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WHETHER AND WHERE TO APPLY*

Information and Discrimination in Matching with Priority Scores

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Abstract

This paper considers a matching market where agents with privately known priority scores apply to acquire one good. The analysis characterizes the Bayes-Nash equilibria, computes welfare ex-ante and interim, and discusses implications for market design. Three main findings emerge. First, the symmetric equilibrium necessarily involves randomization between applications. Second, it exhibits a block structure: agents sort into a finite number of classes of neighboring scores where they use the same strategy. Third, the intermediate-score agents bear most of the inefficiencies, whereas the low-score agents may be better off under private information than under public information. In total, private information mitigates priority-based discrimination.

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1 Whether and Where to Apply

Matching or hunting

This paper studies a simple incentive and welfare problem in a matching market. There are three agents competing for two goods. The goods are indivisible and of heterogeneous qualities. A clearinghouse has forbidden prices and has assigned a priority score to each agent, where each agent observes only her score. The clearinghouse allows each agent to apply to at most one good and assigns each good to the agent with the highest score among the pool of applicants to the good. The questions are: To what goods do agents apply, and what goods do agents receive?

An alternative interpretation of the same problem could be about hunting. There are three predators, meeting two prey at the same time. The prey are of different sizes or caloric values. Each predator is characterized by its privately known strength. It has a time or capacity constraint, implying that it can attack only one prey before all the prey runs away. If two predators attack the same prey, only the stronger predator will earn the prey. The questions are: What prey do predators attack, and what prey do predators eat?

These are simple questions with complex answers. The complexity lies in two features of the problem: truncation on the mechanism (agents can apply to at most one good, predators can attack at most one prey) and private information (on the priority scores, on the strength type).

In the paper, we mostly insist on the matching economic interpretation of the problem. The rest of the introduction aims at emphasizing how prevalent the truncation of the mechanism and the private information on priority scores are in real-life matching problems. The rest of the paper will emphasize how relevant these two frictions are for equilibrium applications and matching.

Matching and truncation

Matching is the formation of productive partnerships, with numerous applications to marriage, labor, housing, college admissions, organ donation, and many more. In centralized matching markets, the mechanism summarizes the rules used to match participants together, in particular, the information that participants have, and the allocation rule used to match participants together. In a direct ordinal rule, agents are asked to report their ordinal preferences; they rank available options on the market. A celebrated direct ordinal rule is deferred acceptance, proposed by Gale and Shapley (1962) [12] and widely used in practice (Abdulkadiroglu and Sonmez (2003) [1]). These rules can be either untruncated or truncated.

In an untruncated allocation rule, participants can rank all available options on the market. Incentives in untruncated allocation rules are generically simple and well-understood. Untruncated deferred acceptance is strategy-proof, meaning that a dominant strategy for any participant is to report one's own preferences.

In a truncated allocation rule, participants can only submit a limited number of options. Incentives in a truncated rule are complex and still not fully understood. Romero-Medina [24] has shown that truncating deferred acceptance sacrifices strategy-proofness. The relevant solution concept becomes Nash equilibrium. At Nash equilibrium, participants may manipulate their preferences. They solve a trade-off between submitting attractive options, options ranked high in their true preferences, and submitting safe options, options where their chances of being accepted are high. The way this trade-off resolves depends finely on the preferences and priorities, and on the information participants have about their preferences and priorities. Under complete information, the Nash equilibrium solution in truncated deferred acceptance is well described by Romero-Medina (1998) [24], and Haeringer and Klijn (2009) [14].

Yet, little is known about equilibrium applications when participants also have limited information about their priorities. Interesting exceptions are Chade Lewis and Smith 2014) [7], Ali and Shorrer (2021) [2], and Avery and Levin (2010) [3]. These papers model a general and exogenous noise on ordinal priorities and study the equilibrium applications and matching. In this paper, we study the same problem when priorities are defined with cardinal scores. This structure of priorities generates an endogenous uncertainty on the ordinal priorities and fits many empirical applications. It yields rich predictions in terms of both the equilibrium applications and the resulting matching.

Priority scores and private information

A participant's priority score summarizes the participant's characteristics in a single real number. A higher score is associated with a higher chance of obtaining a desirable partner. The score defines homogeneous priorities (the score controls the priority of an agent to any good) and cardinal priorities (rather than ordinal). Examples of real-life markets using scores include social housing, where households are assigned a score reflecting the emergency of the housing need; college admissions, where a student is assessed based on her performance in a standardized test; civil servant labor markets such as teacher or doctor allocation, where civil servants accumulate points for experience or performance; or credit, where the credit score measures an individual's creditworthiness. Clearinghouses usually present these scores as tools to finely discriminate between agents on the market.

The issue is that the default information structure on the priorities, absent any intervention by the clearinghouse, is private information. An agent observes her own priority score but is ignorant of other agents' scores. This is because, in general, these scores are computed based on criteria of private circumstances. Participants only know their own status with respect to all the criteria. In social housing, the household knows whether she is homeless, or has health issues, but is uninformed about the competitor households' situations. In college admissions, students receive their exam grades in sealed envelopes.

The empirical matching literature supports this intuition. It provides evidence that matching market participants are bad at comparing themselves with other applicants, and are poorly informed about their priorities (Kapor, Neilson, and Zimmerman (2020) [16], Fabre et al. (2021) [11]).

Efficiency

The mechanism with truncation and private information is obviously suboptimal. Either removing the truncation or switching to public information would restore efficiency. A natural question is then: is it even relevant to study this kind of mechanism? In section §2.1, we argue that it is highly relevant, for both descriptive and normative reasons. On the descriptive side, we provide examples of markets (social housing in Europe) designed exactly in this way. On the normative side, we argue that the solution to such a simple matching or hunting problem would be known as a theoretical benchmark and for market design ambitions.

Overview of model

In this paper, we explicitly model the matching problem with a truncated mechanism and privately known priority scores. We define a stylized two-sided, one-to-one, agent-good matching market, with non-transferable utilities. Preferences and priorities are homogeneous, meaning that each good is characterized by a unique objective value, and conversely, each agent is endowed with a single (privately known) priority score. The allocation occurs through a truncated deferred acceptance mechanism. Agents independently and simultaneously decide to request one or no goods, where application may be costly. Then, each good goes to the highest priority agent in the pool of applicants. We model strategic interactions on the market as a Bayesian Game of incomplete information termed "Application Game". A participant wonders "Whether and Where to Apply?".[3] She solves a trade-off between being "ambitious", targeting high-value goods, or being "pragmatic", targeting under-demanded goods. The solution to this trade-off depends finely on other participants' strategies and scores, so ultimately, on the agent's private information about the scores. We characterize application behaviors as given by the Bayes-Nash equilibria of the Application Game. We also compute the resulting allocation and the welfare, and compare it to the maximal welfare when scores are public information. The welfare analysis characterizes how much each agent

[3] Because the model is static, agents do not wonder *when* to apply.

suffers from the uncertainty, as a function of the score.

Preview of results

The analysis uncovers several salient features of the symmetric equilibrium. In this equilibrium, agents randomize between the various application actions. The equilibrium strategy has a "block structure": agents with scores on a continuous support sort into discrete classes (defined as intervals of priority scores) where they make exactly the same mixed application. At equilibrium, high-score agents are ambitious, and low-score agents are pragmatic, but only in expectation.

The welfare analysis shows that the frictional mechanism is less efficient but more egalitarian than a frictionless mechanism. The intermediate score agents are the ones suffering from most of the burden of the inefficiencies. When applying is costly, low-score agents participate more than under public information and may benefit from private information.

Outline of paper

The rest of the paper is structured as follows. Section §2 models the market and the associated game. Section §3 solves the game and computes the welfare, when applying is free and participants only wonder “where to apply”, and when application is costly and agents also wonder “whether to apply”. Section §4 discusses the main results, a few extensions, and the related literature. Section §5 concludes.

All proofs are available in appendix §A.

2 Application Game

2.1 The market

Matching with privately-known priority scores

We consider a market with n agents and m goods, with congestion ($m \leq n$). Agents (resp. goods) are numbered by $i \in \{1, \ldots, n\}$ ($j \in \{1, \ldots, m\}$).[4]

Goods have heterogeneous qualities, denoted v^j for good $j \in \{1, \ldots, m\}$, and all agents agree on the quality. The convention is that good 1 (resp. m) is the highest (lowest) quality good: $v^m < \ldots < v^1$. Each agent i is characterized by a unique priority score, denoted x_i. Priority scores are private information: an agent i observes her own priority score x_i, but is ignorant of the priority scores x_j of other agents $j \neq i$. Priority scores are independently and identically distributed according to some atom-less distribution with support unit interval with cdf F: $x_i \sim F([0,1])$, $i \in \{1, \ldots, n\}$.

The allocation occurs through a truncated deferred acceptance rule, with truncation $m-1$: agents are asked to independently and simultaneously choose whether to apply to a good and, if yes, to which good. Applying may be costly, it costs $c \geq 0$. Because priorities are common, the mechanism is equivalent to a serial dictatorship where the serial order is given by the score. If a given good receives no application, then it is wasted. If a good receives exactly one application, it goes to the single applicant. If a good receives at least two applications, the mechanism selects the agent with the highest priority score among the pool of applicants and endows this agent with the good.[5] We call "frictional mechanism" and denote ϕ^F the mechanism with private information on the priority scores and truncated deferred acceptance allocation rule.

We say that an agent who applies to a good and gets it succeeds, while an agent who applies and remains unassigned fails. A successful agent receives the value of the good

[4]Notation: In all the following, numerals for agents (goods) are written in subscripts (superscripts).

[5]Due to the continuous prior distribution, ties between two agents with exactly the same score applying to the same good occur with probability 0. The tie-breaking rule has no consequence on payoffs and equilibrium behaviors, and we do not specify it.

she is assigned minus the application cost. An agent who fails just pays the application cost. An agent who does not apply receives a reservation utility of zero.

The mechanism with truncation and private information is obviously suboptimal. Either removing the truncation or switching to public information would restore efficiency. A natural question is then: is it even relevant to study this kind of mechanism? We argue that it is highly relevant for two main reasons.

Descriptive argument and leading example of social housing in Europe

On the descriptive side, these mechanisms are widely used in practice. Truncation is the rule, rather than the exception, in many applications. In school choice or labor markets, students/civil servants can apply to only a limited number of universities/job positions. The private information on priority scores is a consequence of the priority scores being based on private circumstances. The table in Appendix §B.1 provides an example, showing the scoring rule used in London social housing. Households are awarded priority points if they are currently homeless, have large families, or have health issues. A household knows about her status with respect to every criterion and, when given this table, can compute her aggregate priority score. In many circumstances, clearinghouses could fix the private information by explicitly publishing information on the priorities, but they rarely do so.

A prominent example is the assignment of social housing units in Paris. Since 2016, the municipality has allocated around 4,500 housing units a year through an online scheme called "*LOC'annonces*". The allocation occurs in three steps. In the first step, households register as social housing seekers. The market operator performs eligibility checks and places households in rent and bedroom categories depending on their earnings and sizes. Most importantly, households are assigned priority scores based on their circumstances. The computation mode awards points for homelessness or unsuitable current housing (overcrowding), ill-health status, and other criteria. In the second step, households apply for vacant housing units. More specifically, vacancies are advertised on a dedicated website[6] from Tuesday morning until the following Wednesday midnight. Households apply to one housing unit per round or choose not to apply. Very importantly, no precise feedback information is provided on the identities of other applicants to the same housing units. Following the application closing, applicants who have applied and who are on a shortlist of the highest priorities can view the accommodation and decide to maintain or withdraw their bids. In the third and final step, each vacant property goes to the applicant with the highest priority score among those who have applied for it. The whole allocation process, from the application closing to the final allocation, can take a maximum of three months.[7] During the three-month period, households may miss attractive opportunities from the private sector.

The London social housing allocation scheme ("Choice Based Lettings Scheme") much resembles the Paris scheme, except for the information. On the dedicated website,[8] the application period runs from each Thursday morning until the following Monday midnight. During this period, when a household applies, she observes her position in a priority ranking of all current applicants to her targeted housing unit. Through trials and withdrawals, it is then possible to recover common knowledge on (the order of) priority scores.

Why exactly these mechanisms are so popular remains an open question. We argue that clearinghouses may have hidden social objectives. They may not only care about efficiency, but also about participation. This paper will show that the introduction of frictions may improve participation. They may also face operating costs when receiving many applications. In school choice, Pathak (2016) [22] reports that it is costly to review application files. In social housing, it is costly to organize viewings. If the operating cost exceeds the social cost of waste, the mechanism with frictions may become

[6] `https://teleservices.paris.fr/locannonces`.

[7] The example of social housing is, in fact, dynamic, with successive rounds of application. An agent who does not apply or fails in a given round is offered the opportunity to apply again in the next round. With a static model, we certainly miss this aspect. We address this issue in section §4.2 by introducing into the static model an endogenous cost of participation, which captures one of the main effects of the dynamic. The cost decreases with the priority, reflecting that high-priority agents keep their high score in successive rounds.

[8] `www.homeconnections.org.uk`.

optimal. In any case, the popularity of these mechanisms implies economists have a responsibility to understand how they work before even reforming them.

Normative argument

On the normative side, this kind of matching mechanism or the meeting between predators and prey described above generates a simple and general incentive problem "whether and where to apply?". We have described two very different situations where this issue could arise; there are many more. The solution to this problem, as well as the welfare implications, should be known as a theoretical benchmark. In this sense, this work seeks to address a gap in the economic literature. The welfare aspects are also worth investigating, beyond the coarse criterion of ex-ante welfare. As economists, we should study how welfare spreads among different agents. This approach is key for market design ambitions, since it assesses how a new policy (in this case, how a change in the mechanism) could affect each agent.

2.2 The game

Bayesian game

This market defines a symmetric Bayesian Game of incomplete information that we call "Application Game". This game comprises n players, with action space $A_i = \{A^1, \dots, A^m, N\}$ - where A^j denotes the action of applying to good j and N[9] stands for the action of not applying -, privately known i.i.d. types, prior F over $[0,1]$, and ex post payoffs:

$$u_i(a|x) = \begin{cases} v^j - c & \text{if } a_i = A^j \text{ and } \{l \in \{1,\dots,n\}\backslash\{i\} | a_l = A^j,\ x_l > x_i\} = \emptyset,\ j \in \{1,\dots,m\} \\ -c & \text{if } a_i = A^j \text{ and } \{l \in \{1,\dots,n\}\backslash\{i\} | a_l = A^j,\ x_l > x_i\} \neq \emptyset,\ j \in \{1,\dots,m\} \\ 0 & \text{if } a_i = N \end{cases}$$

where $u_i(a|x)$ denotes the payoff of player i under action profile $a = (a_i, a_{-i})$ and score

[9] Occasionally, to ease the notations, we will denote the action N as an additional application action, $A^{m+1} := N$, and define $v^{m+1} := c$.

profile $x = (x_i, x_{-i})$.

The strategies

A pure strategy $s_i : [0,1] \to \{A^1, \dots, A^m, N\}$ in the Application Game is a mapping from the interval of scores to the set of available actions (distributions over actions). A mixed or behavioral strategy $p_i : [0,1] \to \triangle\{A^1, \dots, A^m, N\}$ is a mapping from the support of scores into the simplex of the action set. The notation $p_i^j(x_i),\ j \in \{1, \dots, m+1\}$ stands for the probability that agent i chooses action A^1 when her score is x_i. A strategy is interior whenever the distribution of application probabilities $(p_i^j(x_i))_{j\in\{1,\dots,m+1\}}$ is non-degenerate on more than a finite number of scores x_i, that is when $\exists\ x_i' < x_i'' \in [0,1],\ j \in \{1, \dots, m+1\}$ s.t. $\forall\ x_i \in [x_i', x_i'']:\ 0 < p_i^j(x_i) < 1$.

We introduce below a specific family of strategies.

Definition 1. *A strategy p_i is **block** if $x_i \mapsto p_i(x_i)$ is piece-wise constant, with a finite number of pieces.*

In a block strategy, there exists a finite number of intervals of scores, where the agent applies to the various goods with fixed probabilities.[10] In a block strategy, agents with different scores but belonging to the same interval use exactly the same mixture over application actions. We call these intervals of scores *classes*, the bounds of the classes *thresholds*, and the application probabilities *levels*.

The expected payoffs

We abuse notations and write all expected payoffs for player i with $u_i()$. In particular, the interim expected payoff of player i under strategy profile p, when her priority score is x_i is denoted $u_i(p|x_i) := \mathbb{E}_{x_{-i}}[u_i(p|x)]$. In the Application Game, the interim payoffs depend on the strategy of other agents through the behaviors of higher score agents, yet are independent of the behaviors of lower score agents. Trivially, the interim payoffs also depend only on the agent's strategy through the agent's behavior at the set

[10] A pure strategy is trivially block: within each class, the agent applies with fixed probability one to exactly one good. But block strategies can also be interior.

score and not at higher or lower scores. Thus in the formalism for interim payoffs, it is enough to specify for p just $p_i(x_i)$ for the agent, and $p_{-i}([x_i,1])$ for the other agents: $u_i(p|x_i) = u_i(p_i(x_i), p_{-i}([x_i,1])|x_i)$.

When the agent plays a pure application action, applying to good j at score x_i, the interim payoff writes as a product between good j's value and a probability of success conditional on applying to good j, denoted $q_i^j(x)$, minus the cost:

$$u_i(A_i^j, p_{-i}([x_i,1])|x_i) = q_i^j(x_i)v^j - c \tag{1}$$

For the application to good j to be successful, it must be that no other agent with a higher score applies to good j. The probability of success is:

$$q_i^j(x_i) = \prod_{i' \neq i} \left(1 - \mathbb{P}\big(x_{i'} > x_i, p_{i'}(x_{i'}) = A_{i'}^j\big)\right)$$

The following lemma characterizes the interim payoff (1).

Lemma 1.

(i) <u>*Continuity:*</u> *Interim payoff conditional on any action is continuous in the score.*

(ii) <u>*Monotonicity:*</u>

- *Interim payoff conditional on not applying is constant equal to* 0*.*
- *Interim payoff conditional on any application action is increasing (constant) on any interval of scores where at least one other agent (no other agent) applies to the same good with a positive probability.*

(iii) <u>*Value at highest bound:*</u> *Interim payoff at score 1 conditional on applying to any good, is equal to the value of the good minus the cost.*

Statement (i) says that there is no jump in the interim payoff: one's chance to get a good, hence one's payoff, cannot dramatically change from one score to a nearby score. Statement (ii) says that one's payoff increases when one's score rises if and only if agents with scores slightly higher are applying to the same good. Statement (iii) says that the highest score agent is successful for sure.

These statements will be key in proving the main results. They are direct consequences of the serial dictatorship mechanism and the continuity of the priority score support.

Bayes-Nash equilbrium

Our solution concept is the Bayes-Nash equilibrium. A Bayes-Nash equilibrium of the Application Game is a strategy profile p^* that solves the following equivalent maximisation problems:

$$\textit{Ex ante: } \forall\ i \in \{1,\ldots,n\}:\ p_i^* \in \underset{p_i:[0,1]\to\triangle\{A^1,\ldots,A^m,N\}}{\arg\max}\ u_i(p_i, p_{-i})$$

$$\textit{Interim: } \forall\ i \in \{1,\ldots,n\},\ \forall\ x_i \in [0,1]:\ p_i^*(x_i) \in \underset{p_i(x_i)\in\triangle\{A^1,\ldots,A^m,N\}}{\arg\max}\ u_i(p_i(x_i), p_{-i}([x_i,1])|x_i)$$

We have existence in the generic case:

Lemma 2. *There exists a Bayes-Nash equilibrium of the Application Game.*

The proof is by the Bayes-Nash existence theorem for games with finite action space and independent types (potentially infinite type space) - Milgrom and Weber (1985) [21].

In the following, if the Application game admits a symmetric Bayes Nash equilibrium p^*, we denote $W^F(x_i)$ the interim expected payoff at this equilibrium for an agent with score x_i: $W^F(x_i) := u(p^*|x_i)$. We call the mapping $W^F : [0,1] \to \mathbb{R}$ the interim welfare.

Gross incentive analysis

The formulas for interim payoff and probability of success reveal the incentives that the players face, and how they vary with the score. A player with a high score should feel confident that when applying to any good, she will succeed. She should target high-value goods, fully accepting the prospect of competition. Due to the competition from high-score agents, a player with a low score should expect that when applying, she is likely to fail. She should seek to avoid competition and to coordinate with other agents so as to target under-demanded goods. If there is a positive cost of applying, she may consider not applying at all to avoid paying this cost. Any agent with an intermediate score faces a trade-off between competition and coordination.

We illustrate this discussion in the graph below.

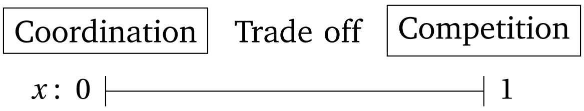

FIGURE I:
Coordination and competition incentives in the Application Game as a function of the priority score

2.3 Social objectives

Efficiency

The mechanism ϕ^F does not maximize the sum of agents' payoffs, due to two interacting frictions: private information on the scores, and truncation on the number of applications.[11]

At equilibrium, agents miscoordinate, being too ambitious or too little ambitious. One source of inefficiency is the waste: some goods remain unassigned in spite of the congestion. If there is a positive cost of application, failure is another source of inefficiency: some agents apply but remain unassigned and pay the cost.

A mechanism with public information (agents have common knowledge on the order of priority scores) and the same allocation rule (truncated deferred acceptance) would implement the efficient allocation. We denote ϕ^E this mechanism. It induces a game of perfect information. For future reference, we solve for the Nash equilibrium of this game below.

Proposition 1. *When agents have common knowledge of the order of priority scores, there exists a unique Nash equilibrium σ^*, and it is symmetric:*

[11] In a sense, there is a third friction arising from the finiteness of the number of agents. If there were many agents, by the Law of Large Numbers, the empirical distribution of scores would match the theoretical distribution of scores, and information on the order of scores would be perfect. That is why this work applies more to local markets, such as social housing.

(i) The agent ranked i^{th}, $2 \leq i \leq m$ in priority applies to and is allocated good ranked i^{th} in value.

(ii) Any agent ranked i^{th}, $m+1 \leq i \leq n$ in priority does not participate if $c > 0$, may participate if $c = 0$ but fails.

At the Nash equilibrium of the efficient benchmark, agents perfectly sort according to their scores. No good is wasted, and no agent fails if $c > 0$. We write below the maximal welfare W^E, and the expected equilibrium payoff $W^E(x)$ of an agent with score x (or interim welfare):

$$W^E := u(\sigma^*) = \frac{1}{n}\sum_{k=1}^{m}(v^k - c)$$
$$W^E(x) := u(\sigma^*|x) = \sum_{j=1}^{m}\binom{n-1}{j-1}(1-F(x))^{j-1}F(x)^{n-j}(v^j - c)$$

$W^E(x)$ is the sum of probabilities that the agent with score x is ranked j^{th}, $j \in \{1,\ldots,m\}$ in priority multiplied by value of good j minus the cost. If the agent is ranked lower (j^{th}, $j \in \{m+1,\ldots,n\}$), she recieves zero payoff.

Discrimination

The role of the priority points and scores is to recognize and quantify that different applicants have different rights to be allocated a good or a high value good. The mechanism maps the different rights into different probabilities of success conditional on application. A mechanism is said to be egalitarian (discriminatory) if quite different scores map into similar (very different) probabilities of success.

Designers have preferences over the allocations, not only in terms of efficiency, but also in terms of discrimination. The fact that they bother to design a priority score system (putting much effort into the design of complex score computation rules) provides evidence of this discrimination objective. If designers cared only about efficiency, they could use any arbitrary priority order to assign the goods, and any resulting allocation would be optimal. However, the precise discrimination objectives of designers often remain unclear. It seems that the objective is to have some, but not maximal discrimination. In public reports on their activities, clearinghouses often positively interpret a large participation rate and the participation of low score agents. They sometimes define proportionality targets: the mechanism should give twice as many chances to an agent with a score $2x$ of getting any good as to an agent with a score x.

In this paper, we do not explicitly model the discrimination objective and rather aim at providing descriptive results on discrimination. We define a partial order on mechanisms based on discrimination at equilibrium.

Definition 2. *An allocation mechanism ϕ with interim welfare $W^{\phi}()$ is more discriminatory (more egalitarian) than another allocation mechanism ψ with interim welfare $W^{\psi}()$ if:*

(i) $\forall\ x \in [0,1] : \frac{\partial W^{\phi}(x)}{\partial x} \geq (\leq) \frac{\partial W^{\psi}(x)}{\partial x}$

(ii) $\exists\ x \in [0,1]\ s.t.\ \frac{\partial W^{\phi}(x)}{\partial x} > (<) \frac{\partial W^{\psi}(x)}{\partial x}$

3 Equilibrium and welfare

We first focus on the case where application is free $c = 0$.[12] When $c = 0$, all expost and interim payoffs are positive. Agents always apply and never use the default no application action N. Agents only wonder "where to apply".

3.1 Example with small dimensions

We first illustrate the equilibria with a market with small dimensions: only $n = 3$ agents and $m = 2$ goods. To have explicit formulas, we also pick the uniform distribution ($F \sim \mathcal{U}([0,1])$) as the prior distribution of scores.

This trade-off between competition and coordination resolves differently in two kinds of equilibria: a pure asymmetric equilibrium and a symmetric mixed equilibrium.

Asymmetric pure equilibrium

On the graph below, each line going from 0 to 1 stands for the score support $[0,1]$, one line for each strategy of the three players, and the letters above stand for the action played at the corresponding scores.

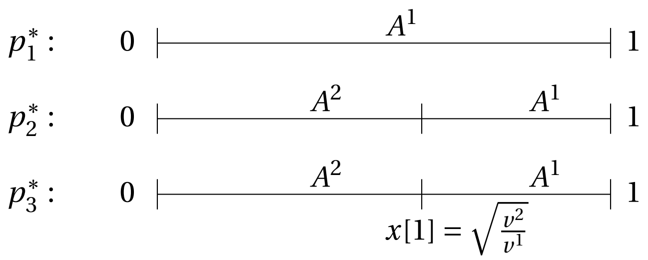

FIGURE II:
Pure (asymmetric) BNE - $n = 3,\ m = 2,\ F \sim \mathcal{U}$

The graph shows two intervals, where the middle bound of the intervals $x[1]$ is an indifference point.

[12]Almost every element in the equilibrium characterization (Theorem 1.) write the same with any $c \geq 0$, with one exception: the behavior of low scores agents can differ; when $c > 0$, they may not apply. For these agents, the case $c = 0$ is not the limit case of $c > 0, c \to 0$. That is why some ex ante equilibrium aggregates, such as the mass of agents applying to a good or the ex ante welfare, can differ slightly for $c = 0$ or $c > 0$. We write all proofs in appendix §A for the case $c > 0$. The proofs follow the same steps.

On an interval of high scores $[x[1], 1]$, all agents apply to good 1. This is because an agent with the highest possible score 1 always gets the good she has applied to. So, she applies to the highest value good and secures the highest possible payoff in the game v^1.

At score $x[1]$, confidence in success when applying to good 1 becomes quite low. By contrast, the probability of success conditional on applying to good 2 is constant, equal to 1. Algebraically, the interim payoff of applying to good 1 hits the value of good 2 minus the cost $v^2 - c$. At $x[1]$, agents become indifferent between applying to goods 1 or 2.

At any lower score in $[0, x[1]]$, two agents (agents 2 and 3) apply to good 2. Because they compete with each other, their interim payoffs steadily decrease from $x[1]$ leftward. The other agent (agent 1) keeps on applying to good 1, because she faces no competition anymore on good 1; her interim payoff is constant on the whole interval $[x[2], x[1]]$.
The fact that agents share roles (with a majority of applicants to good 2 and a minority of applicants to good 1) breaks the possibility of a symmetric equilibrium in pure strategies. To get an intuition on the necessity of the asymmetry, we can consider (by contradiction) a symmetric strategy profile where all players would shift to apply to good 2 below $x[1]$. Then, the interim payoff conditional on playing good 1 would be constant (no competition) as the score decreases below $x[1]$, whereas the interim payoff conditional on playing good 2 would decrease (2 competitors). Consequently, the former would be higher than the latter at any score below $x[1]$, and any player would face a profitable deviation from good 1 to good 2.

Symmetric mixed equilibrium

In the graph below, the horizontal line in the square still represents the score support $[0, 1]$. The vertical line represents the probabilities of each action in the mixed strategy. The notation $p^2[2]$ stands for the probability of applying to good j at scores where the agent applies with positive probabilities to both goods 1 and 2.

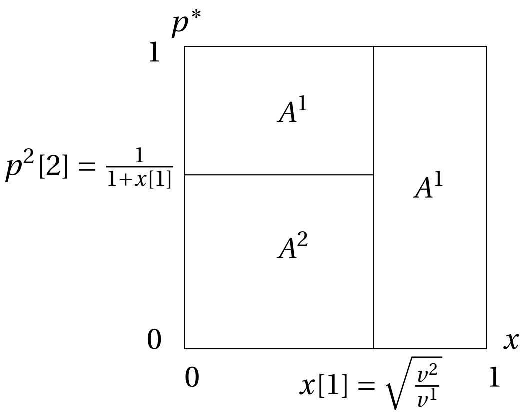

FIGURE III:
Symmetric (interior) BNE - $n = 3,\ m = 2,\ F \sim \mathcal{U}$

As in the pure strategy equilibrium, all agents apply to good 1 at scores belonging to an interval $[x[1], 1]$.

The equilibrium strategy becomes interior on $[0, x[1]]$, where an agent applies with positive probabilities to goods 1 or 2. Those probabilities are constant, implying that the symmetric equilibrium is in block strategies. Intermediate score agents apply more often to good 2 than to good 1 ($p^2[2] > \frac{1}{2}$). We give an intuition for the block structure in section §3.2.

Summary results from example

In both the pure and interior equilibria, confidence in success makes high-priority agents ambitious. At intermediate and low scores, it becomes rewarding to be more pragmatic and to try to coordinate to avoid competition.

In the pure equilibrium, this happens through a sharing of roles between applicants to different goods; in the interior equilibrium, by positive probabilities to apply to both goods. Many intermediate and low score agents settle for the secure option. A remaining smaller group of agents takes advantage of alleviated competition to maintain high ambitions.

3.2 Equilibrium

The lessons from the example smoothly generalize to any number of agents n, goods m, and any priority score distribution F.

Asymmetric pure equilibrium

Proposition 2. *A pure strategy Bayes-Nash equilibrium of the Application Game is necessarily asymmetric.*

The proof is done by contradiction, just as in the example. The short intuition is that competition by high-score agents on the highest-value goods smooths interim payoffs conditional on different applications. Consequently, at any intermediate or low score, several goods of different values are equally attractive. To guarantee the absence of profitable deviation, they all need to be targeted by at least one agent with positive probability. A pure symmetric profile would not allow this.

In the rest of the paper, we discard the Bayes-Nash Equilibrium in pure strategies for two reasons. First, we remain skeptical about the capacity of (ex-ante symmetric) agents to coordinate and share the different roles in an asymmetric profile without any communication. Second, the pure strategy Bayes-Nash Equilibrium structure is not robust. In the general model with any number of agents, goods, and any distribution, it depends finely on the set of parameters of the game.[13]

Symmetric mixed equilibrium

The theorem below describes the symmetric equilibrium.

Theorem 1. *A symmetric (interior) Bayes-Nash equilibrium p^* of the Application Game with $c = 0$:*

(1) Exists and is unique.

(2) Is in block strategies, with:

(i) $k_0 \in \{2, \ldots, m-1\}$ *classes if* $1 + \sum_{l=1}^{k} \left(\frac{v^{k+1}}{v^l}\right)^{\frac{1}{n-1}} \leq k_0 < 2 + \sum_{l=1}^{k-1} \left(\frac{v^k}{v^l}\right)^{\frac{1}{n-1}}$

$k_0 = m$ *classes if* $m < 2 + \sum_{l=1}^{m-1} \left(\frac{v^m}{v^l}\right)^{\frac{1}{n-1}}$

(ii) Classes write $C[1] := [x[1], x[0]], C[2] := [x[2], x[1]], \ldots, C[k_0] := [x[k_0], x[k_0 - 1]],$

[13] We illustrate this lack of robustness in appendix §B.2.

with:

$$x[0] = 1$$

$$\forall\ k \in \{1,\dots,k_0-1\}:\ x[k] = F^{-1}\Big(1-k+\sum_{l=1}^{k}\Big(\frac{v^{k+1}}{v^l}\Big)^{\frac{1}{n-1}}\Big)$$

$$x[k_0] = 0$$

(iii) In class $C[k], k \in \{1,\dots,k_0\}$, agents apply to good $j \in \{1,\dots,k\}$, with probability:

$$p^j[k] := \Big(\sum_{l=1}^{k}\Big(\frac{v^j}{v^l}\Big)^{\frac{1}{n-1}}\Big)^{-1}$$

We illustrate this equilibrium in the figure below:

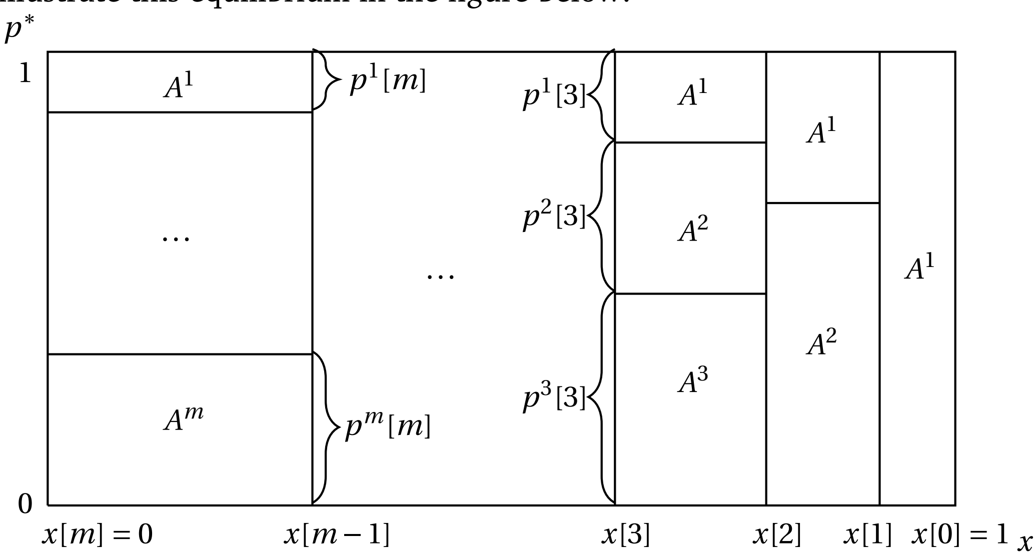

FIGURE IV:
Symmetric (interior) BNE of the Application Game with c=0 – General form

Just as in the example, high-score agents apply to good 1 only. Agents with slightly lower scores apply to goods 1 and 2 with constant probabilities, and a higher probability on good 2. Agents with still lower scores may apply to the three highest-value goods (with larger probabilities on the lowest-value goods within this set). This goes on until (potentially) the lowest score agents apply to all the goods with positive probabilities, including the highest-value goods (and these probabilities decrease with the value of the good).

The proof of Theorem 1. is in three steps. In the first step, we prove that we can divide the score support $[0,1]$ into a finite number of intervals where agents with scores in one interval apply with positive probabilities to only good 1, then both goods 1 and 2 until an interval where they potentially apply to all goods. This step relies heavily on Lemma 1., and the intermediate value theorem applied recursively m times. The proof that $x[1]$ necessarily exists is done by contradiction: if all agents apply to good 1 at all scores, an agent with a score approaching zero would get no chance to get good 1 and all chances to get another good, conditional on application, hence a profitable deviation.

The next step characterizes the probability functions. The proof that they are piecewise constant is done by induction. At inductive step $k \in \{2,\dots,m-1\}$, the strong indifference principle applied at a score $x^* \in (x[k], x[k-1])$ delivers a system of $k-1$ differential equations with $k-1$ unknowns $(p^j(x^*), j \in \{2,\dots,k\})$. Substituting within the equations, we find a relation between the primitives of fp^j and f, hence constant probabilities $\forall\ x \in [x[k], x[k-1]],\ p^j(x) := p^j[k]$. We use the same differential equations to get a recursive relation between all $p^j[k],\ j \in \{1,\dots,k\}$, and the fact that they sum up to one to get the explicit formula. In the third and final step, we use the thresholds' definitions $(x[k],\ k \in \{1,\dots,m\}$ is the highest score where there is indifference between all actions $A^1,\dots,A^{k+1})$ to find their formula.

We discuss three features of the symmetric equilibrium in more detail.

Block strategies

The block structure implies that agents sort according to their priority scores into a finite number of classes. Agents with priority scores belonging to the same class use exactly the same application mixture. Class $C[k]$ is the interval of scores where agents apply to goods $1,\dots,k$ with positive and constant probabilities. Starting from a continuous support, we end up with a discrete number of strategies effectively played.

For an intuition, it is helpful to rewrite the indifference equations as the ratio of probabilities of success is equal to the ratio of good values. For class $C[2]$, this gives:

$$\frac{q^1(x)}{q^2(x)} = \frac{v^2}{v^1}$$

The right-hand side is a constant. Although the cardinal values of these probabilities of success vary with the score, the trade-off that the agent faces at different scores remains the same. Hence the same application probabilities.

Sorting

An important and related question is the sorting question: Is it the case that higher score agents are more ambitious at equilibrium? The answer clearly is non-obvious, as we see that low-score agents apply to the highest value good with positive probabilities. If some assortativity holds, it must hold only in expectation. The sorting question matters in relation to the efficient benchmark, where the equilibrium outcome is the perfectly positive assortative matching. We would like to know whether part of the assortativity remains in the frictional environment. The corollary below provides a positive answer.

Corollary 1. *At the symmetric (interior) Bayes-Nash equilibrium p^* of the Application Game, the vector of application probabilities taken at any score x $(p^1(x), \ldots, p^m(x))$ first-order stochastically dominates the same vector taken at any lower score $x' < x$.*

The probability of application to the best good increases with the score. The sum of the probabilities of application to the two best goods also increases with the score, and so forth.

Conversely, we wonder: is it the case that higher value goods receive more applications, from higher scores? The next corollary also answers positively.

Corollary 2. *At the symmetric (interior) Bayes-Nash equilibrium p^* of the Application Game:*

(i) The ex ante probability of applying to good j $\int_0^1 p^{*j}(x) f(x) dx$ decreases in j.

(ii) The distribution of the scores applying to good j first order stochastically dominates the same distribution for good $j' > j$.

Number of classes

Theorem 1. says that there are between 2 and m classes. How many precisely depends on the set of parameters of the Application Game. Only the top part of the equilibrium structure $[x[1]-\varepsilon, 1]$, can be seen as "robust", arising at the symmetric equilibrium of any Application Game.

When there are only $k_0 < m$ classes, goods $k_0+1, \ldots, m$ receive no application at all and remain unassigned. This is a major contrast with the efficient mechanism, where we know that with congestion, each good receives exactly one application, and no good is wasted. This has important consequences for the coming welfare analysis.

Comparative statics

We describe how a change in the parameters of the Application Game (values, number of agents, number of goods, score distribution) affects the symmetric Bayes-Nash Equilibrium.

The equilibrium is invariant to a rescaling of all good values.

Proposition 3. *In the Application Game with $c = 0$, if we multiply all good values by a constant, the symmetric Bayes-Nash equilibrium remains unchanged.*

The relative position of values affects the equilibrium structure in an intuitive way. If goods are highly homogeneous in values, agents almost perfectly randomize between available goods. There is close to a single class. Coordination is horizontal. When goods are heterogeneous in value, people sort by scores. The strategy is close to a pure assortative strategy. Coordination is vertical. We provide below a graphical illustration in the example with $m = 2,\ n = 3,\ F \sim \mathcal{U}$.

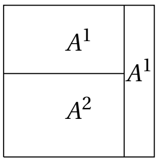

FIGURE V:
Homogeneous values: $v^1 = 3,\ v^2 = 2$

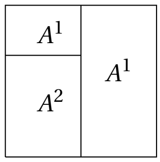

FIGURE VI:
Heterogeneous values: $v^1 = 4,\ v^2 = 1$

Due to the recursive structure of the Application Game, the introduction of an additional good has a very clean effect on the equilibrium structure, as described in the coming proposition.

Proposition 4. *At the symmetric Bayes-Nash equilibrium of the Application Game with $c = 0$, the addition of a new good with value $v^{k'+1} < v^{new} < v^{k'}$:*

(i) Only affects thresholds $x[k]$ and levels $(p^j[k])_{j\in\{1,\dots,k\}}$ on the class just above and on lower classes $C[k], k \in \{k', \dots, k_0\}$.

(ii) Only affects the ex ante probability of applying to good k_0, which decreases.

In other words, the addition of a good does not affect individual behavior at levels of scores where agents were all applying to goods with values larger than $v^{k'}$. The collective behavior remains unchanged in the sense that each good is played as often as before the addition: only the identities of the applicants to goods with values smaller than $v^{k'}$ are modified. Playing the new good only happens at the detriment of good k_0, the lowest value good that was played at equilibrium.

An increase in the number of agents n increases competition as measured by the probability of failure for a fixed strategy. This pushes all agents to be more pragmatic. In the symmetric equilibrium, all thresholds increase, all classes decrease in size, and new classes emerge. The levels for low (high) value goods increase (decrease). In net effect, introducing more agents increases the mass of agents applying to each good.

Another parameter of the application game is the distribution of scores. The whole effect of this distribution is captured in the thresholds, hence in the class sizes.

Proposition 5. *At the symmetric Bayes-Nash equilibrium of the Application Game with $c = 0$, the priority score distribution:*

(i) Does not affect probability levels.

(ii) Only affects thresholds and class sizes, in a way that keeps the mass of each class fixed.

We expect a narrow (wide) class at score levels featuring a high (low) concentration of agents. For example, if we change the priority score distribution to a mean-preserving spread distribution, we will get that the extreme (middle) classes become narrower (wider). These changes in thresholds and class sizes exactly neutralize the change in distribution so that each agent has the same probability of having a score within the class. In total, the probability of applying to each good remains unchanged.

3.3 Welfare

We study the welfare at the symmetric equilibrium of the Application Game, and compare it to the welfare from the efficient mechanism.

Ex ante welfare

The ex-ante expected payoff is:[14]

$$W^F := u(p^*) = \frac{1}{n}\Big(\sum_{k=1}^{m} v^k - (m-1)^n \Big(\sum_{j=1}^{m} (v^j)^{-\frac{1}{n-1}}\Big)^{-(n-1)}\Big)$$

Therefore, the difference in welfare between the efficient and the frictional mechanisms is:

$$W^E - W^F = \frac{1}{n}(m-1)^n \Big(\sum_{j=1}^{m} (v^j)^{-\frac{1}{n-1}}\Big)^{-(n-1)}$$

By definition, the difference is positive. The next proposition provides straightforward comparative statics on this difference.

Proposition 6. *When $c = 0$, the frictional mechanism ϕ^F is inefficient: $W^E - W^F > 0$. The size of the inefficiency increases with the value of any good.*

[14] The formula is written for the case $k_0 = m$ or $x[m] \geq 0$. For the case $k_0 < m$ or $x[m] < 0$, the formula is more sophisticated but Proposition 6. remains valid.

When a good's value increases, waste on this good is more detrimental to welfare.

Interim welfare

We are not only interested in the size, but also in the shape of the inefficiencies. The question is: how do the inefficiencies associated with the private information spread as a function of the score?

To answer this question, we write interim expected payoffs. In the frictional mechanism, we have:

$$W^F(x) := u(p^*|x) = \sum_{j=1}^{k} p^j[k]\Big(\big(\frac{v^k}{v^j}\big)^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} v^j$$

$$x \in C[k],\ k \in \{1,\dots,m\}$$

With probability $p^j[k]$, the agent with score x applies to good j. $q^j(x) = \Big(\big(\frac{v^k}{v^j}\big)^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1}$ is the probability of success. If the agent is successful, she receives the good value v^j.

We study the variations of these interim payoffs.

Lemma 3. *When $c = 0$:*

(E) The interim welfare $W^E()$ is continuous and strictly increasing on $[0,1]$ from 0 to v^1.

(F) The interim welfare $W^F()$ is continuous and strictly increasing on $[0,1]$ from $b > 0$ to v^1.

In both kinds of markets, higher score agents are always better off. There is assortativity in the expected outcome. For the frictional mechanism, it was unobvious, due to the block structure. In the block structure, the class defines the strategy, hence the ambition. Ambition is constant within a class, but jumps between classes. The risk (as measured by the probability of failure) varies within a given class: it is high (low) just above (below) the thresholds. The ambition and risk variations combined give the strict monotonicity of the interim payoff.

This enables us to compare discrimination in the two mechanisms.

Theorem 2. *When $c = 0$, the frictional mechanism ϕ^F is more egalitarian than the efficient mechanism ϕ^E.*

Said differently, the friction of private information introduces noise in the allocation process and smoothens discrimination.

To measure more precisely the individual burden that inefficiencies impose on the various scores, we also compute and characterize the difference in interim payoffs $(W^E - W^F)()$.

Proposition 7. *When $c = 0$, there exist two scores $0 < \underline{x} \leq \overline{x} < 1$ such that the interim welfare difference $(W^E - W^F)()$ is strictly negative (positive) on $[0, \underline{x})$ $((\overline{x}, 1))$.*

We illustrate the proposition below in the example with $m = 2$, $n = 3$, $F \sim \mathcal{U}$.[15] The left figure shows the interim expected payoffs. On the right figure, the filled area displays the difference between interim expected payoffs.

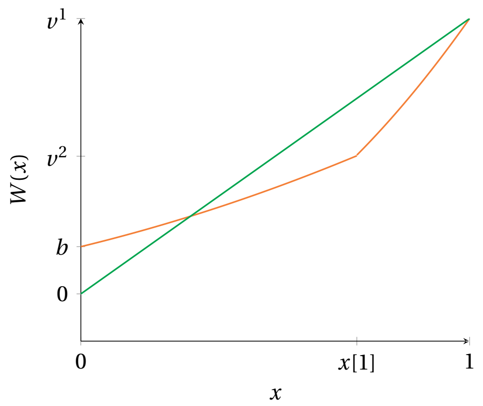

FIGURE VII:
$W^F()$ in orange, $W^E()$ in green with
$m = 2,\ n = 3,\ v^2 = 2,\ v^1 = 4,\ F \sim \mathcal{U}$

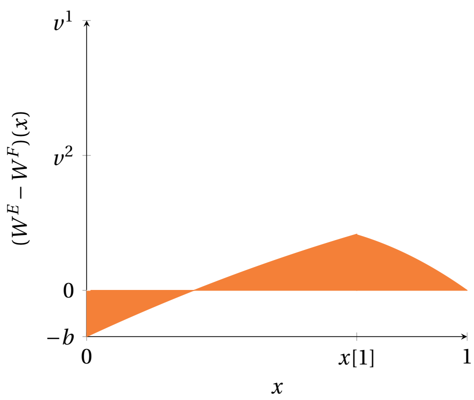

FIGURE VIII:
$(W^E - W^F)()$ in filled orange with
$m = 2,\ n = 3,\ v^2 = 2,\ v^1 = 4,\ F \sim \mathcal{U}$

The intermediate to high score agents benefit from public information on the scores, and the intermediate score agents bear most of the burden associated with the inefficiency of private information. Conversely, the low-score agents are better off with less (private) information than with full (public) information.

One important implication is that we cannot rank the efficient and the frictional

[15] In this example $\underline{x} = \overline{x} = x[1]$.

mechanisms through a Pareto order. The conclusion is that private information mitigates priority-based discrimination. In particular, it favors low-score agents.

Cardinal signal on ordinal ranking

We formulate an intuition. In the Application Game, agents all receive signals of the same nature: they all perfectly observe their own cardinal priority scores. The signal is cardinal, whereas what agents really care about is the ordinal ranking. The cardinality comes with boundary effects, so that the informative value of the signal depends on the score level. A low-score agent, by observing a score at the bottom of the prior distribution support, realizes that she is almost surely the lowest type. By contrast, for an intermediate score agent, observing her score is a poor signal of her rank in the priority order. She holds a quite imprecise posterior on this rank. In terms of information, the low-score agent benefits from a relative competitive advantage.

It translates into actions in the following way. Due to the imprecise posterior, the intermediate score agent is likely to make mistakes (compared to what she would do with perfect information): being too ambitious or too little ambitious, miscoordinating with intermediate-to-high score agents. In total, and due to the recursive structure of the Application Game, these mistakes tend to soften the competition imposed by intermediate score agents.

At equilibrium, a low-score agent is fully aware that competition is moderate. She understands that this leaves some room for her to apply. This more than offsets the fact that, in absolute terms, the private information is also slightly less revealing about her own rank in priority. She applies to various goods and captures a positive expected payoff.[16]

To be more precise about the informative value of the signal, we show explicitly how

[16]A wonder may be: how does private information affect the competition between high-score agents and intermediate-score agents? A high-score agent observes a score close to the higher bound of the support, which is a clear signal that she is likely the highest type, so this agent also benefits from a relative informational competitive advantage. But because she always plays the same action (applying to the best good) in the two markets, she cannot exploit this advantage, and there is no competition easing effect from which the intermediate score agent could benefit.

the score affects an agent's posterior belief about her ranking in priority. Her posterior belief is a probability distribution on support $\{1,\ldots,n\}$. The probability to be ranked j^{th}, $j \in \{1,\ldots,n\}$ is:

$$r^j(x) = \binom{n-1}{j-1}(1-F(x))^{j-1}F(x)^{n-j}$$

The next proposition characterizes the posterior belief variations.

Proposition 8.

(i) $r^1(x)$ *strictly increases in* x *on* $[0,1]$ *from* 0 *to* 1.

(ii) $r^n(x)$ *strictly decreases in* x *on* $[0,1]$ *from* 0 *to* 1.

(iii) $r^j(x)$, $j \in \{2,\ldots,n-1\}$ *strictly increases (decreases) on* $[0,x_0^j]$ $([x_0^j,1])$*, where* $x_0^j = F^{-1}(\frac{n-j}{n-1})$. *In addition:* $r^j(0) = r^j(1) = 0$.

Overall, the probability of being ranked high (low) increases (decreases) with the score. What a high or low rank means depends on the score. At very low scores, a marginal increase in the score increases the probabilities to be ranked $n-1,\ldots,1$, only decreases the probability to be the lowest priority. At very high scores, a marginal increase in the score decreases the probabilities to be ranked $n,\ldots,2$, only increases the probability to be the highest priority.

We use a standard measure of entropy to measure the resulting overall uncertainty in the posterior beliefs. Shannon entropy summarizes how much missing information there is in a probability distribution. When the posterior belief is r, the Shannon entropy is:

$$H[r](x) = -\sum_{j=1}^{n} r^j(x)\ln(r^j(x))$$

At extreme scores $x=0$ or $x=1$, the posterior distribution is degenerate $r^n(0)=1$, $r^1(1)=1$, so the entropy is minimal equal to 0. Starting from these bounds and going towards the interior of the score support, the entropy becomes higher, reflecting the more imprecise posterior.

Proposition 9. *There exist scores* $0 < \underline{x_0} < \overline{x_0} < 1$ *such that the entropy* $H[r]()$ *strictly increases (decreases) on* $[0, \underline{x_0}]$ *(*$[\overline{x_0}, 1]$*).*

For illustration, we plot below the Shannon entropies, in the example with $n = 3, m = 2$ for two prior distributions: the uniform and the triangular distributions.

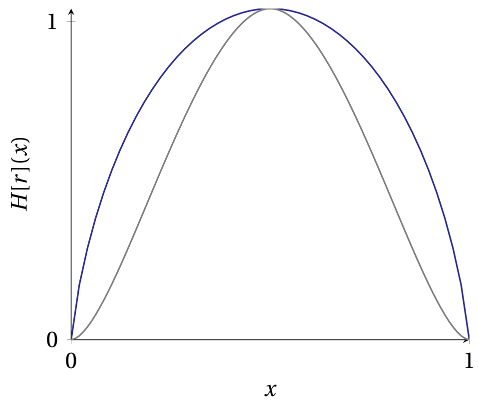

FIGURE IX:
$H[r]()$ in blue for F uniform, in gray for F triangular in $\frac{1}{2}$.

In both cases, the entropy is single-peaked ($\underline{x_0} = \overline{x_0}$). Because the two distributions are symmetric, the maximal entropy is reached at $\frac{1}{2}$. At any score $x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, the entropy when the prior is the triangular distribution is lower. This is because the triangular distribution is more concentrated around its mean. At a given score x, the agent is virtually closer to the bound with the more concentrated distribution, and less uncertain about the ranking of scores.

3.4 "Whether" to Apply

In this section, we consider the case where applying is costly: $c > 0$. The cost c models a monetary application or participation fee that applicants have to pay on the application platform. It can also capture the cost of time and effort dedicated to the application, and the cost of missing opportunities in the private (housing) sector.

With $c > 0$, the ex post payoff can be negative, equal to $-c$, if the applicant applies but fails to get the good she has applied to. If the probability of failure is large, the interim payoff may also be negative. If this is the case, the agent would rather switch to the default action N to secure a zero payoff. This model therefore addresses the full question "whether and where to apply".

Equilibria

The overall structures of the equilibrium set and of each equilibrium remain similar, only the behavior of low score agents is modified. We write below the statements that are modified. To simplify the writing, we define: $v^{m+1} := c$.

Theorem 1. ***(bis)*** *A symmetric (interior) Bayes-Nash equilibrium p^* of the Application Game with $c > 0$:*

(1) Exists and is unique.

(2) Is in block strategies, with:

(i) $k_0 \in \{2, \ldots, m\}$ *classes if* $1 + \sum_{l=1}^{k} \left(\frac{v^{k+1}}{v^l}\right)^{\frac{1}{n-1}} \leq k < 2 + \sum_{l=1}^{k-1} \left(\frac{v^k}{v^l}\right)^{\frac{1}{n-1}}$

$k_0 = m+1$ *classes if* $m < 1 + \sum_{l=1}^{m} \left(\frac{c}{v^l}\right)^{\frac{1}{n-1}}$

(ii) Classes write $C[1] := [x[1], x[0]]$, $C[2] := [x[2], x[1]]$, ..., $C[k_0] := [x[k_0], x[k_0-1]]$ *with:*

$$x[0] = 1$$

$$\forall\ k \in \{1, \ldots, k_0 - 1\} :\ x[k] = F^{-1}\left(1 - k + \sum_{l=1}^{k} \left(\frac{v^{k+1}}{v^l}\right)^{\frac{1}{n-1}}\right)$$

$$x[k_0] = 0$$

(iii) In class $C[k], k \in \{1, \ldots, \min(k_0, m)\}$, *agents apply to good* $j \in \{1, \ldots, k\}$, *with probability:*

$$p^j[k] := \left(\sum_{l=1}^{k} \left(\frac{v^j}{v^l}\right)^{\frac{1}{n-1}}\right)^{-1}$$

If $k_0 = m+1$, *in class* $C[m+1]$, *agents do not apply.*

We illustrate this equilibrium in the figure below:

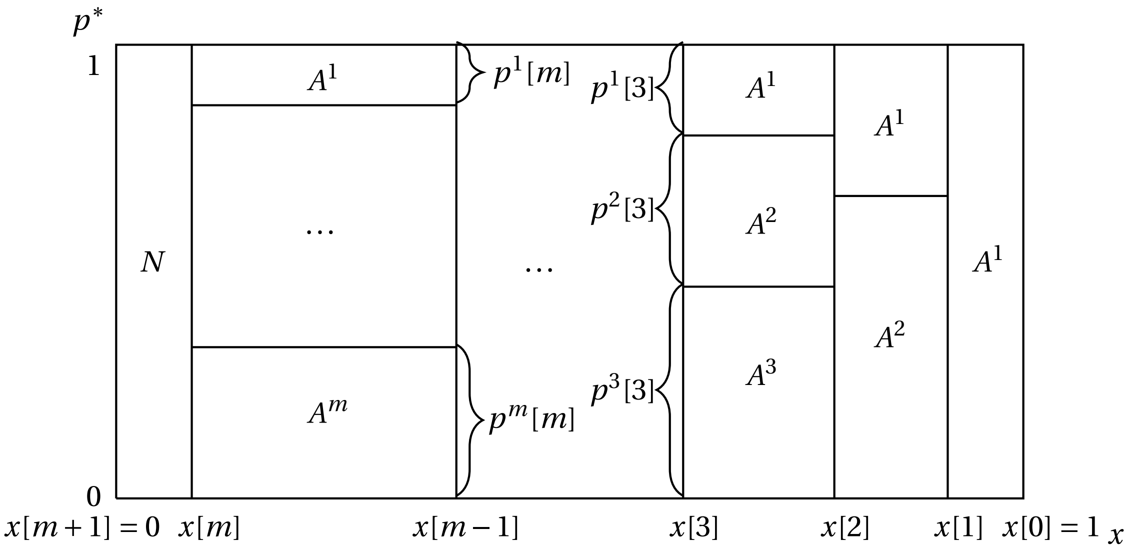

FIGURE X:
Symmetric (interior) BNE of the Application Game with $c > 0$ – General form

Again, the equilibrium is in block strategies, with sorting. The number of classes depends on the parameters of the game; it is between 2 and $m+1$. Only the low score agents may consider not applying. A necessary condition is that the class $C[m]$, where all application actions are played, is realized. Then, if the interim expected payoff hits the zero bound at a positive score, agents with still lower scores will play N with probability one, which makes the class $C[m+1]$.

Comparative statics with respect to good values, number of agents, number of goods, and score distribution are unchanged. Comparative statics with respect to the cost is straightforward.

Proposition 3. *(bis) In the Application Game with $c > 0$, if we multiply all good values and the cost by a constant, the symmetric Bayes-Nash equilibrium remains unchanged.*

When the cost increases, the $m+1$ class appears or expands in size.

Welfare

The ex ante expected payoff is:

$$W^F = \frac{1}{n}\sum_{k=1}^{m}(v^k - c) + \frac{n-1}{n}\left(c^{\frac{n}{n-1}}\sum_{j=1}^{m}(v^j)^{\frac{-1}{n-1}} - cm\right)$$

Therefore, the difference in welfare is:

$$W^E - W^F = \frac{n-1}{n}\Big(cm - c^{\frac{n}{n-1}} \sum_{j=1}^{m} (v^j)^{\frac{-1}{n-1}}\Big)$$

By definition, the difference is positive.

Proposition 6. ***(bis)*** *When $c > 0$, the frictional mechanism ϕ^F is inefficient: $W^E - W^F > 0$. The size of the inefficiency increases with the value of any good and with the cost.*

The interim welfare in the frictional mechanism is given by:

$$W^F(x) = \begin{cases} \sum_{j=1}^{k} p^j[k].\Big(\big(\frac{v^k}{v^j}\big)^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} v^j - c, \\ x \in C[k],\ k \in \{1,\dots,m\} \\ 0,\ x \in C[m+1] \end{cases}$$

Lemma 3. ***(bis)*** *When $c > 0$:*

(E) The interim welfare $W^E()$ is continuous and strictly increasing on $[0,1]$ from 0 to v^1.

(F) If $k_0 \le m$, the interim welfare $W^F()$ is continuous and strictly increasing on $[0,1]$ from $b > 0$ to $v^1 - c$.

If $k_0 = m+1$, the interim welfare $W^F()$ is continuous and constant equal to 0 on $C[m+1]$, strictly increasing on $[x[m],1]$ from 0 to $v^1 - c$.

In both kinds of markets, higher score agents are always better off. The new case when $c > 0$, is when there are $m+1$ classes. Then the interim welfare strictly increases from $x[m]$.

The comparison of the two mechanisms in terms of discrimination generalizes with a positive cost:

Theorem 2. ***(bis)*** *When $c > 0$, the frictional mechanism ϕ^F is more egalitarian than the efficient mechanism ϕ^E.*

We also get reversed welfare effects at low scores, which we can connect with participation:

Proposition 7. ***(bis)*** *When $c > 0$, and $k_0 \leq m$:*

(i) There exist two scores $0 < \underline{x} \leq \overline{x} < 1$ such that the interim welfare difference $(W^E - W^F)()$ is strictly negative (positive) on $[0, \underline{x})$ ($(\overline{x}, 1)$).

(ii) Participation is higher in the frictional mechanism ϕ^F than in the efficient mechanism ϕ^E.

Under the condition $k_0 \leq m$, the intermediate to high score agents would benefit from public information. The low-score agents are better off with private information. The condition $k_0 \leq m$ is quite general. It is satisfied whenever $m \geq 1 + \sum_{l=1}^{m} \left(\frac{c}{v^l}\right)^{\frac{1}{n-1}}$.

This reverse welfare effect can also be termed as participation. In the efficient mechanism, the low score agents do not participate. In the frictional mechanism, when $k_0 \leq m$, all agents, including the low-score agents, participate with probability one. If they participate rather than abstain, this can only be because they expect a positive payoff.

Overall, the conclusion that private information mitigates the ability of the priority system to discriminate between agents remains valid in the environment with costly application. Private information favors low-score agents as soon as the rest of the market parameters incentivize them to participate. The mechanics behind (cardinal signal on ordinal signals giving low score agents a relative competitive advantage) are similar to those in the environment with free application.

4 Discussion

4.1 Implications for real-life markets

Empirical test of equilibrium

The equilibrium analysis derives a number of clear and testable predictions. These predictions could be tested using data on individual applications in real-life markets.

One prediction is that low score agents may be ambitious in their application strategies. If we interpret the mixed equilibrium as the distribution of pure strategies in the population, we should observe that some low score agents apply to the best goods. This prediction is robust to the exact parameters of the market. On the markets where the theory predicts a small number of classes, we should also observe that the lowest value goods receive no application at all, in spite of the congestion.

Another prediction is the block structure. It should materialize in the data in the following way: if we compute the average population strategy at each score, this mixed strategy should change at a finite number of scores or thresholds. The number of thresholds should be bounded by the number of goods. Sorting (in terms of the stochastic dominance criterion) should still be respected: moving one class higher in score should make the population strategy more ambitious.

Treatment analysis could also help. If the designer gives more information online about other agents' scores, application behaviors should update in the direction of more sorting, less waste of goods, and fewer failures for the applicants.

Welfare analysis and market design

Our analysis shows that three elements in the mechanism jointly determine the final allocation and the welfare on markets with priority scores: the allocation rule, the score rule, and the information structure. The question is: how can this be of use for market designers in the field?

Market designers in the field seem to understand pretty well the role of the allocation rule and of the score rule. They do not seem to fully understand the role of information yet. They do not seem to be aware that the information they provide to applicants interacts with the allocation and score rules, and that it also plays a role in the final allocation, through equilibrium play. We mention two clues of this. The first clue is in the public reports that designers (in particular, social housing clearinghouses) have to produce on their activities. In these reports, they suggest that, one, the role of the priority points and score is to recognize differential rights to housing based on different levels of emergency, and two, that the role of the allocation rule is to map those differential rights into different probabilities of satisfaction. Information is rarely mentioned. The second clue is related to shocks in the social objectives of the designer (for example, after a change in the politicians in office in the supervising administration). On these occasions, designers will mostly reform the scoring rule and/or the allocation rule, but leave the information structure unchanged. In contrast, the amount of information available on the allocation platform often changes for exogenous reasons.

Consider, for illustration, a designer with a proportionality objective on discrimination: she would like the mechanism to give twice as many chances to an agent with a score $2x$ of getting any good as to an agent with a score x. This proportionality objective is computed from the point of view of the designer, with public information. Private information will distort the proportionality by increasing the probabilities of low-score agents to get valuable goods.

Hence, an immediate and broad policy recommendation: the allocation rule, the score rule, and the information structure should always be designed jointly. This joint design is a necessary condition to achieve any social objective in terms of the amount or form of priority-based discrimination on the market.

4.2 Extensions

We process three natural extensions of our model, at least on examples with small dimensions. We also discuss more ambitious generalizations, which we plan to address in subsequent papers.

Imperfectly correlated preferences

The assumption that preferences are homogeneous, with each good giving exactly the same value to all agents, is restrictive. In social housing, applicants all value positively the size of the accommodation and the quality of the equipment, but different applicants may value different micro-locations differently due to the different locations of their jobs. In total, individual matching preferences are only imperfectly correlated across different agents.

We model imperfectly correlated preferences in a setting with $n = 2$ agents and $m = 2$ goods, and a uniform prior distribution on the priority scores $F \sim \mathcal{U}([0,1])$. Each agent has exactly one most preferred good with value v and one least preferred good with value u, $v > u > 0$. A preference profile $(y_1, y_2),\ y_i \in \{u, v\}, i \in \{1,2\}$ means that good 1 has value y_1 to agent 1, y_2 to agent 2. The prior distribution over preference profiles is as follows:

$$\mathbb{P}(uu) = \mathbb{P}(vv) = \frac{\theta}{2},\ \mathbb{P}(uv) = \mathbb{P}(vu) = \frac{1-\theta}{2},\ \theta \in [0,1]$$

That is, goods are the same ex-ante (they are equally likely to be each agent's most preferred good), but preferences are correlated. The correlation is positive when $\theta > \frac{1}{2}$. We assume that preferences, just as priority scores, are private information. Therefore, an agent's type is two-dimensional: it specifies the priority score and the individual preference, with independence between the two dimensions.

A strategy is a mapping of the score support into two possible actions: applying to one's most preferred good (denoted $A^{\oplus}$) or to one's least preferred good (denoted $A^{\ominus}$). We characterize the symmetric Bayes-Nash equilibrium of the game.

Proposition 10. *A symmetric (interior) Bayes-Nash equilibrium of the Application Game with $n = 2,\ m = 2,\ F \sim \mathcal{U}$, and $c = 0$:*

(1) Exists and is unique.

(2) Is in block strategies:

(i) With 1 *class if* $\theta \leq \frac{v}{v+u}$, *2 classes if* $\theta > \frac{v}{v+u}$.

(ii) Classes write $C[1] := [x[1], 1]$, $C[2] := [0, x[1]]$ *with* $x[1] := \frac{\theta u-(1-\theta)v}{\theta v-(1-\theta)u}$.

(iii) In class $C[1]$, *agents play action* $A^{\oplus}$ *with probability* 1.

In class $C[2]$, *agents play action* $A^{\oplus}$ *with probability:* $p^{\oplus}[2] := \frac{\theta v-(1-\theta)u}{(2\theta-1)(v+u)}$.

At any equilibrium, and as expected, agents with high scores are ambitious and apply to their most preferred good ($A^{\oplus}$). It may become more profitable at lower scores (below score $x[1]$) to also target one's least preferred good ($A^{\ominus}$) because this good is less likely to receive an application from a higher score. Interestingly, the block structure remains.

The difference with the case of perfect correlation is that the shift at $x[1]$ does not necessarily happen. The fact that all agents play $A^{\oplus}$ combined with the imperfect correlation guarantees that both goods receive applications with positive probabilities. Even when both agents are ambitious, there is partial coordination. Therefore, it can be that all agents stick to the same strategy at low scores. The shift happens if and only if the correlation is sufficiently strong and the gap between the two good values is sufficiently small ($\theta > \frac{v}{v+u}$). The figure below illustrates the two cases:

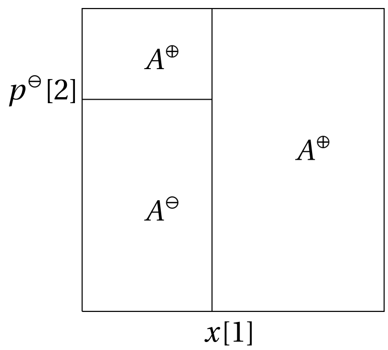

FIGURE XI:
BNE - Correlated preferences
$m = n = 2,\ F \sim \mathscr{U},\ v = 3,\ u = 2,\ \theta = 0.75$

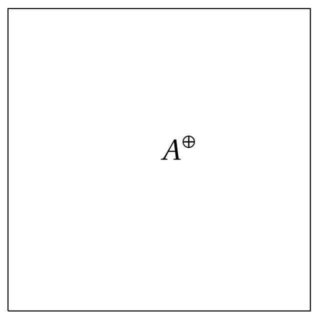

FIGURE XII:
BNE - Correlated preferences
$m = n = 2,\ F \sim \mathscr{U},\ v = 3,\ u = 1,\ \theta = 0.75$

Score-dependent cost

One interpretation of the application cost is as a cost of missing opportunities in the private housing sector. In real life, agents considering applying to social housing may face different outside options and therefore different application costs. The low priority (mostly high income) households are the ones with the most attractive outside options and the higher application costs. To capture this asymmetry, we model a cost that depends negatively on the priority score.

Another virtue of the endogenous cost is that it can capture some of the main effects of a dynamic allocation process in several rounds. For social housing in Paris, every week, a new application round opens and new vacant accommodation becomes available. An agent keeps the same score across different rounds. An agent with a high priority who fails in a given round still holds a high chance of being allocated a good in future rounds. Her continuation value is high; equivalently, her cost is low.

We define the application cost to be a strictly decreasing function of x: $c'(x) < 0$. We set $c(0) < v^m$, implying $\forall\ x \in [0,1], j \in \{1,\dots,m\}$, $c(x) < v^j$ (the cost never exceeds the value of any good).

Proposition 11. *The symmetric (interior) Bayes-Nash equilibrium of the Application Game with score-dependent strictly decreasing cost $c(x)$, $c'(x) < 0$:*

(1) Exists and is unique.

(2) Is similar to the symmetric Bayes-Nash equilibrium of the Application Game with exogenous cost:

(i) If $x[m] \le 0$, the equilibria are exactly the same with exogenous and score-dependent costs.

(ii) If $x[m] > 0$, the equilibria are the same except that $x[m]$ is higher with endogenous cost.

At symmetric equilibrium, the block structure, the thresholds and probability levels in the domain where agents apply for sure are independent of the cost. Only the indifference equations between the application actions $(A^j)_{j\in\{1,\dots,m\}}$ and the no application N at $x[m]$ feature the cost. When the cost depends negatively on the score, class $C[m]$

is narrower, and low score agents apply a little less to all goods. In total, lower value goods suffer the larger decreases in the ex ante probabilities of receiving an application.

For illustration, we display the symmetric equilibrium with $m = 2$, $n = 3$, $F \sim \mathcal{U}$, and a score-dependent (linear) cost: $c(x) = \frac{3}{2} - x$. For comparison, we also display the equilibrium in the same market with exogenous cost $c = 1$.[17] We denote $x[2]'$ the bottom threshold of the second class with score-dependent cost.

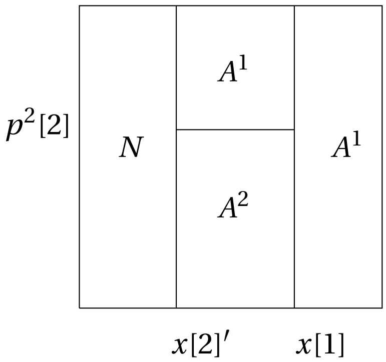

FIGURE XIII:
BNE with cost $c(x) = \frac{3}{2} - x$,
$m = 2,\ n = 3,\ v^2 = 2,\ v^1 = 4,\ F \sim \mathcal{U}$

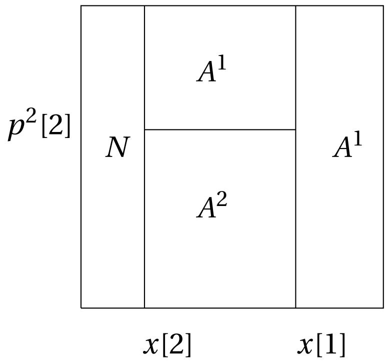

FIGURE XIV:
BNE with cost $c(x) = 1$,
$m = 2,\ n = 3,\ v^2 = 2,\ v^1 = 4,\ F \sim \mathcal{U}$

The welfare analysis gives similar qualitative results.

Corollary 3. *With score-dependent strictly decreasing cost $c(x)$, $c'(x) < 0$, the frictional mechanism ϕ^F is more egalitarian than the efficient mechanism ϕ^E.*

We illustrate this result below, again with $m = 2$, $n = 3$, $F \sim \mathcal{U}$, and $c(x) = \frac{3}{2} - x$:

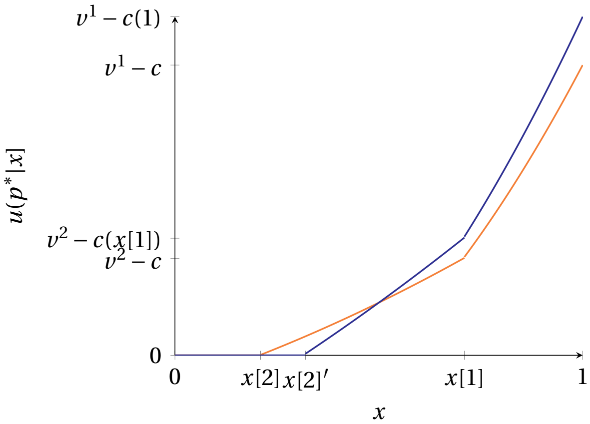

FIGURE XV:
Interim welfare at BNE with score-dependent cost $c(x) = \frac{3}{2} - x$ (blue) vs constant cost $c = 1$ (orange) - $m = 2,\ n = 3,\ v^2 = 2,\ v^1 = 4,\ F \sim \mathcal{U}$

[17] Note that the expected cost is the same in both cases (equals 1).

Quantitatively, the magnitude of the welfare effects identified in section §3.3 decreases. The market outcome is closer to the outcome of the efficient mechanism. Endogenous cost induces more discrimination according to the score.

Smaller or no truncation

In many real-life matching markets, such as school choice or centralized labor markets, agents are allowed to apply to more than one good. Thus, a natural extension of our model consists of relaxing or removing the truncation of the application list.

We process this extension on an example with $n = 3,\ m = 2,\ F \sim \mathscr{U}$. With no truncation, the action space includes an additional action B for "Both". It means submitting a rank-ordered list with good 1 first and good 2 second[18] and paying the application cost twice.

Proposition 12. *A symmetric (interior) Bayes-Nash equilibrium of the Application Game with $n = 3,\ m = 2,\ F \sim \mathscr{U}$ and no truncation:*

(1) Exists and is unique.

(2) (i) If $\frac{v^2}{v^1} + \frac{c}{v^2} > 1$, the symmetric equilibrium is the same as with the truncation. In particular, agents always apply to at most one good.

(ii) If $\frac{v^2}{v^1} + \frac{c}{v^2} < 1$, at symmetric equilibrium, agents with scores higher that $x[1]'$ apply to good 1, agents with scores just lower than $x[1]'$ apply to both goods.

Case (i), where agents disregard the possibility of applying to both goods, arises when the application cost is high relative to the goods' values. In case (ii), agents use the possibility to apply to all goods, aiming for the high-value good, but hedging against the possibility that it may not be available anymore. Below $x[1]$, all interim payoffs strictly decrease due to competition on both goods. What the next shift in action is is non-obvious (and non-robust). In the limit $c = 0$, $x[1]' = 1$ in case (ii) and all agents apply to both goods at any score.

The figure below illustrates the two cases:

[18] Due to homogeneous preferences, applying to good 2 first and good 1 second is dominated.

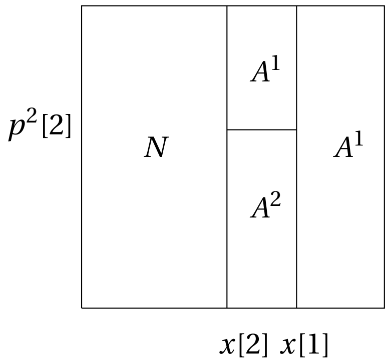

FIGURE XVI:
BNE with no truncation (i)
$m = 2,\ n = 3,\ v^2 = 2,\ v^1 = 4,\ c = 1.5,\ F \sim \mathcal{U}$

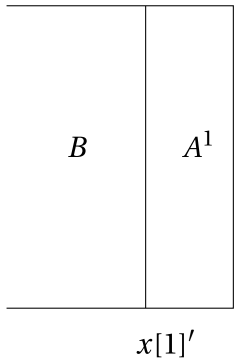

FIGURE XVII:
BNE with no truncation (ii)
$m = 2,\ n = 3,\ v^2 = 2,\ v^1 = 4,\ c = 0.5,\ F \sim \mathcal{U}$

Dynamic mechanisms

In the Application Game, agents miscoordinate with other agents because information on scores is private. Because the mechanism is static, agents cannot get more informed over time. A dynamic mechanism with several deterministic rounds, each asking for truncated rank-ordered lists, could improve efficiency. Between two rounds, the unassigned agents would get two pieces of information. One, they would observe the smaller set of unassigned goods and agents. Second, they would understand they failed in the previous round, which would push down their beliefs about their ranks in the priority order. In the next round, they would apply in a more pragmatic way.

Another way to improve efficiency through the dynamics could be to use hybrid mechanisms with both deterministic and stochastic rounds. There would first be a deterministic round with truncation on the rank-ordered list. In this first round, some goods and agents would be assigned and would leave the market. Then the designer would organize a lottery to assign the remaining goods and agents. This would increase efficiency in a cheap way (no submission of new rank-ordered list, no additional application revision cost), by eliminating one of the two sources of inefficiency: the waste. The designer may consider different truncations in the first deterministic round. With large truncation, the lottery will assign the majority of goods, whereas with small truncation, many goods may already be allocated in the first round. At one extreme, we have the full lottery mechanism; at the other extreme, we have the untruncated deferred acceptance mechanism. Ideally, we would like to span the whole spectrum of hybrid mechanisms and study how efficiency and discrimination vary in between, for intermediate truncations.

Information design

The analysis mostly compares two information structures: private information (for all the agents) and public information (for all the agents) on priority scores. From an information design perspective, this is a highly constrained problem. It would be interesting to study intermediate and more sophisticated information structures. For example, what if the designer provides noisy information, such as the quantiles of the score distribution? In school choice markets, designers often provide historic data about quantiles of applicants' scores from previous years.

It could even be argued that clearinghouses are able to use the online application platforms to provide fully flexible information, including customized information. Ideally, we would explicitly model this problem and solve the resulting information design problem, and show that once we consider a two-fold social objective (efficiency, plus a discrimination or participation objective), the frictional mechanism becomes optimal. This would provide micro-foundations for the widespread use of this mechanism.

Political economy

Another foundation could come from political economy. In our leading example of social housing, the frictions give a chance to low priority applicants (mostly middle class households) to be allocated social housing. Sticking to the frictional mechanism could be a way for a greedy politician to earn these households' votes for reelection.

Strategic score reporting

The main model considers that the designer knows the true score. In reality, though, the agents report their status with respect to each criterion in the scoring rule, and the designer computes the aggregate score. It may be costly for the designer to verify that every report for every criterion and agent is truthful. We are interested in the resulting communication problem, and the interaction with the Application Game.

4.3 Literature review

Matching and game theories

This paper belongs to the theory literature on matching with uncertainty. In most of this literature, the uncertainty applies to individual preferences, that is, to one's own ex-post payoff in the match. Most papers do not investigate the effect of privately known priority scores. They model ordinal priorities and ordinal mechanisms (where agents are asked to rank available options) rather than cardinal ones (where agents reveal how much they like each available option). With respect to information, they mostly either assume perfect information (which is unrealistic) or set strategy-proof mechanisms (hence no incentive to know about other agents).

There are a few interesting exceptions, where the uncertainty applies to the priorities, that is, to the probability of success. Chade Lewis and Smith 2014) [7] is about college admissions with noisy assessment of students' skills. Ali and Shorrer (2021) [2] models the decision problems of students who are uncertain about their (correlated) priorities, and the resulting signaling effects. Avery and Levin (2010) [3] is about early college admissions when students are differentiated in their (unknown) academic abilities and in their fits for different schools.

In the present paper, the uncertainty on priorities endogenously emerges from a lack of information about competitors. It therefore connects with the literature on agent-agent matching with uncertainty on other agents' preferences. Roth (1989) [26] revises standard results about dominance and equilibrium when there is private information on preferences on both sides of a matching market. Kloosterman and Troyan (2020) [17] shows that when preferences are uncertain but correlated, the deferred acceptance mechanism is no longer strategy-proof or stable, and less informed students are worse off due to a winner's curse. Gleyze and Pernoud (2023) [13] models the information acquisition problem of agents who can learn about their own preferences and about other agents' preferences at a cost before matching. The main technical difference is that in these papers, an agent's payoff depends on the types of other players only indirectly through the actions of the players. In agent-object matching with uncertainty on priorities, the payoff depends both on the actions of the other players and directly on their types.

This paper more generally relates to the literature on frictional matching. The "whether and where to apply" dilemma stems from the fact that truncating the deferred acceptance mechanism sacrifices strategy-proofness, an observation made by Romero-Medina [24].[19] The model also bears similarities with auctions, in particular all-pay auctions (Baye, Kovenock, and de Vries (1996) [4]), and multi-item auctions with common preferences and distribution of a single item for each bidder (Demange, Gale, and Sotomayor (1986) [9]). The main difference is that auctions allow for monetary transfers, hence endogenous priorities, whereas our priorities are exogenous, and agents can only adjust their applications.

On the game theory side, this paper relates to the literature on contest games (Tullock (1980) [27], Lazear and Rozen (1981) [18]) where players spend a scarce resource (effort, capital, time) to win valuable prizes. In general, players are heterogeneous in skills, and only the skill distribution (not the realization) is common knowledge. The application game can be viewed as a contest game with a discrete effort space, where applying is exerting effort.

Closest related papers

The closest related papers are Chade, Lewis, and Smith (2014) [7] and Peters (2010) [23]. Both are with two-sided strategic interactions and continuums (which

[19] In truncated deferred acceptance, Roth (1982) [25] and Ehlers and Masso (2007) [10] study the interactions between stability and strategy-proofness. Haeringer and Klijn (2009) [14] identify conditions under which Nash equilibrium outcomes are stable. In our model with homogeneous preferences, the set of stable matchings is trivially the whole set of full matchings.

allows for symmetric equilibrium in pure strategies), whereas our model features strategic interactions within one side of the market and a finite number of players.

In Chade, Lewis, and Smith (2014), inspired by school choice, strategic interactions happen within the college side and between students and colleges, whereas in our model, strategic interactions happen only within the agent side. The uncertainty also differs. They set common knowledge of students' priority types but exogenous noise on the allocation, whereas the uncertainty on the priority order in our model endogenously arises from private information on priority scores. They solve their model for only two heterogeneous colleges, whereas we characterize the symmetric Bayes-Nash equilibrium for any number of goods. Their analysis finds that student-college sorting may fail with weaker students applying more aggressively and weaker colleges imposing higher standards at equilibrium. Our analysis finds the opposite and more standard pattern: at the symmetric equilibrium of the Application Game, higher-value goods are played more often and at higher scores.

In Peters (2010), inspired by labor markets, the workers' subgame is like a continuum version of the Application Game, focusing on the "where to apply" question, abstracting from the "whether" question.

Analog results in different environments

The block structure of the symmetric equilibrium in the Application Game is reminiscent of the "class segregation result" in dynamic search problems. The dynamic search literature (McNamara and Collins (1990) [19], Burdett and Coles (1997) [6], Bloch and Ryder (2000) [5], Jacquet and Tan (2007) [15]) studies dynamic two-sided agent-agent markets where each agent is characterized by a value distributed on a continuous support. At each time period, agents are tentatively matched, they observe each other's values, and decide to accept or reject the proposed match. At equilibrium, agents sort into a finite number of classes (value intervals) where all agents use exactly the same acceptance cutoff and match within classes. In their case, the equilibrium with class segregation is in pure strategies. In our case, the block structure is even more surprising as agents use mixed (interior) strategies.

The result that informational frictions favor participation and equity is very general. The short intuition is that the lack of information pools the various incentive constraints. Economists are still trying to understand in what kind of economic settings this arises and the consequences. For example, Mekonnen (2019) [20] compares random and directed searches on an agent-object market with common preferences but homogeneous agents. This is equivalent to comparing the no information design to the full information design. At equilibrium, an agent is better off under the random search because she benefits more from the ease of congestion on high-value objects than she suffers from not being able to target objects accurately. Che and Tercieux (2021) [8] study the optimal design of a queueing system when agents' arrival and servicing are governed by a general Markov process. They show that the optimal information is no information (beyond recommendations to join, stay in, or leave the queue). This ensures more participation, which, in the queue environment, increases efficiency.

5 Conclusion

This paper models a stylized market where agents with homogeneous preferences and privately known priority scores can apply (at a cost) to at most one good, and each good is assigned to its highest priority applicant. In this market, participants wonder "Whether and Where to Apply?". The frictions (private information on the priority score and truncation of the application list) ask for a trade-off between ambition (application to high-value goods) and pragmatism (application to under-demanded goods). At equilibrium, high score agents are ambitious, and low score agents are practical, but only in expectation.

The analysis uncovers three salient features of the symmetric equilibrium. One, agents randomize between (potentially many different) applications. Second, agents with scores on a continuous support sort into discrete classes, defined as intervals of priority scores, where they adopt exactly the same strategy. Third, the frictional mechanism is less efficient but more egalitarian than the mechanism with public information. Low-score agents may even be better off with private information than with public information because they benefit from a relative informational competitive advantage. These lessons seem to generalize in various directions: when preferences are only imperfectly correlated, when the cost is endogenous to the score, and when the truncation on the mechanism is less extreme.

We view the value of this paper as twofold. First, it illustrates the role of information on priorities matching markets, whereas the economic literature has mostly focused on uncertain preferences. The analysis emphasizes how the various frictions interact: they interplay to create non-trivial strategic interactions within the agent side of the market and distort the allocation, resulting in counter-intuitive welfare effects. This results in a clear and broad policy recommendation. A designer aiming for some control over the amount of discrimination in the market should jointly design the scoring rule, the allocation rule, and the information structure. Overall, we believe that these results have the potential to shed light on numerous empirical applications in economics and beyond. We have insisted on the matching application to social housing, and beyond, to any market where priority is defined by a point system (teacher allocation, college admissions in many countries). We have also illustrated the same problem with a hunting application.

We think there is a methodology contribution too. To the best of our knowledge, the model is one of the first to combine centralization on the good side (coordination in the timing of applications)[20] and decentralization on the agent size (private information, agents cannot communicate scores). The Application Game displays a novel and rich mode of strategic interactions arising within the agent side of a matching market, resulting in an equally novel equilibrium structure (the block structure in mixed strategies). Last but not least, this paper questions the value of cardinal signals when only ordinality matters and highlights boundary effects. We believe these issues would gain to be formalized in more general informational contexts in the coming years.

We still consider completing three main tasks so that matching operators on the field can take the design aspects of this work seriously. The first one is an empirical test of the equilibrium, using the multiple testable predictions our theory provides. If agents behave as expected, we should observe that the low score agents apply to the full variety of goods, including, for some of them, to the very best goods. By the block structure, we should also observe that agents with quite different scores apply similarly on average, at the population level. The second task would be to provide micro-foundations for the use of a sub-optimal mechanism. One could solve a general, fully flexible information design problem with hybrid social objectives, including designers' political motives. Finally, the existing welfare analysis of the social objectives beyond efficiency (discrimination, participation) remains mostly qualitative. We plan to complement it with quantitative measures such as the distribution of the probabilities (or ratios of probabilities) of successes as a function of the scores.[21]

[20]In the predators and prey example, coordination may happen because prey gathers around water sources at specific times of the day, and predators learn this.

[21]One could also draw standard measures of discrimination from the large economic literature on discrimination.

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A Proofs

Proof of Lemma 1.

- By definition, $\forall\ x_i \in [0,1],\ u_i(N_i|x_i) = 0$. This proves (i) and (ii) for $a_i = N_i$.
- For $a_i = A_i^j,\ j \in \{1,\dots,m\}$, we have by (1):

$$u_i(A_i^j, p_{-i}([x_i,1])|x_i) = q_i^j(x)v^j - c$$

$$q_i^j(x) = \prod_{i' \neq i}\Big(1 - \mathbb{P}\big(x_{i'} > x_i, p_{i'}(x_{i'}) = A_{i'}^j\big)\Big)$$

$$= \prod_{i' \neq i}\Big(F(x_i) + \int {x_i}^1 (1 - p_{i'}^j(x)) f(x) dx\Big)$$

$\forall\ i' \in \{1,\dots,n\} \backslash \{i\},\ j \in \{1,\dots,m\},\ F(x_i) + \int {x_i}^1 (1 - p_{i'}^j(x)) f(x) dx$ is continuous.

So the probability of success $x_i \mapsto q_i^j(x)$ and the interim payoff $x_i \mapsto u_i(A_i^j, p_{-i}([x_i,1])|x_i)$ are also continuous. This proves (i).

Set $x_- < x_+$.

$$F(x_+) + \int {x_+}^1 (1 - p_{i'}^j(x)) f(x) dx = F(x_-) + \int {x_-}^{x_+} f(x) dx + \int {x_+}^1 (1 - p_{i'}^j(x)) f(x) dx$$

$$> F(x_-) + \int {x_-}^1 (1 - p_{i'}^j(x)) f(x) dx$$

So:

$$q_i^j(x_+) > q_i^j(x_-) \text{ and } u_i(A_i^j, p_{-i}([x_i,1])|x_+) > u_i(A_i^j, p_{-i}([x_i,1])|x_-)$$

This proves (ii).

In addition:

$$\lim_{x_i \to 1} q_i^j(x) = 1 \implies \lim_{x_i \to 1} u_i(A_i^j, p_{-i}([x_i,1])|x_i) = v^j - c \ (iii)$$

Proof of Lemma 2.

- Action spaces $\forall\ i \in \{1,\dots,n\},\ A_i = \{A^1,\dots,A^m,N\}$ are finite. So by Proposition 1 in Milgrom and Weber (1985) [21], payoffs are equicontinuous (R1).

- Types $x_i,\ i \in \{1,\dots,n\}$ are independent. So by Proposition 3 in Milgrom and Weber (1985) [21], information is absolutely continuous (R2).

Finally, by Theorem 1 in Milgrom and Weber (1985) [21] applied to the AG satisfying R1 and R2, there exists a BNE in the AG.

Proof of Proposition 1. (bis)

We label agents by their (commonly known) order of priority: $x_1 > \dots > x_n$. We denote $\mu_i \in \{1,\dots,m,\emptyset\}$ the Nash equilibrium allocation of agent $i \in \{1,\dots,n\}$ in ϕ^E.

(i) We prove an induction statement: $H(i):\ \mu_i = i,\ i \in \{1,\dots,m\}$

- Initial step

  $$\forall\ \sigma_{-1} \in \Delta(\{A^1,\dots,A^m,N\})^{n-1}: \begin{cases} u_1(A_1^1,\sigma_{-1}) = v^j - c \\ u_1(N_1,\sigma_{-1}) = 0 \end{cases} \implies BR_1(\sigma_{-1}) = \{v_1^j\}$$

  So: $\sigma_1 = v_1^j,\ \mu_1 = 1 - H(1)$.

- Inductive step

  Set $i \in \{1,\dots,m-1\}$ s.t. $H(1),\dots,H(i)$ true.

  $$\forall\ (\sigma_{i+2},\dots,\sigma_n) \in \Delta(\{A^1,\dots,A^m,N\})^{n-i-1}: \begin{cases} \forall\ j \in \{1,\dots,i\},\ u_{i+1}(A_i^j,\sigma_{-(i+1)}) = -c \\ \forall\ j \in \{i+1,\dots,n\},\ u_{i+1}(A_i^j,\sigma_{-(i+1)}) = v^j - c \\ u_i(N_i,\sigma_{-(i+1)}) = 0 \end{cases}$$

  $$\implies BR_{i+1}(\sigma_{-(i+1)}) = \{A_{i+1}^{i+1}\}$$

  So: $\sigma_{i+1} = A_{i+1}^{i+1},\ \mu_{i+1} = i+1 - H(i+1)$.

(ii) We set $m+1 \le i \le n$.

$$\forall\ (\sigma_{m+1},\dots,\sigma_n) \in \Delta(\{A^1,\dots,A^m,N\})^{n-m}: \begin{cases} \text{By } (i):\ \forall\ j \in \{1,\dots,m\},\ u_i(A_i^1,\sigma_{-i}) = -c \\ u_i(N_i,\sigma_{-i}) = 0 \end{cases}$$

$$\implies BR_i(\sigma_{-i}) = \{N_i\}$$

So: $\sigma_i = N_i,\ \mu_i = \emptyset$.

Proof of Proposition 2. (bis)

We define the "interim action set" at score x as the subset of actions that are played with positive probabilities at score x: $v^j \in IAS(x)$ if $\exists\ i \in \{1,\dots,n\}$ s.t. $p_i^j(x) > 0$.

We state and prove a lemma characterizing interim action sets at any BNE of the AG (symmetric or asymmetric).

Lemma 4.

(i) At any BNE p of the AG:

- $\exists\ 1 = x[0] > x[1] > x[2] \geq 0$ *s.t.* : $\begin{cases} IAS((x[1], x[0])) = \{A^1\} \\ IAS((x[2], x[1])) = \{A^1, A^2\} \end{cases}$.
- $\forall\ k \in \{3, \ldots, m\}$, *if* $\exists\ i \in \{1, \ldots, m\},\ x \in [0,1] | p_i^k(x) > 0$, *then:*

  $\exists\ 1 = x[0] > x[1] > ... > x[k] \geq 0$ *s.t.:* $\forall\ j \in \{1, \ldots, k\},\ IAS((x[j], x[j-1])) = \{A^1, \ldots, A^j\}$.
- *If* $\exists\ i \in \{1, \ldots, m\},\ x \in [0,1] | p_i^{m+1}(x) > 0$, *then:*

  $\exists\ 1 = x[0] > x[1] > \ldots > x[m] > x[m+1] = 0$ *s.t.* :
  $\begin{cases} \forall\ j \in \{1, \ldots, m\}, IAS((x[j], x[j-1])) = \{A^1, \ldots, A^j\} \\ \{N\} \subseteq IAS((0, x[m])) \end{cases}$

(ii) Moreover: $x[1]$ *is the same in all multiple BNE of a given AG:* $x[1] = F^{-1}\left(\left(\frac{v^2}{v^1}\right)^{\frac{1}{n-1}}\right)$.

Proof of Lemma 4.

Set a BNE p of the AG.

- By Lemma 1., (iii):

$$\lim_{x_i \to 1} u_i(A_i^1, p_{-i}|x_i) = v^1 - c > \begin{cases} v^j - c = \lim_{x_i \to 1} u_i(A_i^j, p_{-i}|x_i), \forall\ j \in \{2, \ldots, m\} \\ 0 = \lim_{x_i \to 1} u_i(N, p_{-i}|x_i) \end{cases}$$

  And by Lemma 1., (i): $\exists\ x[1]$ s.t. $IAS((x[1], 1)) = \{A^1\}$.

- Suppose all players play A^1 at all scores: $IAS([0,1]) = \{A^1\}$. Then:

$$\lim_{x_i \to 0} q_i^j(x_i) = 0 \implies \lim_{x_i \to 0} u_i(A_i^1, p_{-i}|x_i) = -c < 0$$

$$\forall\ j \in \{2, \ldots, m\}: \lim_{x_i \to 0} q_i^j(x) = 1 \implies \lim_{x_i \to 0} u_i(A_i^j, p_{-i}|x_i) = v^j - c$$

$$\lim_{x_i \to 0} u_i(N_i, p_{-i}|x_i) = 0$$

  Then, by Lemma 2. again, (i) and (ii):

  - $x_i \mapsto u_i(A_i^1, p_{-i}|x_i)$ is continuous and strictly increasing on $[0,1]$ from $-c$ to $v^1 - c$.
  - $x_i \mapsto u_i(A_i^j, p_{-i}|x_i),\ j \in \{2, \ldots, m\}$ is constant on $[0,1]$ and equal to $v^j - c$.
  - $x_i \mapsto u_i(N_i, p_{-i}|x_i),\ j \in \{2, \ldots, m\}$ is constant on $[0,1]$ and equal to 0.

By the bijection theorem, $\exists\ x[1] \in [0,1]$ s.t.:

$$u_i(A_i^1, p_{-i}|x_i) = u_i(A_i^2, p_{-i}|x_i) = v^2 - c > \begin{cases} v^j - c = u_i(A_i^j, p_{-i}|x_i), j \in \{3,\dots,m\} \\ 0 = u_i(N_i, p_{-i}|x_i) \end{cases}$$

And by Lemma 4., (i) again: $\exists\ 0 < x^2 < x^1$ s.t. $IAS((x[2], x[1])) \subseteq \{A^1, A^2\}$.

- Suppose no agent plays A^k, $k \in \{1,\dots,2\}$ on $(x[2], x[1])$. Set $k' \neq k \in \{1,2\}$.

  Then, by Lemma 1., (ii) again:

  - $x_i \mapsto u_i(A_i^k, p_{-i}|x_i)$ is constant on $(x[2], x[1])$ equals to $v^2 - c$.
  - $x_i \mapsto u_i(A_i^{k'}, p_{-i}|x_i)$ is strictly increasing on $(x[2], x[1])$.

  So: $u_i(A_i^k, p_{-i}|x_i) > u_i(A_i^{k'}, p_{-i}|x_i)$, and playing A^k is a profitable deviation. So $A^k \in IAS((x[2], x[1]))$. In total: $\{A^1, A^2\} \subseteq IAS((x[2], x[1]))$.

  Combining both inclusions, we get: $IAS((x^1, x^2)) = \{A^1, A^2\}$.

- The proof for the intervals $(x[k], x[k-1])$, $k \in \{3,\dots,m\}$ below is similar. The inclusion $IAS((x[k], x[k-1])) \subseteq \{A^1,\dots,A^k\}$ relies on the continuity and monotonicity of interim payoffs from Lemma 1., and the intermediate value theorem. The reverse inclusion comes from indifference at $x[k-1]$, the monotonicity of interim payoffs from Lemma 1., and the absence of profitable deviations below $x[k-1]$.

- If $x[m-1]$ exists and $\forall\ j \in \{1,\dots,m\}$ s.t. $\lim_{x_i \to 0} u_i(A_i^j, p_{-i}|x_i) < 0$, then by the intermediate value theorem again, $\exists\ 0 < x[m] < x[m-1]$ s.t. $\{N\} \subseteq IAS((0, x[m])) \subseteq \{A^1,\dots,A^m, N\}$.

All the preceding proves (i). It also characterizes a unique $x[1]$:

$$\begin{aligned} & x[1] := \sup\left\{x \in [0,1] | u_i(A_i^1, p_{-i}([x,1])|x) = u_i(A_i^2, p_{-i}([x,1])|x)\right\} \\ \iff & F(x[1])^{n-1} v^1 - c = v^2 - c \\ \iff & x[1] = F^{-1}\Big(\Big(\frac{v^2}{v^1}\Big)^{\frac{1}{n-1}}\Big) \end{aligned}$$

This proves (ii).

Main proof:

Lemma 3. shows that both actions A^1 and A^2 are played with positive probabilities on $(x[2], x[1])$. At a pure BNE, this implies that different players play different actions, and the equilibrium profile is asymmetric.

Proof of Theorem 1. (bis)

Lemma 3. gives the interim action sets.

Then, the proof proceeds in four steps:

1. Constant probabilities

   We locate on the interval $(x[k], x[k-1])$ and prove that the probability functions $x \mapsto p^j(x),\ j \in \{1, m\}$ are constant on each interval $(x[k], x[k-1])$. To ease notations, we write down explicitly the differential equations characterizing probabilities in the class $C[2]$, then we generically describe the equations for the lower classes.

   - Set $k = 2$, and consider the $(x[2], x[1])$ interval of scores.

     The strong indifference principle applied at a score $x^* \in (x[2], x[1])$ writes as the following differential equation:

$$
\begin{aligned}
&(E^1_{[1]}) : u_i(A^1_i, p_{-i}([x^*, 1])|x^*) = u_i(A^2_i, p_{-i}([x^*, 1])|x^*) \\
&\iff \left(1 - \int_{x[1]}^{1} f(x)dx - \int_{x^*}^{x[1]} p^1(x)f(x)dx\right)^{n-1} v^1 - c = \left(1 - \int_{x^*}^{x[1]} p^2(x)f(x)dx\right)^{n-1} v^2 - c \\
&\iff \left(1 - F(1) + F(x[1]) - F(x[1]) + F(x^*) + \int_{x^*}^{x[1]} p^2(x)f(x)dx\right)^{n-1} v^1 \\
&\qquad = \left(1 - \int_{x^*}^{x[1]} p^2(x)f(x)dx\right)^{n-1} v^2 \\
&\iff F(x^*) + \int_{x^*}^{x[1]} p^2(x)f(x)dx = \left(\frac{v^2}{v^1}\right)^{\frac{1}{n-1}} \left(1 - \int_{x^*}^{x[1]} p^2(x)f(x)dx\right) \\
&\iff (1 + F(x[1])) \int_{x^*}^{x[1]} p^2(x)f(x)dx = F(x[1]) - F(x^*)
\end{aligned}
$$

     Set G^2, a primitive of $p^2 f$. Then:

$$
(E^1_{[1]}) \iff G^2(x[1]) - G^2(x^*) = \frac{F(x[1]) - F(x^*)}{1 + F(x[1])}
$$

     We derive on both sides and get a necessary condition on the probability functions:

$$
\begin{aligned}
(E^1_{[1]}) &\implies p^2(x^*)f(x^*) = \frac{-f(x^*)}{1 + F(x[1])} \\
&\implies \begin{cases} p^2(x^*) = \dfrac{1}{1 + F(x[1])} \\ p^1(x) = \dfrac{F(x[1])}{1 + F(x[1])} \end{cases}
\end{aligned}
$$

We check that those constant probability functions are sufficient, that they indeed verify equation $(E^1_{[1]})$:

$$(1+F(x[1]))\int_{x^*}^{x[1]}\frac{1}{1+F(x[1])}f(x)dx = F(x[1]) - F(x^*)$$
$$\Longleftrightarrow \frac{(1+F(x[1]))}{(1+F(x[1]))}(F(x[1])-F(x^*)) = F(x[1])-F(x^*) \checkmark$$

- Set $k \in \{2,\ldots,m-1\}$, and suppose $H(1),\ldots,H(k)$ hold. Consider the interval $(x[k+1],x[k])$. The strong indifference principle applied at a score $x^* \in (x[k+1],x[k])$ writes as a system of k differential equations with $k+1$ unknowns. We denote those equations $(E^j_{[k+1]}),\ j \in \{1,\ldots,k\}$. Each of them is given by:

$$(E^j_{[k+1]}): u_i(A^1_i, p_{-i}([x^*,1])|x^*) = u_i(v^{k+1}_i, p_{-i}([x^*,1])|x^*)$$
$$\Longleftrightarrow \Big(1-\int_{x^*}^{x[k]} p^j(x)f(x)dx - \sum_{l=j}^{x[k]}\int_{x[l]}^{x[l-1]} p^j(x)f(x)dx\Big)^{n-1} v^j - c$$
$$= \Big(1-\int_{x^*}^{x[k]} p^{k+1}(x)f(x)dx\Big)^{n-1} v^{k+1} - c$$

We replace $p^1(x^*)$ by $1-\sum_{j=2}^{k+1} p^j(x^*)$. We end up with only k unknowns, hence a Cramer system. We use the substitution method, and get a relation between (for instance) only $\int_{x^*}^{x[k]} p^{k+1}(x)f(x)dx$ and $p^k[k]\int_{x[k]}^{x[k-1]} f(x)dx$. We define primitives and derive the whole and get constant probabilities $(p^j_{[k+1]})_{j\in\{1,\ldots,k+1\}}$.

2. Probability levels

We set $k \in \{2,\ldots,m-1\}$, and further exploit the differential equations:

$$(E^j_{[k+1]}): u_i(A^1_i, p_{-i}([x^*,1])|x^*) = u_i(A^{k+1}_i, p_{-i}([x^*,1])|x^*)$$
$$\Longleftrightarrow \Big(1-\int_{x^*}^{x[k]} p^j(x)f(x)dx - \sum_{l=j}^{x[k]}\int_{x[l]}^{x[l-1]} p^j(x)f(x)dx\Big)^{n-1} v^j - c$$
$$= \Big(1-\int_{x^*}^{x[k]} p^{k+1}(x)f(x)dx\Big)^{n-1} v^{k+1} - c$$
$$\Longleftrightarrow \Big(1-\int_{x^*}^{x[k]} p^j(x)f(x)dx - \sum_{l=j}^{x[k]}\int_{x[l]}^{x[l-1]} p^j(x)f(x)dx\Big)$$
$$= \Big(\frac{v^{k+1}}{v^j}\Big)^{\frac{1}{n-1}}\Big(1-\int_{x^*}^{x[k]} p^{k+1}(x)f(x)dx\Big)$$

$$\iff \Big(1 - p^j_{[k+1]}(F(x^*) - F(x[k])) - \sum_{l=j}^{x[k]} p^j_{[l]}((F(x[l]) - F(x[l-1]))\Big)$$
$$= \Big(\frac{v^{k+1}}{v^j}\Big)^{\frac{1}{n-1}}\Big(1 - p^{k+1}_{[k+1]}(F(x^*) - F(x[k]))\Big)$$

We derive on both sides and get:

$$(E^j_{[k+1]}) \implies -p^j_{[k+1]} f(x^*) = -\Big(\frac{v^{k+1}}{v^j}\Big)^{\frac{1}{n-1}} p^{k+1}_{[k+1]} f(x^*)$$
$$\implies p^j_{[k+1]} = \Big(\frac{v^{k+1}}{v^j}\Big)^{\frac{1}{n-1}} p^{k+1}_{[k+1]}$$

This is a recursive formula for the probability levels within class $C[k+1]$. To find the explicit formulas, we use the fact that the $k+1$ probability levels sum up to one:

$$\sum_{j=1}^{k+1} p^j_{[k+1]} = 1 \implies p^{k+1}_{[k+1]} \sum_{l=1}^{k+1} \Big(\frac{v^{k+1}}{v^l}\Big)^{\frac{1}{n-1}} = 1$$
$$\implies p^{k+1}_{[k+1]} = \Big((v^{k+1})^{\frac{1}{n-1}} \sum_{l=1}^{k+1} (v^l)^{\frac{-1}{n-1}}\Big)^{-1}$$
$$\implies p^j_{[k+1]} = \Big((v^j)^{\frac{1}{n-1}} \sum_{l=1}^{k+1} (v^l)^{\frac{-1}{n-1}}\Big)^{-1}$$
$$\implies p^j_{[k+1]} = \Big(\sum_{l=1}^{k+1} \Big(\frac{v^j}{v^l}\Big)^{\frac{1}{n-1}}\Big)^{-1} \text{ (decreasing in } v^j)$$

Finally, on the bottom interval $(0, x[m])$, it cannot be that players apply to $j \in \{1, \ldots, m\}$. Otherwise, by Lemma 1. (ii), $x \mapsto u_i(A^1_i, p_{-i}|x)$ would be increasing so strictly negative at some score $\in (0, x[m])$, hence a profitable deviation to action N. So: $x \mapsto p^{m+1}(x)$ is constant equal to 1 on $(0, x[m])$.

3. Thresholds

   The remaining task is to characterize the thresholds $x[k]$, $k \in \{2, \ldots, m\}$.

   - $k \in \{2, \ldots, m-1\}$.

     By definition:

     $$x[k] := \inf\{x^* \in [0,1] |\ \forall\ x > x^*,\ \min_{l \in \{1,\ldots,k\}} u_i(A^l_i, p_{-i}([x,1])|x) \geq u_i(A^{k+1}_i, p_{-i}([x,1])|x)\}$$

     By the strong indifference principle, we have that all $u_i(A^l_i, p_{-i}([x,1])|x)$, $l \in \{1, \ldots, k\}$

are equal on $(x[k], x[k-1])$. So:

$$\begin{aligned}
&x[k] = \inf\{x^* \in [0,1] |\ \forall\ x > x^*,\ u_i(A_i^k, p_{-i}([x,1])|x) \geq u_i(A_i^{k+1}, p_{-i}([x,1])|x)|x]\} \\
&\Longrightarrow u_i(A_i^k, p_{-i}([x,1])|x[k]) = u_i(A_i^{k+1}, p_{-i}([x,1])|x[k]) \\
&\Longrightarrow \Big(1 - \int_{x[k]}^{x[k-1]} p^k(x) f(x) dx\Big)^{n-1} v^k - c = v^{k+1} - c \\
&\Longrightarrow \big(1 - p^k[k](F(x[k-1]) - F(x[k])\big)^{n-1} v^k - c = v^{k+1} - c \\
&\Longrightarrow p^k[k](F(x[k-1]) - F(x[k])) = 1 - \Big(\frac{v^{k+1}}{v^k}\Big)^{\frac{1}{n-1}} \\
&\Longrightarrow F(x[k]) = F(x[k-1]) + \frac{1}{p^k[k]}\Big(-1 + \Big(\frac{v^{k+1}}{v^k}\Big)^{\frac{1}{n-1}}\Big)
\end{aligned}$$

Plugging in the formula for the probability levels, we get:

$$F(x[k]) = F(x[k-1]) + (v^k)^{\frac{1}{n-1}} \sum_{l=1}^{k} (v^l)^{\frac{-1}{n-1}} \Big(-1 + \Big(\frac{v^{k+1}}{v^k}\Big)^{\frac{1}{n-1}}\Big)$$
$$F(x[k]) = F(x[k-1]) + ((v^{k+1})^{\frac{1}{n-1}} - (v[k])^{\frac{1}{n-1}}) \sum_{l=1}^{k} (v^l)^{\frac{-1}{n-1}}$$

This is a recursive formula. From this, we get the explicit formula for the thresholds:

$$F(x[k]) = F(x[0]) + \sum_{j=1}^{k} ((v^{j+1})^{\frac{1}{n-1}} - (v^j)^{\frac{1}{n-1}}) \sum_{l=1}^{k} (v^l)^{\frac{-1}{n-1}}$$

The second term is close to being a telescopic sum. It writes $\sum_{j=1}^{k} \alpha_j (v^j)^{\frac{1}{n-1}}$ with:

- $j = 1$: $\alpha_1 (v^1)^{\frac{1}{n-1}} = -(v^1)^{\frac{1}{n-1}} (v^1)^{\frac{-1}{n-1}} = -1$
- $2 \leq j \leq k$:
  $\alpha_j (v^j)^{\frac{1}{n-1}} = \Big(\sum_{l=1}^{j-1} (v^l)^{\frac{1}{n-1}} - \sum_{l=1}^{j} (v^l)^{\frac{-1}{n-1}}\Big)(v^j)^{\frac{-1}{n-1}} = -(v^j)^{\frac{1}{n-1}} (v^j)^{\frac{-1}{n-1}} = -1$
- $j = k+1$: $\alpha_{k+1} (v^{k+1})^{\frac{1}{n-1}} = \sum_{l=1}^{k} (v^l)^{\frac{-1}{n-1}} (v^{k+1})^{\frac{1}{n-1}} = \sum_{l=1}^{k} \Big(\frac{v^{k+1}}{v^l}\Big)^{\frac{1}{n-1}}$

We get:

$$F(x[k]) = 1 - k + \sum_{l=1}^{k} \Big(\frac{v^{k+1}}{v^l}\Big)^{\frac{1}{n-1}}$$

- $k = m$

$$\begin{aligned}
& u_i(A_i^m, p_{-i}([x[m],1])|x[m]) = u_i(N, p_{-i}([x[m],1])|x[m]) \\
\iff & \big(1 - p_{[m]}^m (F(x[m]) - F(x[m-1]))\big)^{n-1} v^m - c = 0 \\
\iff & F(x[m]) = F(x[m-1]) + \frac{1}{p_{[m]}^m}\Big(-1 + \big(\frac{c}{v^m}\big)^{\frac{1}{n-1}}\Big) \\
\iff & F(x[m]) = 2 - m + \sum_{l=1}^{m-1} \big(\frac{v^m}{v^l}\big)^{\frac{1}{n-1}} + \sum_{l=1}^{m} \Big(\frac{v^m}{v^l}\Big)^{\frac{1}{n-1}} \Big(-1 + \big(\frac{c}{v^m}\big)^{\frac{1}{n-1}}\Big) \\
\iff & F(x[m]) = 1 - m + \sum_{l=1}^{m} \big(\frac{c}{v^l}\big)^{\frac{1}{n-1}}
\end{aligned}$$

4. Number of classes

$$\begin{aligned}
k_0(p^*) = k \in \{1,\dots,m\} \iff & F(x[k]) \le 0 < F(x[k-1]) \\
\iff & 1 - k + \sum_{l=1}^{k} \Big(\frac{v^{k+1}}{v^l}\Big)^{\frac{1}{n-1}} \le 0 < 2 - k + \sum_{l=1}^{k-1} \Big(\frac{v^k}{v^l}\Big)^{\frac{1}{n-1}} \\
\iff & 1 + \sum_{l=1}^{k} \Big(\frac{v^{k+1}}{v^l}\Big)^{\frac{1}{n-1}} \le k < 2 + \sum_{l=1}^{k-1} \Big(\frac{v^k}{v^l}\Big)^{\frac{1}{n-1}} \\
k_0(p^*) = m+1 \iff & F(x[m]) > 0 \\
\iff & 2 - m + \sum_{l=1}^{m} \Big(\frac{c}{v^l}\Big)^{\frac{1}{n-1}} > 0 \\
\iff & 2 + \sum_{l=1}^{m} \Big(\frac{c}{v^l}\Big)^{\frac{1}{n-1}} > m
\end{aligned}$$

Proof of Corollary 1. (bis)

We denote $\gamma(x)$ the distribution with support $\{v^j, \dots, A^m\}$ and probabilities: $\mathbb{P}(\gamma(x) = A^j) := p^j(x),\ j \in \{1,\dots,m\}$.

By Theorem 1. (bis), we have for $x \in C[k] = (x[k], x[k-1])$:

$$\sum_{l=1}^{j} p^l(x) = \sum_{l=1}^{j} p^l[k] = \begin{cases} \sum_{l=1}^{j} \Big((v^j)^{\frac{1}{n-1}} \sum_{l=1}^{k} (v^l)^{\frac{-1}{n-1}}\Big)^{-1} & \text{if } j < k \\ 1 & \text{if } j \ge k \end{cases}$$
$$= \begin{cases} \dfrac{\sum_{l=1}^{j} (v^l)^{\frac{-1}{n-1}}}{\sum_{l=1}^{k} (v^l)^{\frac{-1}{n-1}}} & \text{if } j < k \\ 1 & \text{if } j \ge k \end{cases}$$

We set $0 \le x' < x \le 1$.

We want to prove $\forall\ j \in \{1,\dots,m\}$:

$$(\star)_j : \sum_{l=1}^{j} p^l(x') \leq \sum_{l=1}^{j} p^l(x)$$

- If x', x belong to the same class $x', x \in C[k]$, then by the block structure:

$$\sum_{l=1}^{j} p^l(x') = \sum_{l=1}^{j} p^l(x)$$

  So $(\star)_j,\ j \in \{1,\dots,m\}$ trivially holds.

- If x', x belong to different classes, $x' \in C[k'],\ x \in C[k],\ k < k'$, then there are two subcases:
  - If $k' = m+1$ then $\forall\ j \in \{1,\dots,m\}$, $\sum_{l=1}^{j} p^l(x') = 0$ and $(\star)_j$ is trivially verified.
  - If $k' \leq m$ then:

    For $j \leq k$, then: $\sum_{l=1}^{j} p^l(x) = 1$ and $(\star)_j$ is trivially verified.

    For $j < k < k'$, then:

$$\frac{\sum_{l=1}^{j} p^l(x)}{\sum_{l=1}^{j} p^l(x')} = \frac{\sum_{l=1}^{k'} (v^l)^{\frac{-1}{n-1}}}{\sum_{l=1}^{k} (v^l)^{\frac{-1}{n-1}}} > 1 \implies (\star)_j$$

Proof of Corollary 2. (bis)

(i) The ex ante probability of applying to good j is:

$$\begin{aligned} p^j := & \int_0^1 p^j(x) f(x) dx \\ = & \sum_{k=j}^{m} (F(x[k-1]) - F(x[k])) p^j[k] \end{aligned}$$

For $1 \leq k \leq m$, by Theorem 1. we have:

$$\begin{aligned} (F(x[k-1]) - F(x[k])) p^j[k] &= \frac{1}{p^k[k]} \Big(1 - (\frac{v^{k+1}}{v^k})^{\frac{1}{n-1}}\Big) p^j[k] \\ &= \sum_{l=1}^{k} (\frac{v^k}{v^l})^{\frac{1}{n-1}} \frac{1}{\sum_{l=1}^{k} (\frac{v^j}{v^l})^{\frac{1}{n-1}}} \Big(1 - (\frac{v^{k+1}}{v^k})^{\frac{1}{n-1}}\Big) \\ &= (\frac{v^k}{v^j})^{\frac{1}{n-1}} \Big(1 - (\frac{v^{k+1}}{v^k})^{\frac{1}{n-1}}\Big) \\ &= \frac{(v^k)^{\frac{1}{n-1}} - (v^{k+1})^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}} \quad (\star) \end{aligned}$$

Summing up, we recognize a telescopic sum and we get:

$$p^j = \frac{(v^j)^{\frac{1}{n-1}} - (v^{m+1})^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}}$$

$$p^j = 1 - (\frac{c}{v^j})^{\frac{1}{n-1}} \text{ (increasing in } v^j\text{)}$$

(ii) We denote γ^j the distribution of scores applying to good j. The support is $[x[m], x[j-1]]$.

The cdf is:

$$\begin{aligned}
\mathbb{P}(\gamma^j \leq x) &= \frac{\int_0^x p^j(t) f(t) dt}{p^j} \\
&= \frac{1}{p^j}\Big(\sum_{k+1}^{m}(F(x[l-1]) - F(x[l]))p^j[l] + (F(x) - F(x[k])p^j[k]\Big) \\
&= \frac{1}{p^j}\Big(\sum_{k+1}^{m}\frac{(v^l)^{\frac{1}{n-1}} - (v^{l+1})^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}} + \alpha(F(x[k-1]) - F(x[k])p^j[k]\Big) \text{ (α constant, by block structure)} \\
&= \frac{1}{p^j}\Big(\frac{(v^{k+1})^{\frac{1}{n-1}} - c^{\frac{1}{n-1}}}{(v^j)\frac{1}{n-1}} + \alpha\frac{(v^k)^{\frac{1}{n-1}} - (v^{k+1})^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}}\Big) \\
&= \frac{(v^j)^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}} - c^{\frac{1}{n-1}}}\Big(\frac{(v^{k+1})^{\frac{1}{n-1}} - c^{\frac{1}{n-1}}}{(v^j)\frac{1}{n-1}} + \alpha\frac{(v^k)^{\frac{1}{n-1}} - (v^{k+1})^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}}\Big) \\
&= \frac{\alpha(v^k)^{\frac{1}{n-1}} + (1-\alpha)(v^{k+1})^{\frac{1}{n-1}} - c^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}} - c^{\frac{1}{n-1}}}
\end{aligned}$$

The cdf decreases in v^j, increases in j, $\forall\ j \in \{1,\ldots,m\}$.

Proof of Proposition 3. (bis)

By Theorem 1. (bis), the formulas for probability levels and thresholds depend only on the ratios between good values or on the ratios between the cost and a good value.

Proof of Proposition 4. (bis)

By Theorem 1. (bis), the threshold formula $x[k]$ only depends on $v^j, j \in \{1,\ldots,k+1\}$. The probability level formulas only depends on $v^j, j \in \{1,\ldots,k\}$. This proves (i).

By Corollary 2. (bis), $p^j, j > k_0$ only depends on v^j. This prooves (ii).

Proof of Proposition 5. (bis)

By Theorem 1. (bis), probability levels do not depend on F. This proves (i).

By Theorem 1. (bis), the mass of class $C[k]$ is:

$$F(x[k-1]) - F(x[k]) = 1-(k-1) + \sum_{l=1}^{k-1}\Big(\frac{v^k}{v^l}\Big)^{\frac{1}{n-1}} - \Big(1-k+\sum_{l=1}^{k}\Big(\frac{v^{k+1}}{v^l}\Big)^{\frac{1}{n-1}}\Big)$$
$$= 1-\frac{v^{k+1}}{v^k}$$

It does not depend on F, which proves (ii).

Proof of Proposition 6. (bis)

- To compute the ex ante welfare W^F, we use the formula of the interim welfare (see proof of Proposition 7. (bis) below):

$$W^F(x) = \sum_{j=1}^{k} p^j[k].\Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} v^j - c,\ \ x\in C[k] = [x[k], x[k-1]]$$

The ex ante welfare aggregates interim welfares, taking into account the distribution of scores:

$$W^F = \int_0^1 W^F(x) f(x)dx$$
$$= \sum_{k=1}^{m}\Big[\int_{x[k]}^{x[k-1]} \sum_{j=1}^{k} p^j[k].\Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} v^j f(x)dx - c(F(x[k-1]) - F(x[k]))\Big]$$

We denote, for $k\in\{1,\dots,m\}$:

$$I[k] := \int_{x[k]}^{x[k-1]} \sum_{j=1}^{k} p^j[k].\Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} v^j f(x)dx$$
$$= \sum_{j=1}^{k} p^j[k]v^j.\int_{x[k]}^{x[k-1]} \Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} f(x)dx$$

We denote, for $j\in\{1,\dots,k\}$:

$$L^j[k] := \int_{x[k]}^{x[k-1]} \Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} f(x)dx$$
$$= \Big[\frac{1}{np^j[k]}\Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n}\Big]_{x[k-1]}^{x[k]}$$
$$= \frac{1}{np^j[k]}\Big[(\frac{v^k}{v^j})^{\frac{n}{n-1}} - \Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x[k]))p^j[k]\Big)^{n}\Big]$$
$$\overset{(\star)}{=} \frac{1}{np^j[k]}\Big[(\frac{v^k}{v^j})^{\frac{n}{n-1}} - \Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - \frac{(v^k)^{\frac{1}{n-1}} - (v^{k+1})^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}}\Big)^{n}\Big]$$

$$= \frac{1}{np^j[k]}\Big[\big(\frac{v^k}{v^j}\big)^{\frac{n}{n-1}} - \Big(\big(\frac{v^{k+1}}{v^j}\big)^{\frac{1}{n-1}}\Big)^n\Big]$$
$$= \frac{1}{np^j[k]}\Big[\frac{(v^k)^{\frac{n}{n-1}} - (v^{k+1})^{\frac{n}{n-1}}}{(v^j)^{\frac{n}{n-1}}}\Big]$$

We substitute the $(L^j[k])_{j\in\{1,\dots,k\}}$ in $I[k]$ and get:

$$I_k = \sum_{j=1}^{k} p^j[k]v^j.\frac{1}{np^j[k]}\Big[\frac{(v^k)^{\frac{n}{n-1}} - (v^{k+1})^{\frac{n}{n-1}}}{(v^j)^{\frac{n}{n-1}}}\Big]$$
$$= \frac{1}{n}((v^k)^{\frac{n}{n-1}} - (v^{k+1})^{\frac{n}{n-1}})\sum_{j=1}^{k}(v^j)^{\frac{-1}{n-1}}$$

We substituting the $(I[k])_{k\in\{1,\dots,m\}}$ in W^F and get:

$$W^F = \frac{1}{n}\Big(\sum_{k=1}^{m}((v^k)^{\frac{n}{n-1}} - (v^{k+1})^{\frac{n}{n-1}})\sum_{j=1}^{k}(v^j)^{\frac{-1}{n-1}}\Big) - c\sum_{k=1}^{m}(F(x[k-1]) - F(x[k]))$$

The second term is a telescopic sum, and the first term is almost a telescopic sum.

- $k=1$: $(v^j)^{\frac{n}{n-1}}.(v^j)^{\frac{-1}{n-1}} = v^j$
- $2 \le k \le m$: $(v^k)\frac{n}{n-1}\big(\sum_{j=1}^{k}(v^j)\frac{-1}{n-1} - \sum_{j=1}^{k-1}(v^j)\frac{-1}{n-1}\big) = (v^k)^{\frac{n}{n-1}}(v^k)^{\frac{-1}{n-1}} = v^k$
- $k=m+1$: $-(v^{m+1})^{\frac{n}{n-1}}\sum_{j=1}^{m}(v^j)\frac{-1}{n-1}$

In total, we get:

$$W^F = \frac{1}{n}\big(\sum_{k=1}^{m} v^k - (v^{m+1})^{\frac{n}{n-1}}\sum_{j=1}^{m}(v^j)^{\frac{-1}{n-1}}\big) - c(1-F(x[m]))$$
$$= \frac{1}{n}\big(\sum_{k=1}^{m} v^k - c^{\frac{n}{n-1}}\sum_{j=1}^{m}(v^j)^{\frac{-1}{n-1}}\big) - c\Big(m - c^{\frac{1}{n-1}}\sum_{l=1}^{m}(v^l)^{-\frac{1}{n-1}}\Big)$$
$$= \frac{1}{n}\sum_{k=1}^{m}(v^k - c) - \frac{c^{\frac{n}{n-1}}}{n}\sum_{j=1}^{m}(v^j)^{\frac{-1}{n-1}} + c^{\frac{n}{n-1}}\sum_{j=1}^{m}(v^j)^{\frac{-1}{n-1}} - c(m - \frac{m}{n})$$
$$= \frac{1}{n}\sum_{k=1}^{m}(v^k - c) + \frac{n-1}{n}\Big(c^{\frac{n}{n-1}}\sum_{j=1}^{m}(v^j)^{\frac{-1}{n-1}} - cm\Big)$$

- The ex ante welfare in the efficient mechanism is given by:

$$W^E := u(\sigma^*) = \frac{1}{n}\sum_{k=1}^{m}(v^k - c)$$

- The welfare gap is given by:

$$W^E - W^F = \frac{n-1}{n}\Big(cm - c^{\frac{n}{n-1}} \sum_{j=1}^{m} (v^j)^{\frac{-1}{n-1}}\Big)$$

$$= c\frac{n-1}{n}\Big(m - \sum_{j=1}^{m} (\frac{c}{v^j})^{\frac{1}{n-1}}\Big)$$

$$\forall j \in \{1,\dots,m\},\ c < v^j \Rightarrow (\frac{c}{v^l})^{\frac{1}{n-1}} < 1 \Rightarrow m > \sum_{l=1}^{m} (\frac{c}{v^l})^{\frac{1}{n-1}} \Rightarrow W^E - W^F > 0$$

$W^E - W^F$ increases in v^j, $\forall\ j \in \{1,\dots,m\}$.

Proof of Lemma 3. (bis)

(i) By Proposition 1., at the NE of the game with perfect information, an agent receives utility $(v^i - c)$ iff she is ranked i^{th} in priority. For an agent with score x, this happens with probability:

$$\mathbb{P}(x \text{ ranked } i) = \binom{n-1}{i-1}(1-F(x))^{i-1}F(x)^{n-i}$$

In total, we have:

$$W^E(x) = \sum_{i=1}^{m} \binom{n-1}{i-1}(1-F(x))^{i-1}F(x)^{n-i}(v^i - c)$$

F cdf continuous $\Rightarrow$ W^E continuous.

By definition:

$$\sum_{i=1}^{n} \binom{n-1}{i-1}(1-F(x))^{i-1}F(x)^{n-i} = 1$$

When x increases, $\mathbb{P}(x \text{ ranked } i)$ increases (decreases) for small (large) i - associated to high (low) utilities $(v^i - c)$. So $W^E(x)$ strictly increases with x.

We compute:

$$W^E(0) = 0$$

$$W^E(1) = v^1 - c$$

(ii) – Within class: By Theorem 1. (bis), for $x \in C[k],\ k \in \{1,\dots,m\}$:

$$W^F(x) := u(p^*|x) = \sum_{j=1}^{k} p^j[k]q^j(x)v^j - c$$
$$q^j(x) = \Big(1 - \sum_{l=j}^{k-1}(F(x[l-1]) - F(x[l]))p^j[l] - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1}$$

From Corollary 1., we know:

$$(F(x[l-1]) - F(x[l]))p^j[l] = \frac{(v^l)^{\frac{1}{n-1}} - (v^{l+1})^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}} \quad (\star)$$

Summing up, we recognize a telescopic sum and we get:

$$\sum_{l=j}^{k-1}(F(x[l-1]) - F(x[l]))p^j[l] = \frac{(v^j)^{\frac{1}{n-1}} - (v^k)^{\frac{1}{n-1}}}{(v^j)^{\frac{1}{n-1}}} = 1 - (\frac{v^k}{v^j})^{\frac{1}{n-1}}$$

We substitute into the probability of success, and get:

$$q^j(x) = \Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1}$$
$$W^F(x) := u(p^*|x) = \sum_{j=1}^{k} p^j[k]v^j q^j(x) - c$$

By Theorem 1. again, for $x \in C[m+1]$: $W^F(x) = 0$ (constant).

F cdf continuous $\Rightarrow W^F$ continuous.

For the monotonicity, we differentiate:

$$\frac{\partial q^j(x)}{\partial x} = p^j[k]f(x)(n-1)\Big((\frac{v^k}{v^j})^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-2} > 0$$
$$\frac{\partial W^F(x)}{\partial x} = \sum_{j=1}^{k} p^j[k]v^j \frac{\partial q^j(x)}{\partial x} > 0$$

This proves the monotonicity within each class.

– Across classes: Indifference at $x[k]$ gives the continuity.

For the monotonicity, we consider two adjacent classes $C[k]$ and $C[k+1]$, $k \in\in \{1,\dots,m-1\}$. By indifference again, the two formulas for $W^F(x[k])$ are equal. By monotonicity within classes: $W^F(x) < W^F(x[k]), x \in C[k+1]$ and $W^F(x[k]) < W^F(x'), x' \in C[k]$, so $W^F(x) < W^F(x')$.

If $k_0 \le m$, we compute:

$$W^F(0) = b,\ b > 0 \text{ (sum of positive terms)}$$
$$W^F(1) = v^1 - c$$

If $k_0 = m+1$, $W^F(x) = 0\ \forall\ x \in C[m+1]$ by definition. And we compute:

$$W^F(1) = v^1 - c$$

Proof of Theorem 2. (bis)

By Lemma 3. (bis) and Definition 2..

Proof of Proposition 7. (bis)

(i) By Lemma 3. (bis): $W^E(0) = 0,\ W^F(0) = b > 0$ so $(W^E - W^F)(0) = -b < 0$.

By continuity, $\exists\ \underline{x} > 0$ s.t. $\forall\ x \in (0, \underline{x})(W^E - W^F)(x) < 0$.

By Proposition 6. (bis), for $x \in C[1]$:

$$W^E(x) = \sum_{i=1}^{m} \binom{n-1}{i-1}(1-F(x))^{i-1}F(x)^{n-i}(v^i - c)$$
$$W^F(x) = F(x)^{n-1}(v^1 - c)$$
$$(W^E - W^F)(x) = \sum_{i=2}^{m} \binom{n-1}{i-1}(1-F(x))^{i-1}F(x)^{n-i}(v^i - c) > 0$$
$$(W^E - W^F)(1) = 0$$

By continuity, $\exists\ \overline{x} < x[1]$ s.t. $\forall\ \overline{x} < x < 1:\ (W^E - W^F)(x) > 0$

(ii) By Theorem 1. (bis), in ϕ^F with $k_0 < m$, all scores participate with probability 1.

By Propositon 1. (bis), in ϕ^E, $n - m$ agents do not participate.

Proof of Proposition 8. (bis)

(i) We have:

$$r^1(x) = (F(x))^{n-1}$$
$$r'^1(x) = (n-1)f(x)F(x)^{n-2} > 0$$
$$r^1(0) = 0$$
$$r^1(1) = 1$$

(ii)

$$r^n(x) = (1-F(x))^{n-1}$$
$$r'^n(x) = -(n-1)f(x)(1-F(x))^{n-2} > 0$$
$$r^n(0) = 1$$
$$r^n(1) = 0$$

(iii) We have, for $j \in \{2,\dots,m\}$:

$$r^j(x) = \binom{n-1}{j-1}(1-F(x))^{j-1}F(x)^{n-j}$$
$$r'^j(x) = \binom{n-1}{j-1}f(x)(1-F(x))^{j-2}F(x)^{n-j-1}\big(-(j-1)f(x)+(n-j)(1-F(x))\big)$$
$$r'^j(x) = 0 \iff -(j-1)f(x)+(n-j)(1-F(x)) = 0$$
$$\iff x < F^{-1}(\frac{n-j}{n-1})$$
$$r^j(0) = 0$$
$$r^j(1) = 0$$

Proof of Proposition 9. (bis)

- We set: $g^j : x \mapsto -r^j(x)(1+ln(r^j(x)).$

  We have:

$$\frac{\partial g^j(x)}{\partial x} = -\frac{\partial r^j(x)}{\partial x}(2+ln(r^j(x))$$

The two derivatives $\frac{\partial g^j(x)}{\partial x}$ and $\frac{\partial r^j(x)}{\partial x}$ are of the same sign iff $-(2+ln(r^j(x))>0 \iff r^j(x)<e^{-2}$.

So $g^j()$ and $r^j()$ have the same (opposite) variations on domains where $r^j()<e^{-2}$ ($r^j()>e^{-2}$).

- By Proposition 8. (bis):
  $\begin{cases} \forall\ j\in\{1,\ldots,n-1\}:\ \exists\ x_0^j\in(0,1] \text{ s.t. } r^j() \text{ strictly increases on } [0,x_0^j] \text{ from } r^j(0)=0. \\ \exists\ x_0^n\in(0,1) \text{ s.t. } r^n(x) \text{ strictly decreases on } [0,x_0^n] \text{ from } r^n(0)=1. \end{cases}$

  We have: $\forall\ j\in\{1,\ldots,n-1\},\ r^j(0)=0<e^{-2}<1=r^n(0)$.

  So $\begin{cases} \forall\ j\in\{1,\ldots,n-1\},\ \exists\ \underline{x_0^j}\in(0,1) \text{ s.t. } r^j() \text{ strictly increases and is } <e^{-2} \text{ on } [0,\underline{x_0^j}]. \\ \exists\ \underline{x_0^n}\in(0,1) \text{ s.t. } r^n() \text{ strictly decreases and is } >e^{-2} \text{ on } [0,\underline{x_0^n}]. \end{cases}$

  So $\forall\ j\in\{1,\ldots,n\},\ \exists\ \underline{x_0^j}\in(0,1)$ s.t. $g^j()$ strictly increases on $[0,\underline{x_0^j}]$.

  We set: $\underline{x_0}:=\min(\underline{x_0^1},\ldots,\underline{x_0^n})$.

  Then $\forall\ j\in\{1,\ldots,n\},\ g^j()$ strictly increases on $[0,\underline{x_0}]$.

  So $H[r]()=\sum_{j=1}^n g^j()$ strictly increases on $[0,\underline{x_0}]$.

- By Proposition 8. (bis):
  $\begin{cases} \forall\ j\in\{2,\ldots,n\}:\ \exists\ x_0^j\in(0,1] \text{ s.t. } r^j() \text{ strictly decreases on } [0,x_0^j] \text{ until} r^j(1)=0. \\ \exists\ x_0^1\in(0,1) \text{ s.t. } r^1(x) \text{ strictly increases on } [0,x_0^n] \text{ until } r^n(1)=1. \end{cases}$

  We have: $\forall\ j\in\{2,\ldots,n\},\ r^j(1)=0<e^{-2}<1=r^1(0)$.

  So $\begin{cases} \forall\ j\in\{2,\ldots,n\},\ \exists\ \overline{x_0^j}\in(0,1) \text{ s.t. } r^j() \text{ strictly decreases and is } <e^{-2} \text{ on } [0,\overline{x_0^j}]. \\ \exists\ \overline{x_0^n}\in(0,1) \text{ s.t. } r^n() \text{ strictly increases and is } >e^{-2} \text{ on } [0,\overline{x_0^n}]. \end{cases}$

  So $\forall\ j\in\{1,\ldots,n\},\ \exists\ \overline{x_0^j}\in(0,1)$ s.t. $g^j()$ strictly decreases on $[0,\overline{x_0^j}]$.

  We set: $\overline{x_0}:=\max(\overline{x_0^1},\ldots,\overline{x_0^n})$.

  Then $\forall\ j\in\{1,\ldots,n\},\ g^j()$ strictly decreases on $[0,\overline{x_0}]$.

  So $H[r]()=\sum_{j=1}^n g^j()$ strictly decreases on $[0,\overline{x_0}]$.

Proof of Proposition 10.

The interim payoffs at score 1 are:

$$u_i(A_i^{\oplus})|1)=v-c$$

$$u_i(A_i^{\ominus})|1)=u-c$$

So $p^*(1) = A^\oplus$. By continuity: $\exists\, x[1] < 1$ s.t. $p^*((x[1], 1]) = A^\oplus$.

The interim payoffs at score $x[1] < x < 1$ are:

$$u_i(A_i^\oplus, p_{-i}^* | x) = (1 - (1 - x)\theta)\, v - c$$
$$u_i(A_i^\ominus, p_{-i}^* | x) = (1 - (1 - x)(1 - \theta))\, u - c$$

The agent starts being indifferent between the two actions at score $x[1]$:

$$u_i(A_i^\oplus, p_{-i}^* | x[1]) = u_i(A_i^\ominus, p_{-i}^* | x[1]) \iff x[1] = \frac{\theta u - (1-\theta) v}{\theta v + (1-\theta) u}$$

We have: $x[1] > 0 \iff \theta > \frac{v}{v+u}$, and $\frac{v}{v+u} > \frac{1}{2}$ so the condition is non trivial.

The interim payoffs at score $x < x[1]$ are:

$$u_i(A_i^\oplus, p_{-i}^* | x) = [x + ((1 - x[1]) + (x[1] - x)p^\oplus[2])(1 - \theta) + (x[1] - x)(1 - p^\oplus[2])\theta]\, v - c$$
$$u_i(A_i^\ominus, p_{-i}^* | x) = [x + ((1 - x[1]) + (x[1] - x)p^\oplus[2])\theta + (x[1] - x)(1 - p^\oplus[2])(1 - \theta)]\, u - c$$

We compute the probability level $p^\oplus[2]$ that makes the agent indifferent between the two actions:

$$u_i(A_i^\oplus, p_{-i}^* | x) = u_i(A_i^\ominus, p_{-i}^* | x)$$

We differentiate this equation and get:

$$[1 - p^\oplus[2](1 - \theta) - (1 - p)\theta]\, v = [1 - p\theta - (1 - p)(1 - \theta)]\, u$$
$$\iff p^\oplus[2] = \frac{\theta u - (1-\theta) v}{(2\theta - 1)(u + v)}$$

We check that this constant level indeed verifies the indifference equation.

Proof of Proposition 11.

The proof is similar to the proof of Theorem 1. (bis).

- In classes $C[1]$ to $C[m]$, the endogenous cost simplifies in the differential equations, and we get the same system as with exogenous cost.

- In the equation for threshold $x[m]$, the cost does not simplify:

$$
\begin{aligned}
& u_i(A_i^m, p_{-i}([x[m],1])|x[m]) = u_i(N_i, p_{-i}([x[m],1])|x[m]) \\
\iff & \big(1 - p^m[m](F(x[m]) - F(x[m-1]))\big)^{n-1} v^m - c(x[m]) = 0 \\
\iff & F(x[m]) = 1 - m + \sum_{l=1}^{m} \Big(\frac{c(x[m])}{a^l}\Big)^{\frac{1}{n-1}}
\end{aligned}
$$

The difference in $F(x[m])$ in the endogenous cost model vs the exogenous cost model is:

$$
(c(x[m]) - c) \sum_{l=1}^{m} \Big(\frac{1}{a^l}\Big)^{\frac{1}{n-1}} > 0 \iff c(x[m]) > c
$$

Proof of Corollary 3.

The interim payoff with endogenous cost writes ($x \in C[k],\ k \in \{1, \dots, m\}$):

$$
\begin{aligned}
W_e^F(x) &:= \sum_{j=1}^{k} p^j[k].\Big(\Big(\frac{v^k}{v^j}\Big)^{\frac{1}{n-1}} - (F(x[k-1]) - F(x))p^j[k]\Big)^{n-1} - c(x) \\
\frac{\partial W_e^F(x)}{\partial x} &= \frac{\partial W^F(x)}{\partial x} - \frac{\partial c(x)}{\partial x}
\end{aligned}
$$

$c()$ decreasing with x ($\frac{\partial c(x)}{\partial x} < 0$) implies $\frac{\partial W_e^F(x)}{\partial x} > \frac{\partial W^F(x)}{\partial x}$.

Proof of Proposition 12.

The interim payoffs at score 1 are:

$$
\begin{aligned}
u_i(A_i^1|1) &= v^1 - c \\
u_i(A_i^2|1) &= v^2 - c \\
u_i(B_i|1) &= v^1 - 2c \\
u_i(N_i|1) &= 0
\end{aligned}
$$

So $p_i^*(1) = A^1$. By continuity, $\exists\ x[1] < 1$ s.t. $p_i^*((x[1],1]) = A^1$.

The interim payoffs at lower scores are:

$$u_i(A_i^1, p_{-i}^*([x,1])|x) = x^2 v^1 - c$$
$$u_i(A_i^2, p_{-i}^*([x,1])|x) = v^2 - c$$
$$u_i(B_i, p_{-i}^*([x,1])|x) = x^2 v^1 + (1-x^2) v^2 - 2c$$
$$u_i(N_i, p_{-i}^*([x,1])|x) = 0$$

We solve the indifference equations:

$$u_i(A_i^1, p_{-i}^*([x,1])|x) = u_i(A_i^2, p_{-i}^*([x,1])|x) \iff x = \sqrt{\frac{v^2}{v^1}}$$
$$u_i(A_i^1, p_{-i}^*([x,1])|x) = u_i(B_i, p_{-i}^*([x,1])|x) \iff x = \sqrt{1 - \frac{c}{v^1}}$$
$$\sqrt{\frac{v^2}{v^1}} < \sqrt{1 - \frac{c}{v^1}} \iff \frac{v^2}{v^1} + \frac{c}{v^2} > 1$$

We get two cases:

(i) $\frac{v^2}{v^1} + \frac{c}{v^2} > 1$: Below a threshold $x[1] = \sqrt{\frac{v^2}{v^1}}$, the agent plays A^1 and A^2 with indifference. Similarly to the case with truncation, she plays A^2 with probability $p^2[2] = \frac{1}{1+x[1]}$.

At scores $x < x[1]$, the interim payoffs are:

$$u_i(A_i^1, p_{-i}^*([x,1]))|x) = u_i(A_i^2, p_{-i}^*([x,1]))|x) = \big(1 - (1 - x[1]) - (x[1] - x) p^1[2]\big)^2 v^1 - c$$
$$u_i(B_i, p_{-i}^*([x,1]))|x) = \big(1 - (1 - x[1]) - (x[1] - x) p^1[2]\big)^2 v^1 + \big(1 - (x[1] - x) p^2[2]\big) v^2 - 2c$$
$$\Delta(x) := u_i(A_i^1, p_{-i}^*([x,1]))|x) - u_i(B_i, p_{-i}^*([x,1]))|x) = c - \big(1 - (x[1] - x) p^2[2]\big)^2 v^2$$
$$\frac{\partial \Delta(x)}{\partial x} = -2p^2[2]\big(1 - (x[1] - x) p^2[2]\big)^2 v^2 < 0$$

$\Delta()$ strictly decreases until $x[1]$. And by definition of this case: $\Delta(x[1]) = 0$. So on the left of $x[1]$, $\Delta(x) > 0$. The agent does not switch to B. Just as in the model with truncation, she randomizes between A^1 and A^2 until potentially a threshold $x[2]$ where she starts playing N. Below $x[2]$, all interim payoffs stay constant, so the agent keeps on playing N until score 0.

- $\frac{v^2}{v^1} + \frac{c}{v^2} < 1$: Below a threshold $x[1]' = \sqrt{1 - \frac{c}{v^2}}$, players switch to playing B.

At scores $x < x[1]'$, the interim payoffs are:

$$u_i(A_i^1, p^*_{-i}([x,1])|x) = x^2 v^1 - c$$

$$u_i(B_i, p^*_{-i}([x,1])|x) = x^2 v^1 + \big((1-x[1]')^2 + 2x(1-x)\big) v^2 - 2c$$

$$\Delta(x) := u_i(B_i, p^*_{-i}([x,1])|x) - u_i(A_i^1, p^*_{-i}([x,1])|x) := \big((1-x[1]')^2 + 2x(1-x)\big) v^2 - c$$

$$\frac{\partial \Delta(x)}{\partial x} = 2(1-2x)v^2 > 0 \iff x < \frac{1}{2}$$

$\Delta(x)$ strictly decreases on the left neighborhood of $x[1]$. And by definition of this case: $\Delta(x[1]') = 0$. So on the left neighborhood of $x[1]'$, $\Delta(x) > 0$, and the agent does not immediately switch to another action.

B Supplements

B.1 London social housing - Scoring rule

City of London Allocations Scheme		Secondary Points																	
		Overcrowding		Wellbeing						Unsuitable Housing Conditions						Housing Management		
Primary Group	Primary Points	Per room lacking	Mixed sharing	Medical			Welfare			Sharing			Lack of tenancy	Bedroom Cap	Long TA stay	Advice & Engagement	Intentionality	Decant Urgency
				S	M	L	S	M	L	F	1-4	5+						
Management Transfer	800																	
Under-occupation	400			50	25	10	50	25	10					50				100 / 200
Severe Medical / Welfare	275	25	10	50	25	10	50	25	10	5	10	15	5			15	minus 50	
Severe Overcrowding	250	25	10		25	10		25	10	5	10	15	5			15	minus 50	
Studio Upgrade	250	25			25	10		25	10									
Decants	225	25	10	50	25	10	50	25	10									100 / 200
Moderate Medical / Welfare	225	25	10		25	10		25	10	5	10	15	5				minus 50	
Moderate Overcrowding	200		10			10			10	5	10	15	5				minus 50	
Homeless	140	25	10	50	25	10	50	25	10						150		minus 50	
Lower Income City Connection	100					10			10	5	10	15	5					
Sons and Daughters	50					10			10									
Low Priority	1					10			10	5	10	15	5					

FIGURE XVIII:
Scoring rule in "Choice Based Lettings" - London, UK

B.2 Pure (asymmetric) equilibrium

The next theorem only partially characterizes the pure Bayes-Nash equilibrium of the Application Game:

Theorem 3.

A pure strategy Bayes-Nash equilibrium of the Application Game with $c > 0$:

(i) Exists and is unique up to strategies on the $(0, x[m])$ interval, and payoff-unique.

(ii) Exhibits a finite number of intervals of scores where the interim action sets are constant.

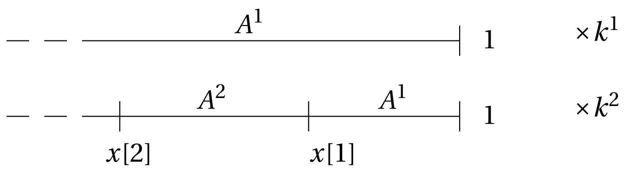
The proof is in two steps. First, we use Lemma 4 characterizing the interim action sets. Second, for each interval of score with a constant interim action set, we characterize the number of agents playing each action in the interim action set. For example, on the interval $[x[2], x[1]]$, we determine the pair(s) of two integers (k^1, k^2), $k^1 + k^2 = n$, where k^1 (k^2) is the number of agents playing A^1 (A^2). We find that the no profitable deviation inequalities between payoffs always define a (unique) pair (k^1, k^2). We proceed similarly at lower scores.

The theorem still allows many different patterns within the intervals where the interim action set is constant. Whenever we introduce $n \geq 4$ agents, the equilibrium pattern depends finely on the parameters of the Application Game, hence a low robustness.[22] We illustrate this lack of

[22] In this respect, the robustness of the pure equilibrium in section §3.1 was a special feature of the example.

robustness below with an example when $n = 4$, $F \sim \mathcal{U}$. There are two cases:

- Case 1: $a^1 > 8a^2 + 7c \rightarrow (k^1, k^2) = (1, 3)$
- Case 2: $a^1 < 8a^2 + 7c \rightarrow (k^1, k^2) = (2, 2)$



In general, the pure Bayes-Nash equilibrium can support quite odd strategy profiles, where some strategies exhibit no sorting (an agent plays higher value goods at lower scores), or sorting with jumps (an agent plays high value goods at high scores, low value goods at intermediate scores, but never plays the intermediary value goods). In these profiles, each strategy is virtually unique and highly sophisticated. The profiles are "very asymmetric". This questions the ability of players to coordinate on these equilibria.