$$\boldsymbol{S} = [(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{x} \quad (\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{x} \quad (\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}^2\boldsymbol{x}]$$

and the matrix of explanatory variables

$$\widetilde{\boldsymbol{X}} = [(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{y} \quad (\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{x} \quad (\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{x}]$$

The estimators in the first step are

$$\begin{pmatrix}\hat{\beta}_1\\ \hat{\gamma}_1\\ \hat{\delta}_1\end{pmatrix} = \left(\widetilde{\boldsymbol{X}}'\boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'\widetilde{\boldsymbol{X}}\right)^{-1}\widetilde{\boldsymbol{X}}'\boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{y}$$

In the second step, we construct a new instrument

$$\boldsymbol{Z} = [\mathrm{E}[(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{y}|\boldsymbol{G},\boldsymbol{x}] \quad (\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{x} \quad (\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{x}]$$

To calculate $\mathrm{E}[(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{y}|\boldsymbol{G},\boldsymbol{x}]$, we generalize equation (9) to describe the whole sample

$$(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{y} = (\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}(\gamma\boldsymbol{I}_n + \delta\boldsymbol{G})(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{x} + (\boldsymbol{I}_n - \beta\boldsymbol{G})^{-1}(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{\varepsilon}_l$$

multiply it by the matrix $\boldsymbol{G}$ and apply the expectation operator.

$$\mathrm{E}[(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{G}\boldsymbol{y}|\boldsymbol{G},\boldsymbol{x}] = \boldsymbol{G}\left(\boldsymbol{I}_n - \hat{\beta}_1\boldsymbol{G}\right)^{-1}\left(\hat{\gamma}_1\boldsymbol{I}_n + \hat{\delta}_1\boldsymbol{G}\right)(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{x}$$

Note that we are using estimators calculated in the first step. The peer effects estimators in the second step are given as

$$\begin{pmatrix}\hat{\beta}_2\\ \hat{\gamma}_2\\ \hat{\delta}_2\end{pmatrix} = \left(\boldsymbol{Z}'\widetilde{\boldsymbol{X}}\right)^{-1}\boldsymbol{Z}'\boldsymbol{y}$$

For the estimation of the variance of the parameters in the second step, we need to calculate residuals $(\boldsymbol{I}_n - \boldsymbol{G})\boldsymbol{\varepsilon}$ from equation (10) and allocate them on the diagonal of matrix $\boldsymbol{D}$. The variance estimator of the parameters is

$$\widehat{\boldsymbol{V}} = \left(\boldsymbol{Z}'\widetilde{\boldsymbol{X}}\right)^{-1}\boldsymbol{Z}'\boldsymbol{D}\boldsymbol{Z}\left(\widetilde{\boldsymbol{X}}'\boldsymbol{Z}\right)^{-1}$$

Table 2 presents our peer effects estimators calculated using data from CLOSE. We observe negative and significant endogenous peer effects and positive and significant exogenous peer effects in Math, Reading, and English. On the contrary, both peer effects estimators for Learning Skills are not significant.

Surprisingly, peer effects estimators are unrealistically high in comparison with previous studies (typically smaller than one in absolute value). First, we should examine if our variance estimator for the calculation of standard errors performs well in this setting. If we calculate significance levels for peer effects in Math, we receive a 99.76% probability that $\hat{\beta}_2$ is negative and almost 100% probability that $\hat{\delta}_2$ is positive. Therefore, we conduct bootstrap simulation using 1,000 repetitions and resampling at the class levels.

**Table 2: Estimation of peer effects on 9th grade score using 6th grade score as explanatory variable**

The table shows the efficient estimators of endogenous peer effects $\hat{\beta}_2$, the estimators of exogenous peer effect $\hat{\delta}_2$, and coefficient of explanatory variable $\hat{\gamma}_2$. Outcome variable is test score in 9th grade from particular subject and explanatory variable is test score in 6th grade from the same subject. Standard deviations implied by the estimator of variance matrix are in parenthesis. We also provide confidence intervals (with 95% probability) and significancy levels calculated using bootstrap simulations.

| | $\hat{\beta}_2$ | $\hat{\gamma}_2$ | $\hat{\delta}_2$ |
|---|---|---|---|
| **Math** | −31<br>(11) | 0.47<br>(0.10) | 14.6<br>(3.0) |
| Bootstrap CI | [−62 ; 12] | [0.24 ; 0.69] | [−8.3 ; 30.6] |
| Bootstrap $p$-value | 96.1% | 98.7% | 95.9% |
| **Reading** | −26.2<br>(5.0) | 0.276<br>(0.048) | 7.7<br>(2.5) |
| Bootstrap CI | [−56.4 ; 18.7] | [0.060 ; 0.757] | [−1.0 ; 21.9] |
| Bootstrap $p$-value | 95.2% | 98.6% | 97.4% |
| **English** | −19.8<br>(1.8) | 0.497<br>(0.036) | 10.3<br>(1.4) |
| Bootstrap CI | [−80.7 ; 18.1] | [0.015 ; 0.889] | [−17.0 ; 45.0] |
| Bootstrap $p$-value | 94.6% | 97.8% | 93.4% |
| **Learning Skills** | −9.2<br>(9.6) | 0.16<br>(0.32) | −3<br>(12) |
| Bootstrap CI | [−176 ; 225] | [−5.11 ; 4.18] | [−259 ; 175] |
| Bootstrap $p$-value | 70.3% | 66.0% | 56.6% |

Then we calculate bootstrap significance levels and obtain a 96.1% probability that $\hat{\beta}_2$ is negative and a 95.9% probability that $\hat{\delta}_2$ is positive. We can see that bootstrap significance levels are lower than significance levels calculated using standard errors. We can also see that bootstrap significance levels are closer to the standard boundary 95% for the statistically significant result. However, the results in table 2 show that peer effects in Math and Reading remain statistically significant even after computing bootstrap significant levels and we will discuss it in further analysis. Since the bootstrap errors seem to be more realistic, we use them for the calculation of bootstrap confidence intervals (CI) that are also reported in table 2. Bootstrap confidence intervals are computed ordering bootstrap estimators in increasing order obtaining sequence $\{\theta_i^b\}_{i=1}^{1,000}$. Then the lower bound of our bootstrap CI is $\theta_{25}^b$ and the upper bound is $\theta_{975}^b$.

**Table 3: Estimation of peer effects on 9th grade score using SES and the average of 6th grade scores as explanatory variables**

The table shows the efficient estimators of endogenous peer effects $\hat{\beta}_2$, the estimators of exogenous peer effect $\hat{\delta}_2$, and coefficient of explanatory variable $\hat{\gamma}_2$. Outcome variable is test score in 9th grade from particular subject and explanatory variables are SES and the average of 6th grade scores from all subjects. Standard deviations implied by the estimator of variance matrix are in parenthesis.

| | SES as Explanatory Variable | | |
|---|---|---|---|
| | $\hat{\beta}_2$ | $\hat{\gamma}_2$ | $\hat{\delta}_2$ |
| **Math** | −22.6<br>(8.2) | 0.064<br>(0.090) | 1.6<br>(2.5) |
| **Reading** | −19.6<br>(1.3) | 0.103<br>(0.049) | 2.1<br>(1.1) |
| **English** | −18.54<br>(0.57) | 0.076<br>(0.046) | 1.34<br>(0.94) |
| **Learning Skills** | −9.5<br>(9.6) | −0.00<br>(0.20) | −1.5<br>(5.7) |

| | The Average of Test Scores in 6th Grade as Explanatory Variable | | |
|---|---|---|---|
| | $\hat{\beta}_2$ | $\hat{\gamma}_2$ | $\hat{\delta}_2$ |
| **Math** | −38<br>(99) | 0.37<br>(0.66) | 15<br>(28) |
| **Reading** | −21.3<br>(5.1) | 0.396<br>(0.042) | 8.8<br>(2.6) |
| **English** | −33.2<br>(7.8) | 0.47<br>(0.12) | 16.3<br>(2.6) |
| **Learning Skills** | −8<br>(46) | 0.25<br>(0.80) | −1<br>(40) |

To examine the robustness of our results, we repeat our estimation for two alternative explanatory variables: SES and average test scores in 6th grade from Math, Reading, English and Learning Skills. The peer effects estimators are in table 3. In the case of SES as the explanatory variable, we see that exogenous peer effects estimators are not statistically significant, which is probably caused by the small explanatory power of SES as we showed in table 1. Regarding endogenous peer effects, they are negative and statistically significant in Math, Reading, and English, which is consistent with the results in table 2. Peer effects estimators in Learning Skills are not statistically significant, which is also consistent with the results in table 2.

In the case of the averages of test scores in 6th grade as the explanatory variable, we can see that peer effects in Reading and English are consistent with the results in table 2. However, the results from Math are significantly different and endogenous and exogenous peer effects estimators are not statistically significant since their standard errors are relatively high. Peer effects estimators in Learning Skills are not statistically significant, which is consistent with the results in table 2.

Summarising the results from table 2 and table 3, peer effects in Learning Skills are not significant for all types of explanatory variables. Regarding peer effects in Math, using the average of test scores in $6^{th}$ grade as an explanatory variable leads to the not statistically significant estimators therefore, we conclude, that peer effects in Math are not robust for different specifications of explanatory variables. Regarding peer effects in English, both are statistically significant; however, note that bootstrap simulations suggest that estimators are noisy and peer effects estimators are not statistically significant in real. Finally, regarding peer effects in Reading, the estimators are statistically significant and robust to different specifications of explanatory variables. Before we discuss the significant result for peer effects in Reading, we will conduct a placebo check to understand deeper the properties of our peer effects estimation method.

## 3.3 Placebo Test

The peer effects estimators from the previous part show another interesting pattern – all endogenous peer effects estimators are negative. Therefore, arises the possibility that peer effects estimators are jointly significant. However, it could imply that peer effects estimators are also unrealistically high in comparison with previous research. Hence, we will examine if our estimators are biased.

To examine the bias of peer effects estimators, we permutate vectors of explanatory variables $\boldsymbol{x}$ and outcome variables $\boldsymbol{y}$. Importantly, we permutate $\boldsymbol{x}$ and $\boldsymbol{y}$ together, and hence the value of the explanatory variable $x_i$ of individual pupil remains connected to her outcome variable $y_i$. Since we do not change the network matrix, the permutation of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be interpreted as a random distribution of pupils among classes. Consequently, peer effects estimators with permutated characteristics of pupils should be zero.

**Table 4: Placebo Analysis of the Bias of Peer Effects Estimators**

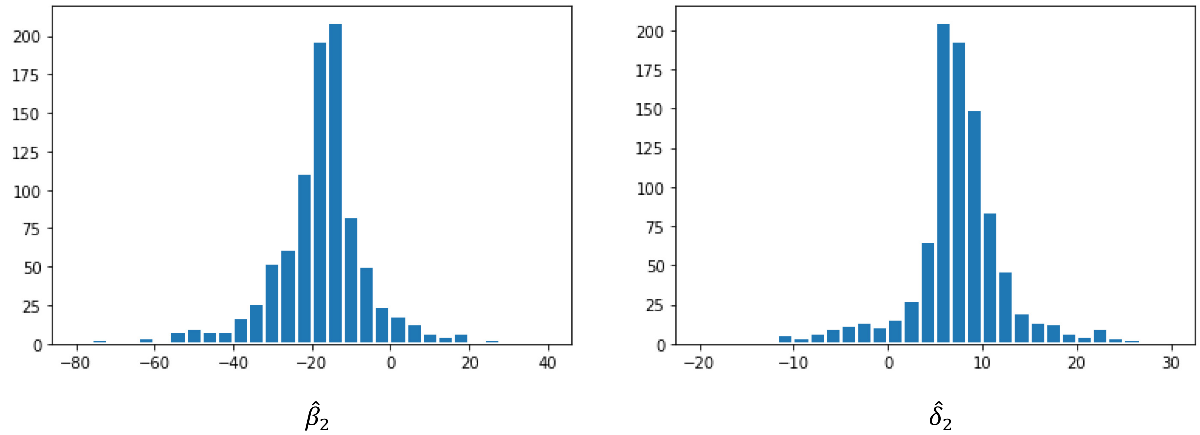
The table shows the efficient estimators of endogenous peer effects $\hat{\beta}_2$, the estimators of exogenous peer effect $\hat{\delta}_2$, and coefficient of explanatory variable $\hat{\gamma}_2$ in the sample of pupils randomly allocated between classes. Outcome variable is test score in 9th grade from particular subject and explanatory variable is 6th grade score from the same subject. Standard deviations implied by the estimator of variance matrix are in parenthesis. Expected bias of peer effects estimators is used for the correction of peer effects estimators from table 2 examining if the estimators are significantly different from corresponding bias.

| | $\hat{\beta}_2$ | $\hat{\gamma}_2$ | $\hat{\delta}_2$ |
|---|---|---|---|
| **Math** | −13<br>(136) | 0.55<br>(1.49) | 7.5<br>(82.4) |
| **Reading** | −18<br>(166) | 0.39<br>(1.46) | 7.4<br>(91.2) |
| **English** | −20<br>(216) | 0.58<br>(1.50) | 12<br>(100) |
| **Learning Skills** | −29<br>(204) | −0.57<br>(1.14) | 17<br>(138) |
| | **Updated bootstrap p-values of peer effects estimators from table 2** | | |
| **Math** | 91.2% | – | 86.1% |
| **Reading** | 66.9% | – | 48.1% |
| **English** | 53.8% | – | 29.4% |
| **Learning Skills** | 19.2% | – | 19.8% |

For placebo analysis, we use test scores from 6th grade as the explanatory variable and test scores from 9th grade in the same subject as the outcome variable. The results are in table 4 and we can observe several patterns. First, all results are statistically insignificant and standard errors are relatively high in the comparison with standard errors from previous analysis of peer effects (in table 2 and table 3). Standard errors are high due to heavy tails of the distribution of peer effects and we have observed it in the case of bootstrap simulation despite we have not reported them in table 2. To stress that the variance of most of the estimators is lower, we report histograms of peer effects estimators in Reading in figure 4. We can see that the mean values of estimators correspond to median values and it seems that in Reading, endogenous peer effects are downwardly biased for 18 points and exogenous peer effects are upwardly biased for 7 points.

**Figure 4: Distribution of Peer Effects Estimators in Reading from Placebo Analysis**

Endogenous peer effects estimators are $\hat{\beta}_2$ and exogenous peer effects estimators are $\hat{\delta}_2$.




Taking observed biases into account significantly lowers p-values of estimated peer effects and the updated values are in the second part of table 4. Specifically, on the example of Reading, we measure the probability that endogenous peer effects estimator is less than 18 and exogenous peer effects estimator is higher than 7.4. To conclude, it is probable that significant peer effects estimators in our previous analysis were caused by the bias of the used estimators.

We hypothesize that both, underestimation of the standard errors of peer effects estimators and the bias of peer effects estimators can be caused by small sample size and relatively low class-size variation. New estimators are usually tested on appropriate samples and our sample size is more than ten times lower than the sample size used by Boucher et al. (2014). Boucher et al. (2014) also show that the theoretical condition of having at least three class sizes is not practically sufficient for the identification of peer effects and one also needs sufficient class size variation. Therefore, the hypothesis that standard estimators do not perform well in our setting is, in our opinion, the most realistic explanation of observed results.

## 3.4 Simple Analysis of Social Effects

As we argued, peer effects estimators seem to be insignificant in our analysis. The theoretical part of our thesis suggests that the estimation of endogenous and exogenous peer effects is rather difficult and requires a rich data sample. Boucher et al. (2014) use a similar method and their data sample contains 116,534 students in contrast with 1,407 students in our case. However, pupils do not take all tests, and therefore sample sizes vary across different subjects. However, they conduct Monte-Carlo simulations with a sample size of approximately 42,000 pupils, which can be regarded as a sufficient sample size for the analysis of peer effects. Nevertheless, we would like to analyse peer effects within our possibilities, for example, to provide some advice for policymakers of parents of pupils. Therefore, we ask whether it is "better" to attend class with high-achieving peers or low-achieving peers. This means that we want to measure the total effect of peers on the expected score of individuals.

We propose a comparison of class averages in 6th and 9th grade. If peer effects are positive, then the class with a relatively high average test score will increase its average relative to the class with a low average test score. Since aggregated test scores are normalised to have zero mean and unit variance the changes in class averages are related to the specific mechanism within the classes. If we conduct OLS regression

$$\bar{y}_i^{9th} = a + b\bar{y}_i^{6th} + \varepsilon_i$$

where $\bar{y}_i^{9th}$ is class average score in 9th grade and $\bar{y}_i^{6th}$ is class average score in 6th grade. Consequently, $b > 1$ implies that the high-achieving class increases its average score relative to the low-achieving class. We expect that the potential effect could be caused by peer effects and the effect of class sizes since we observe the selection of pupils according to class sizes (figure 3). Therefore, we include also the effect of class size

$$\bar{y}_i^{9th} = a + b\bar{y}_i^{6th} + cN_i + \varepsilon_i$$

where $N_i$ is the number of pupils in a class. The regression coefficients are reported in table 5. The results show that coefficient $\hat{b}$ is less than one for all four subjects. Moreover, coefficient $\hat{b}$ is significantly different from one for Reading, English, and Learning Skills. In the case of Math, coefficient $\hat{b}$ is statistically indistinguishable from one. Regarding the coefficient $\hat{c}$ describing the effect of class size, we observe a positive and statistically significant effect on Learning Skills. Finally, we calculate the average of coefficients across all four subjects using the delta method for the calculation of standard errors. The coefficient $\hat{b}$ across all subjects is significantly less than one and the coefficient $\hat{c}$ is significantly positive.

**Table 5: Regression of the class average score in 9th grade on class average score in 6th grade and class size.** Estimator $\hat{a}$ estimates constant term, estimator $\hat{b}$ estimates the effect of 6th grade, and $\hat{c}$ estimates the effect of class size.

| | $\hat{a}$ | $\hat{b}$ | $\hat{c}$ |
|---|---|---|---|
| **Math** | 0.08<br>(0.33) | 0.961<br>(0.099) | −0.003<br>(0.013) |
| **Reading** | −0.29<br>(0.36) | 0.549<br>(0.090) | 0.012<br>(0.014) |
| **English** | −0.36<br>(0.28) | 0.761<br>(0.074) | 0.014<br>(0.011) |
| **Learning Skills** | −1.00<br>(0.52) | 0.700<br>(0.091) | 0.040<br>(0.020) |
| **Total Effect** | −0.39<br>(0.19) | 0.743<br>(0.044) | 0.0156<br>(0.0075) |

To interpret observed results, if peer effects are a significant mechanism affecting test scores of pupils, then these peer effects are negative. However, since we conduct a simple OLS regression omitting many possibly relevant variables, there can be a different mechanism causing observed effects. For example, if we had two types of classes with normally distributed pupils test scores with means -1 and 1, then an increase of variance of the distribution of test scores would reduce the difference between class averages since we always normalise aggregated scores to have unit variance. Consequently, we would observe coefficient $\hat{b}$ smaller than one even in the absence of true peer effects. On the other hand, the observed pattern can be still relevant for parents of policymakers. Class averages converge together over time, and hence it may not be so important whether an individual attends a low- or high-achieving class.

# 4 Further Research

The linear in means peer effects model is the simplest model describing peer interactions, and hence do not capture several phenomena that could occur in real world. First, pupils are affected by the mean characteristics of their peers regardless of other characteristics of the peers' distribution, for example, variance. Second, the aggregated effect of the reallocation of pupils on their educational outcomes is zero. Third, all pupils are affected by their peers in the same magnitude regardless their individual characteristics. We know that the above-mentioned limitations of the linear-in-means peer effects model should be relevant thanks to the experiments conducted by Duflo et al. (2011) and Carrell et al. (2013). Both studies showed that the manipulations with class compositions have a measurable impact on the pupils educational outcomes.

Note that all three limitations of the linear-in-means peer effects model are tightly connected since, for example, including the effect of the variance of the educational outcomes of peers would imply that manipulating homogeneity of classes would have a non-zero aggregate effect. Similarly, including heterogeneity of peer effects would predict that reallocation of pupils could have a non-zero aggregate effect. Recall the linear-in-means peer effects model [equation (2)]

$$y_i = \alpha + \beta \frac{1}{n_i} \sum_{j \in P_i} y_j + \gamma x_i + \delta \frac{1}{n_i} \sum_{j \in P_i} x_j + \varepsilon_i$$

Now we can add two terms to describe the effect of the variance of peers' outcomes and characteristics.

$$y_i = \alpha + \beta_1 \frac{1}{n_i} \sum_{j \in P_i} y_j + \beta_2 \frac{1}{n_i} \sum_{j \in P_i} y_j^2 + \gamma x_i + \delta_1 \frac{1}{n_i} \sum_{j \in P_i} x_j + \delta_2 \frac{1}{n_i} \sum_{j \in P_i} x_j^2 + \varepsilon_i$$

Regarding the heterogeneity of peer effects, we could add the interaction of peer effects with the characteristic of individual

$$y_i = \alpha + \beta_1 \frac{1}{n_i} \sum_{j \in P_i} y_j + \beta_2 \frac{1}{n_i} \sum_{j \in P_i} y_j^2 + \beta_3 \frac{1}{n_i} \sum_{j \in P_i} y_j \, x_i + \gamma x_i + \delta_1 \frac{1}{n_i} \sum_{j \in P_i} x_j + \delta_2 \frac{1}{n_i} \sum_{j \in P_i} x_j^2 + \delta_3 \frac{1}{n_i} \sum_{j \in P_i} x_j x_i + \varepsilon_i$$

Obviously, this is one of many ways how it is possible to extend the peer effect model. We could, for example, substitute the second moment $\sum_{j \in P_i} y_j^2$ by the estimator of variance $\sum_{j \in P_i} \left( y_j^2 - \frac{1}{n_i} \sum_{j \in P_i} y_j \right)$. Different specifications would change the interpretation of estimators; however, the described information would be the same.

Despite, it is relatively simple to extend the linear-in-means peer effects model, it is much more difficult to discuss identification in the extended model. Our theoretical analysis in section 2 uses mathematical tricks which cannot be applied in the extended model. It is also out of the scope of our thesis to describe the identification of the extended model and we leave it for further research.

Finally, we would like to discuss the role of endogenous network formation. For example, Carrell et al. (2013) observe that pupils create an endogenous network of friends within their classes. They hypothesise that this endogenous friendship network may be more relevant for peer interactions than the interactions within the whole class. Therefore, the research of peer effects should take friendship networks into account directly observing network or simulating it based on the observed characteristics. The simulations of endogenous network formation were examined for example by Hsieh and Lee (2016) and Goldsmith-Pinkham and Imbens (2013). To conclude, promising ways for further research of peer effects include, in our opinion, the description of non-linearities, heterogeneities, and more detailed friendship networks

# Summary

Research on peer effects is a current and ever-expanding field in human sciences. Our literature review supports its importance and provides numerous examples of real-life environments where peer effects are present. In the second part, we provide a theoretical analysis of the widely used linear-in-means peer effects models. Although the linear-in-means peer effects model has already been analysed in previous studies, often the analyses are rather brief and sometimes lack proofs of particular arguments. Hence, we contribute to the field by providing step-by-step derivations of different functional forms of the peer effects model, motivating crucial definitions using simple examples and intuition, and proving particular statements that have not been proven in previous studies.

In the empirical part, we analyse peer effects in education among pupils in junior secondary schools in the Czech Republic. We use a subsample of Czech pupils since the data allow us to observe the explanatory variable for test scores that is not-affected by peer effects. We find negative and statistically significant endogenous peer effects and positive and statistically significant exogenous peer effects in Math, Reading, and English. However, our estimates are point unrealistically high, which may be explained by several factors. We conduct bootstrap simulation for the calculation of standard errors and significance levels and we show that the estimators of standard errors are likely beign underestimated. We also conduct a placebo check randomly distributing pupils among classes and show that our peer effects estimators are slightly biased. Both, the bootstrap simulations and a placebo check suggest that our peer effects estimators are in reality statistically insignificant. We hypothesise that possibly incorrect results from standard peer effects estimation may be caused by a small sample size and a relatively small class-size variation.

Finally, we provide a summary of further extensions of peer effects models that can be used in peer effects research. We discuss the extensions along two main dimensions. We suggest introducing non-linearities into the peer effects mode, which means that peer effects are not linear in means and can depend on higher powers of explanatory variables. However, it increases the demand on the sample sizes of the data. The second dimension is to introduce heterogeneity in

peer effects. This means that pupils do not affect all other pupils in their classroom in the same magnitude and peer effects should also depend on the characteristics of the affected individual.

# References


Ali, M.M., Dwyer, D.S., 2011. Estimating Peer Effects in Sexual Behavior Among Adolescents. Journal of Adolescence 34, 183–190. https://doi.org/10.1016/j.adolescence.2009.12.008

Banerjee, A., Chandrasekhar, A.G., Duflo, E., Jackson, M.O., 2013. The Diffusion of Microfinance. Science 341, 1236498–1236498. https://doi.org/10.1126/science.1236498

Betts, J.R., n.d. Peer Groups and Academic Achievement: Panel Evidence from Administrative Data 35. Unpublished manuscript.

Boucher, V., Bramoullé, Y., Djebbari, H., Fortin, B., 2014. Do Peers Affect Student Achievement? Evidence from Canada Using Group Size Variation: Do Peers Affect Student Achievement? J. Appl. Econ. 29, 91–109. https://doi.org/10.1002/jae.2299

Bramoullé, Y., Djebbari, H., Fortin, B., 2009. Identification of Peer Effects Through Social Networks. Journal of Econometrics 150, 41–55. https://doi.org/10.1016/j.jeconom.2008.12.021

Burke, M.A., Sass, T.R., 2008. Classroom Peer Effects and Student Achievement 47. Working paper.

Card, D., Giuliano, L., 2013. Peer Effects and Multiple Equilibria in the Risky Behavior of Friends. The review of Economics and Statistics 95(4), 1130–1149.

Carrell, S., Sacerdote, B., West, J., 2013. From Natural Variation to Optimal Policy? The Importance of Endogenous Peer Group Formation. Econometrica 81, 855–882. https://doi.org/10.3982/ECTA10168

CLOSE, Institute for Research and Development of Education [ online ]. 2014, 2021 [Retrieved on a date]. Available from: https://pages.pedf.cuni.cz/uvrv-en/close-czech-longitudinal-study-in-education/

Duflo, E., Dupas, P., Kremer, M., 2011. Peer Effects, Teacher Incentives, and the Impact of Tracking: Evidence from a Randomized Evaluation in Kenya. American Economic Review 101, 1739–1774. https://doi.org/10.1257/aer.101.5.1739

Duflo, E., Saez, E., 2002. Participation and Investment Decisions in a Retirement Plan: The Influence of Colleagues' Choices. Journal of Public Economics 85, 121–148.

Eisenberg, D., Golberstein, E., Whitlock, J.L., 2014. Peer Effects on Risky Behaviors: New Evidence from College Roommate Assignments. Journal of Health Economics 33, 126–138. https://doi.org/10.1016/j.jhealeco.2013.11.006

Feld, J., Zölitz, U., 2017. Understanding Peer Effects: On the Nature, Estimation, and Channels of Peer Effects. Journal of Labor Economics, 2017, vol. 35, no. 3 387–428.

Goldsmith-Pinkham, P., Imbens, G.W., 2013. Social Networks and the Identification of Peer Effects. Journal of Business & Economic Statistics 31, 253–264. https://doi.org/10.1080/07350015.2013.801251

Gershgorin Circle Theorem, Wikipedia [ online ]. 2021, 2021 [Retrieved on a date]. Available from: https://en.wikipedia.org/wiki/Gershgorin_circle_theorem

Hsieh, C.-S., Lee, L.F., 2016. A Social Interactions Model with Endogenous Friendship Formation and Selectivity: A Social Interactions Model. J. Appl. Econ. 31, 301–319. https://doi.org/10.1002/jae.2426

Lee, L., 2007. Identification and estimation of econometric models with group interactions, contextual factors and fixed effects. Journal of Econometrics 140, 333–374. https://doi.org/10.1016/j.jeconom.2006.07.001

Lefgren, L., 2004. Educational peer effects and the Chicago public schools. Journal of Urban Economics 56, 169–191. https://doi.org/10.1016/j.jue.2004.03.010

Manski, C., 1993. Identification of endogenous social effects: The reflection problem. Review of Economic Studies 60 (3), 531–542.

Miguel, E., Kremer, M., 2004. Worms: Identifying Impacts on Education and Health in the Presence of Treatment Externalities. Econometrica 72, 159–217. https://doi.org/10.1111/j.1468-0262.2004.00481.x

Moffitt, R., 2001. Policy interventions low-level equilibria, and social interactions. In: Durlauf, Steven, Young, Peyton (Eds.), Social Dynamics. MIT Press.

Nakajima, R., 2007. Measuring Peer Effects on Youth Smoking Behaviour. The Review of Economic Studies 74, 897–935. https://doi.org/10.1111/j.1467-937X.2007.00448.x

Neidell, M., Waldfogel, J., 2010. Cognitive and Noncognitive Peer Effects in Early Education. The review of Economics and Statistics, 92(3), 562–576.

Nicoletti, C., Salvanes, K.G., Tominey, E., 2018. The Family Peer Effect on Mothers' Labor Supply. American Economic Journal: Applied Economics 10, 206–234. https://doi.org/10.1257/app.20160195

Sacerdote, B., 2011. Peer Effects in Education: How Might They Work, How Big Are They and How Much Do We Know Thus Far? Handbook of the Economics of Education. Elsevier, pp. 249–277. https://doi.org/10.1016/B978-0-444-53429-3.00004-1

Trogdon, J.G., Nonnemaker, J., Pais, J., 2008. Peer effects in adolescent overweight. Journal of Health Economics 27, 1388–1399. https://doi.org/10.1016/j.jhealeco.2008.05.003

Vigdor, J.L., Nechyba, T.S., 2007. Peer effects in North Carolina public schools. In: Woessmann, L., Peterson, P.E. (Eds.), Schools and the Equal Opportunity Problem. The MIT Press, pp. 73–101.