A Traffic Jam Theory of Recessions*

Jennifer La'O

Columbia University and NBER

September 15, 2015

`Incomplete. Please see updated version at http://tinyurl.com/qg5rzmv`

Abstract

I construct a dynamic economy in which agents are interconnected: the output produced by one agent is the consumption good of another. I show that this economy can generate recessions which resemble traffic jams. At the micro level, each individual agent waits for his own income to increase before he increases his spending. However, his spending behavior affects the income of another agent. Thus, the spending behavior of agents during recessions resembles the stop-and-go behavior of vehicles during traffic jams. Furthermore, these traffic jam recessions are not caused by large aggregate shocks. Instead, in certain parts of the parameter space, a small pertubation or individual shock is amplified as its impact cascades from one agent to another. These dynamics eventually result in a stable recessionary equilibrium in which aggregate output, consumption, and employment remain low for many periods. Thus, much like in traffic james, agents cannot identify any large exogenous shock that caused the recession. Finally, I provide conditions under which these traffic jam recessions are most likely to occur.

**Email:* `jenlao@columbia.edu`. I thank Jingyi Jiang at the University of Minnesota for invaluable research assistance. For insightful comments and excellent feedback, I thank Fernando Alvarez, Marios Angeletos, Saki Bigio, Chris Carroll, Vasco Carvalho, John Cochrane, Jordi Gali, Veronica Guerrieri, Bob Hall, Boyan Jovanovic, Pat Kehoe, Ellen McGrattan, Herakles Polemarchakis, Ricardo Reis, Jose Scheinkman, Nancy Stokey, and Mike Woodford.

1 Introduction

Are recessions similar to traffic jams? This paper puts forth the idea that these two phenomena may resemble one another. Consider the following two observations.

First, driver behavior seems similar to that of economic agents. In traffic jams, one often gets the feeling that if all cars just drove forward at a slow but steady pace, we would all get out of the traffic jam. However, this takes coordination and it is in fact not a good description of how drivers actually behave. Instead, in traffic jams, we observe what is known as "stop-and-go" behavior. An individual driver waits for the car in front to move forward before he moves forward. This opens up space for the car behind him, in which case that car moves forward. Hence, in traffic jams, all cars are simply waiting for the space to open up ahead of them before they move. One sees clearly that the actions of these drivers are not based on the entire state of the highway[1], but instead are based on their own very local conditions. Similarly, in recessions we observe another form of "stop-and-go" behavior. Households wait for their income to increase before they increase their consumption spending. Firms wait for sales to pick up before they increase production or employ more workers. It seems as though the actions of economic agents, too, are not based on the entire state of the aggregate economy, but instead are based on their own individual situations or constraints. And again, one gets the feeling that if all households simply spent more and if all firms simply employed more workers, the recession would come to an end. Yet, this takes coordination; instead, for each individual economic agent and for each individual driver, local interactions matter first and foremost.

Second, traffic jams, like recessions, do not seem to always be driven by large exogenous shocks. Sometimes traffic jams are caused by something fundamental–an obstruction on the road or a car crash. However, more often than not traffic jams seem to occur spontaneously, or at least without any underlying cause–perhaps due to some slight, unobserved perturbation.[2] The traffic engineers call these "phantom jams" as drivers in the jam cannot seem to identify any particular cause of the jam. Furthermore, these phantom jams seem more likely to occur when traffic dense.

Similiarly, the underlying causes of business cyles seem to be equally elusive. While the standard approach to modelling business cycles is to build dynamic models of rational agents and then to analyze the model's equilibrium response to exogenous aggregate shocks, this approach is in some ways unsatisfactory. As John Cochrane (1994) writes, "it is difficult to

[1]This could be due either to the fact that drivers don't know what's going on in the entire highway, or the simple physical constraint that they can't hit the car ahead of them.

[2]In fact, this has also been shown in some experiments.

find large, identifiable, exogenous shocks" in the data. Modigliani (1977) and Hall (1980) contend that standard equilibrium models may leave too much unexplained. Furthermore, during and after actual recessions, it is not as if firm executives, consumers, central bankers, or even economists are easily able to identify the large aggregate shocks driving each episode.[3] Thus, although standard general equilibrium models rely on aggregate shocks as the main drivers of fluctuations, it is difficult both through introspection and by observation of the data to be fully satisfied with this modelling approach. Much like traffic jams, macroeconomic recessions are often "phantom".

In this paper I construct a model in which recessions resemble traffic jams in these two respects. Agents are arranged in a network such that the output produced by one agent is the consumption good of another. During normal times agents receive steady streams of income and as a result their consumption is a steady flow. However, during recessions, agents exhibit stop-and-go behavior: each agent i waits for his own income to increase before increasing his spending. But, this implies that agent $i-1$, who produces the consumption good for agent i, is experiencing a drop in income, and hence also not spending. If agent $i-1$ isn't spending, this affects the income of agent $i-2$, and so on. Thus, agents are all locally waiting for their prospects to improve, while their non-spending behavior is affecting the income of others. Thus, the model in some way shares the same spirit of the earlier literature on Keynesian coordination failures, but through a very different mechanism and modeling technique.

Second, in this model recessions are driven not by large aggregate shocks, but instead by small perturbations, or local shocks. These individual-specific or local shocks may have reverberating effects so that the economy eventually finds itself in a recession. However, these perturbations could be so small that they would not be identified as aggregate shocks in the data, nor would all the agents in the model be aware of them. Furthermore, in this model small perturbations do not always lead to recessions. Under certain conditions, these perturbations die out and the equilibrium converges back to the "normal times" equilibrium. Under certain other conditions, however, these perturbations are amplified, leading to prolonged traffic jam recessions. Thus, in sharp contrast to standard equilibrium models, this model could potentially identify conditions under which recessions are more likely to occur, rather than simply attributing them to unpredictable exogenous shocks. Furthermore, this model may allow for new policy insights designed to end the traffic-jam recession and bring

[3] Sure, for certain recessions, such as the oil price recessions of the 70s or the Volcker recession in the early 80s, we have some idea of the large aggregate shocks behind these aggregate declines. However, I would argue that for most business cyles fluctuations this is not the case. As Hall (1977) points out, only rarely do we find obvious candidates such as the oil shocks in the 70s. Even if we consider the latest recession, the fall in the value of the housing market was only a negligible fraction of total U.S. GDP.

the economy back to the normal times regime.

Framework. First, I draw on the literature on traffic flow in engineering. In this literature one of the most successful and widely accepted models of simulating traffic is called the Optimal Velocity Model introduced by Bando et al (1995). This is a car-following model in which N cars follow each other on a circular road of length L; car i follows car $i + 1$. The bumper-to-bumper distance between car i and car $i + 1$ is called car i's "headway". In car-following models, cars are given a behavioral equation which dictates their acceleration or speed as a function of nearest-neighbor stimuli (see survey of the literature by Orosz et al 2006). The innovation in Bando et al (1995) is the introduction of a particular form for this behavioral equation–it imposes that each car's acceleration is an increasing function of its headway. If a car's headway is very large, the car speeds up, if it is too small, the car slows down and potentially comes to a stop.

The results of this simple model are quite striking. This model can produce both uniform traffic flow as well as a stop-and-go waves which resemble traffic jams. In the uniform-flow equilibrium, all cars follow each other around the circle at equal velocity and at equal speed. This equilibrium is unique and globally stable in a particular region of the parameter space, implying that the effects of any small perturbation eventually die out and the system converges back to uniform flow. The uniform flow equilibrium, however, loses stability when a certain parameter is varied; at this point a Hopf bifurcation of the dynamical system occurs meaning that an individual vehicle limit cycle becomes stable.[4] Here, what emerges instead are travelling waves which resemble the stop-and-go behavior in traffic jams. Individual cars converge to a limit cycle: cars oscillate between facing low headway and slowing down to a stop (entering a traffic jam), and facing large headway and speeding up (exiting the traffic jam). In this equilibrium, there are many cars sitting in the traffic jam, waiting for their headway to increase before moving forward, implying that aggregate velocity has decreased relative to that in the uniform-flow. Furthermore, due to the instability of the uniform-flow equilibrium and the stability of the stop-and-go solution, the transition path seems compelling: small perturbations develop into large traffic jams as their effects cascade down the line of cars.

With this model in mind, I then build a similar model within an economic environment. I construct a dynamic economy in which agents are inter-connected: the output produced by one agent is the consumption good of another. I then show how this environment is similar to that in the traffic model. In this analogy, the expenditure of each agent is similar to their velocity. Given this interpretation, I show that headway in the model is equal to cash-on-

[4] However, note that the aggregate behavior is not in a limit cycle. Only that of individual cars.

hand at the beginning of the period. Thus, the resources an agent spends on consumption in a given period becomes the income for the next agent (the producer of that good) the following period. This increases the latter agent's cash-on-hand in the following period, which he may then choose to spend on consumption, therefore moving those resources to the next agent. And so on. This is analagous to the idea that whenever a car moves forward, this increases the headway for the car behind him, in which case that car may move forward.

Now, in the traffic model there is a behavioral equation which dictates the behavior of cars–cars are supposed to accelerate when headway is large, and decelerate when headway is low. The next step in the economic model then is to see whether the behavior of the agents in the model can match the behavior of cars in the traffic model. Here, I take two approaches. First, in the economic model I start by allowing for arbitrary consumption functions and then derive under what conditions these functions can lead to traffic jam recessions. To understand this, note that in the traffic model, depending on the parameters of the behavioral equation, either the uniform flow equilibrium or the stop-and-go solution is stable. In particular, what matters is the slope of the acceleration of the car with respect to the headway. When this slope is sufficiently low, uniform flow is stable; when this slope is sufficiently high, uniform flow loses stability and the traffic jam occurs. This slope is analogous in the economic model to the marginal propensity to consume out of current cash-in-hand. I formalize this condition, and show that when the marginal propensity to consume out of cash-in-hand is very high, the economy can fall into a traffic jam recession. I then simulate the economy and analyze the transitional paths. I find this preliminar exercise useful–once one understands the general properties consumption functions must have in order to generate traffic jams recessions, I can then provide guidance as to what conditions in terms of microfoundations: preferences, information, constraints, etc. would allow for policy functions of this shape as an optimal response to the household's problem.

Second, I then attempt to construct from micro-foundations optimal household policy functions such that the consumption function satisfies these properties. The starting point is a model without any credit or borrowing constraints. I show that with permanent income consumers, one can acheive a policy function which is similar to the behavior equation in the traffic model. This is because whenever an agent observes an income shock, if he believes income is a random walk, his consumption will also increase as an optimal response to the increase in his permanent income. As in Hall (1977), under certain preferences, this implies that his own consumption follows a random walk, which therefore implies that the income of the following agent is a random walk. In this model, however, the slope of this consumpion policy function is not high enough to generate traffic jams. In order to generate traffic jams, a higher marginal propensity to consume is needed. I thus explore the case of quasi-hyperbolic agents. In this case, I show that depending upon parameters, one can obtain a high enough marginal propensity to consume such that a traffic jam recession occurs.

Finally, I consider a variant with borrowing constraints. In my opinion, this is the most natural microfoundation, as we well know that this leads to high marginal propensities to consume when agents are close to their borrowing constraints. The model here is similar to a consumption savings model with idiosyncratic income (labor) risk, as in Aiyagari, Huggett, Bewley. However, in contrast to these papers, the income risk here is endogenous–the income of one agent depends on the consumption behavior of another. In this version of the model the state space unfortunately blows up as agents are trying to forecast the shocks of all other agents and must keep track of entire distributions. Hence, in order to simplify the problem, I assume that agents have a constrained information capacity as in Sims (2003), Gabaix (2011), Woodford (2012). Households thus cannot keep track of entire state of the world, and instead can only keep track and form expectations over a finite number of moments. I thus define an approximate equilibrium as in Krussell-Smith () and then simulate the economy with borrowing constraints. I show that this environment can easily lead to traffic jam recessions.

Related literature. This paper is firstly related to the engineering literature on traffic flow. Finally, in terms of the traffic literature, I borrow the models of Traffic Bando et. al. (1995). This model has been used extensively throught that literature. See, e.g. Gasser et. al. (2004), Orosz Stepan (2006), Orosz et. al. (2009) In car-following models, discrete entities move in continuous time and continuous space[5]

In economics, my paper is most closely related to Jovanovic (1987 and working paper 1983) and the "sandpile" models Scheinkman and Woodford (1994) and Bak, Chen, Scheinkman, Woodford (1993). In fact, in his 1983 working paper version, Jovanovic explores an environoment very similar to this one: agents are arranged in a circle and each agent consumes the good produced by the agent to his left. Jovanovic shows that with independent agent-specific preference shocks and without any aid of aggregate shocks, in this economy he can produce aggregate fluctuations!

This paper is also related to the self-organized criticality literature. The "Sandpile Model" of Scheinkman and Woodford (1994) and Bak, Chen, Scheinkman, Woodford (1993). In these models there is some low frequency movement that takes you into the Bifurcation range. Stresses the importance of supply chain linkages.

Furthermore, the results of this model have the flavor of Keynesian Coordination Failures; it thus complements the literature on multiple equilibria and sunspot fluctuations. See, e.g.

[5] There is another literature called continuum or macroscopic models. These models characterize traffic in terms of density and velocity fields use partial differential equations.

Shell (1977), Azariadis (1981), Azariadis and Guesnerie (1986), Benhabib and Farmer (1994, 1999), Cass and Shell (1983), Cooper and John (1988), Farmer (1993), Farmer and Woodford (1997), and Woodford (1991). The results of the traffic model can be interpreted as a coordination failure: the network structure and decentralized trading prevents households from coordinating on spending more and generating more income. However, unlike this previous literature, the coordination failure does not originate from any of the familiar sources (externalities and non-convexities), nor is there ever more than one stable equilibria.

Furthermore, the methodology used in this paper is that of dynamical systems, limit cycles and Hopf Bifurcations; it is thus partly related to an older literature in dynamic general equilibrium theory, studying whether rational behavior can give rise to endogenous aggregate fluctuations. See, for example, Magill (1979), Boldrin and Montrucchio (1986), Scheinkman (1984) Boldrin and Deneckere (1987). Turnpike theorem. This work is surveyed in Boldrin and Woodford (1990). These papers look at representative agent growth models with a unique perfect-foresight equilbrium. They find that deterministic dynamical systems can generate both periodic limit cycles as well as chaotic dynamics that can look very irregular. In this model, rather, on the aggregate there are no endogenous fluctuations–there are limit cycles only at the individual level.

Finally, in this paper fluctuations are driven by small shocks to individual agents, rather than aggregate shocks. In this sense, this paper shares the spirit of the early literature on real business cycles and the role of intersectoral linkages and sectoral shocks. Beginning with Long and Plosser's (1983) multi-sectoral model of real business cycles, a debate then ensued between Horvath (1998, 2000) and Dupor (1999) over whether sectoral shocks could lead to strong observable aggregate TFP shocks. More recently, this work has been extended and generalized by Acemoglu et al. (2011), for arbitrary production networks. Finally, the results of the Acemoglu et. al. paper are related to that of Gabaix (2011), who shows that firm level shocks may translate into aggregate fluctuations when the firm size distribution is power law distributed, i.e. sufficiently heavy-tailed. La'O and Bigio (2013) build on the production network literature and show how financial frictions within firms affect other firms within the network. Finally, there is the Credit Chains model of kiyotaki moore.

Layout. This paper is organized as follows. Section 2 first introduces the basic workhorse traffic model from the traffic literature. Section 3 then sets up the economic environment with the goal of reproducing traffic-jam recessions. Section 4 partially characterizes the competitive equilibrium within this environment. Section 5 relates the economic model to the traffic model and explores the implications of an exogenously imposed behavioral equation on households. Section 6 considers a variant of the model with borrowing constraints and demonstrates how one may obtain expenditure policy functions for individual households in this environment. Section 7 then concludes. All proofs are in the Appendix.

2 The Traffic Model

In this section I present the simple traffic model that can produce both uniform flow and stop-and-go traffic. There are two general approaches to modeling traffic. One is continous models in which traffic is described via a continuous density distribution and a continuous velocity distribution over location and time.[6] The other method of modelling traffic is to consider a car-following model. In car-following models, discrete entities move in continuous time and continuous space. I follow the latter approach. The rest of this section mirrors the exposition on car-following models found in Orosz et al (2006, 2009).

Consider a model of N cars indexed by $i \in \{1, 2, \ldots, N\}$. Here, car i follows car $i+1$. Let $x_{i,t}$ denote the position of car i at time t, let $v_{i,t}$ denote the velocity of car i at time t and let $\dot{v}_{i,t}$ denote the acceleration of car i at time t. Finally, let $h_{i,t}$ be bumper-to-bumper distance between car i and car $i+1$, also calle dthe headway:

$$h_{i,t} = x_{i+1,t} - x_{i,t} - l$$

where l is length of car. For simplicity and without loss of generality, we take $l \to 0$. See Figure 1.

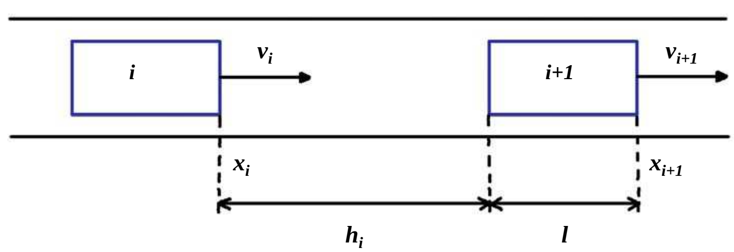

Figure 1: Diagram of Car-Following Model

One must also specify boundary conditions. For simplicity, we place these N cars on a circular road of length L. This yields the following equation $\sum_{i=1}^{N} h_{i,t} = L$.

Finally, to complete the model we need a car-following rule, that is, the velocity or the acceleration of each car has to be given as the function of stimuli–these are usually headway,

[6] See, e.g. Lighthill & Whitham (1955).

the velocity difference, or the vehicle's own velocity. As economists, we can think of this as simply a behavioral equation for each car. Here, I will follow a class of models that has been extensively studied and widely accepted in the traffic literature called the "Optimal Velocity Model" (Bando et. al, 1995). See (Bando et al. 1998, Gasser et al 2004, Orosz et al. 2004) In this class of models, the acceleration of vehicle i is given by

$$\dot{v}_{i,t} = \alpha \left(V \left(h_{i,t} \right) - v_{i,t} \right) \tag{1}$$

where $\alpha > 0$ is a constant, and V is a continuous, monotonically increasing function of vehicle i's headway $h_{i,t}$.[7] This equation was proposed by Bando et al (1995) and has proved quite successful. Despite its simplicity, this model can produce qualitatively almost all kinds of traffic behaviour, including uniform traffic flow as well as stop-and-go waves.

Equation (1) deserves some comment. First, the assumption here is that the acceleration of vehicle i is a function only of nearby stimuli–the vehicle's own velocity and its headway (its distance to the nearest car). These are called nearest-neighbor interactions. That is, each car's individual state is strictly smaller than the aggregate state.[8]

Next, this model is entitled the optimal velocitiy model (OVM) and $V(\cdot)$ is called the optimal velocity function. However, note that in the usual economic sense, there is nothing necessarily "optimal" about it. That is, equation (1) is *not* the result of any optimization problem on the part of the agents nor a planner; instead, this behavior is simply imposed. The reason one might call it optimal is that $V(h_{i,t})$ can be thought of as the "optimal velocity" a driver would like to have given its current headway $h_{i,t}$. If this optimal velocity $V(h_{i,t})$ is greater than the car's current velocity $v_{i,t}$, the car speeds up. Conversely, if $V(h_{i,t})$ is less than the car's current velocity $v_{i,t}$, the car slows down. Finally, $\alpha > 0$ is called the relaxation parameter; it dictates how sensitive the driver's acceleration is to this difference in optimal and current velocity.

Finally, the optimal velocity function V satisfies the following properties: (i) it is contiuous, non-negative, and monotonically increasing, (ii) it approaches a maximum velocity for large headway $\lim_{h\to\infty} V(h) = v^0$ where v^0 acts as a desired speed limit, and (iii) it is zero for small headway. A simple example of the optimal velocity function is given by the

[7] A more general version often studied is given by $\dot{v}_{i,t} = \alpha \left(V \left(h_{i,t} \right) - v_{i,t} \right) + W \left(\dot{h}_{i,t} \right)$. Here, I follow Bando et. al. 1995 and Gasser et al. 2004 and set $W = 0$.

[8] There exist extensions in which stimuli also include next-nearest neighbour interactions (Wilson et al 2004). In multi-look-ahead models, drivers respond to the motion of more than one vehicle ahead. These can increase the linear stability of the uniform flow.

following specification, used in Orosz et. al (2009)

$$V(h) = \begin{cases} 0 & \text{if} \quad h \in [0,1) \\ \frac{(h-1)^3}{1+(h-1)^3} & \text{if} \quad h \in [1,\infty) \end{cases}$$

This is rescaled by v^0. Figure 2 plots this function and its first derivative. Note that the rescaled speed limit is 1.

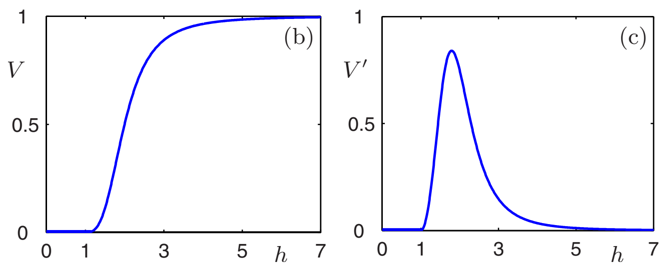

Figure 2: Optimal Velocity Function

Therefore, equilibrium of this traffic model is given by the following set of ODEs

$$h_{i,t} = x_{i-1,t} - x_{i,t}, \quad \forall i \in \{1, \dots N\} \tag{2}$$

$$v_{i,t} = \dot{x}_{i,t} \tag{3}$$

$$\dot{v}_{i,t} = \alpha \left[V(h_{i,t}) - v_{i,t} \right], \quad \forall i \in \{1, \dots N\} \tag{4}$$

where $\sum_{i=1}^{N} h_{i,t} = L$. The first equation simply describes the relation between positions and headway, the second condition gives us periodic boundary conditions, and the third equation are the behavioral equations for the cars. Finally, as mentioned before This behavioral equation is useful as it can produce both uniform flow and stop-and-go traffic, which I will describe next.

Uniform Flow Equilibrium. This system admits a uniform flow equilibrium. The definition of the uniform flow equilibrium is an equilibrium which satisfies (2)-(4) in which the velocities and the headways of all cars are constant (time-independent): $h_{i,t} = h^*$ and $v_{it} = v^*, \forall i \in \{1, \dots N\}$. In this equilibrium, all cars travel at same velocity, equally spaced. Characterizing the uniform flow is quite simple. If all cars are equally spaced, then $h^* = L/N$. Furthermore, in order for all cars to be travelling at constant velocity, in order for equation (4) to hold, we must have that $0 = V(h^*) - v^*$. Thus, the uniform flow equilibrium is characterized by

$$h_{i,t} = h^* = L/N, \qquad v_{it} = v^* = V\left(L/N\right), \qquad \forall i \in \{1, \ldots N\}$$

As will be discussed next, the uniform flow equilibrium is unique and globally stable in part of parameter space. This implies that one may start cars in any position and at any velocity, and as long as they behave according to the optimal velocity equation, over time these cars will converge to the uniform flow equilibrium. This is convergence is demonstrated in Figure 3.

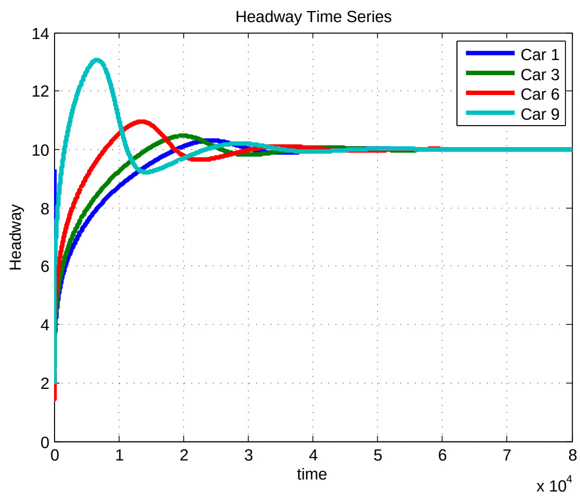

Figure 3: Convergence to the Uniform Flow Equilibrium

Bifurcations of the Uniform Flow. We now consider the stability of the uniform flow equilibrium. We find that the uniform flow equilibrium is stable in part of parameter space, however the uniform equilibrium may lose stability when the parameter h^* is varied. In order to see this, one needs to linearize the system around uniform-flow equilibrium and consider the eigenvalues $\lambda \in \mathbb{C}$. To conserve on space, the linear stability analysis is restricted to the appendix; here, I will simply present the result.

Proposition 1. *The uniform flow equilibrium is stable if and only if*

$$V'\left(h^*\right) < \frac{1}{2}\alpha$$

The proof is in the Appendix. In the terminology of dynamical systems, when crossing the stability curve at $V'\left(h^*\right) = \frac{1}{2}\alpha$, a (subcritical) Hopf bifurcation takes place. At this point a pair of complex conjugate eigenvalues cross the imaginary axis, $\lambda = i\omega$. Once this occurs, the uniform flow becomes unstable and instead, travelling waves with frequency ω appear. That is, the stable equilibrium is a the limit cycle for each vehicle.

Figure 4 summarizes this information by ploting the linear stability diagrams. Figure 4 plots the stability diagram in terms of the $(V'(h^*), \alpha)$ space. The domain in which the uniform flow is linearly stable is shaded. When $V'(h^*) < \frac{1}{2}\alpha$ the uniform flow equilibrium loses stability and a Hopf bifurcation occurs; the arrows represent the increase in wave number k. Using the derivative of the optimal velocity function, one may transform the

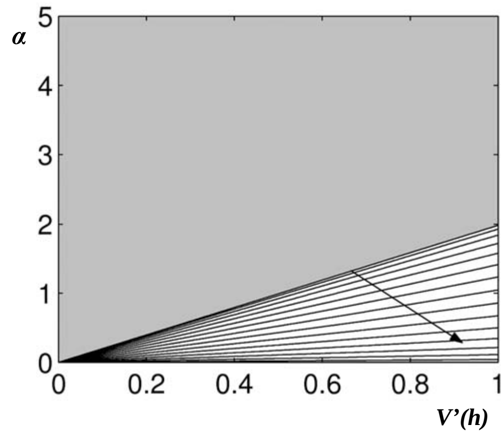

Figure 4: Bifurcation Diagram

stability diagrams from the $(V'(h^*), \alpha)$ plane to the (h^*, α) plane, thus Figure 5 plots the linear stability in terms of the (h^*, α) space. From this, we see that when traffic is sufficiently dense, i.e. when h^* is low enough (approaching from above), the uniform flow equilibrium loses stability.

Stop-and-Go Waves. Thus, when $V'(h^*)$ is sufficiently high relative to α, the uniform flow equilibrium loses its stability. When this occurs, what emerges instead are travelling waves which resemble the stop-and-go behavior in traffic jams. Individual cars converge to a limit cycle, an oscillatory solution. See Figure 6. Cars oscillate between facing low headway and slowing down to a very low speed or to a stop, sitting in a traffic jam waiting for their headway to increase, and then facing large headway and speeding up until they hit the traffic jam again. Furthermore, this limit cycle is stable in this region, hence any small perturbation thus takes cars into the oscillatory solution.

Finally Figure 7 plots the trajectories of multiple vehicles. The y-axis is the position of each vehicle, plotted as a function of time t. Each blue line is the trajectory of an individual vehicle. The vehicle enters the traffic jam, is stuck there for a while, and then when its headway opens up, the car speeds up. The red line indicates the stop-front of the jam and

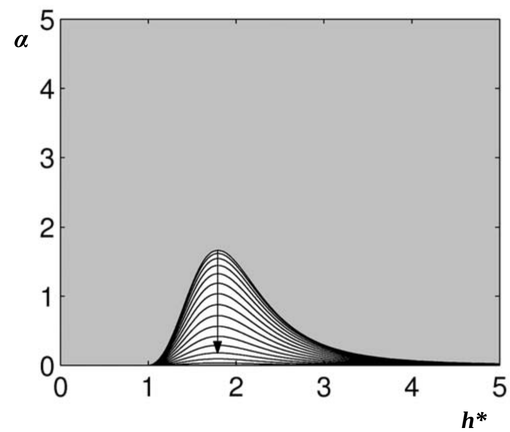

Figure 5: Bifurcation Diagram

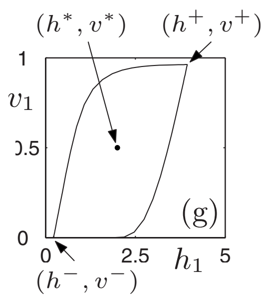

Figure 6: Limit Cycle of the Traffic Jam

the green line indicates the go-front of the jam.

To summarize, when $V'(h^*)$ is sufficiently high, or when traffic is sufficiently dense, a traffic jam can emerge. At the micro level, individual cars enter a traffic jam in which they wait for their headway to increase before moving. At the macro level, aggregate velocity and headway have fallen relative to the uniform flow equilibrium. Furthermore, small perturbations develop into large traffic jams; "tiny fluctuations may develop into stop-and-go waves as they cascade back along the highway, i.e. 'tiny actions have large effects'" (Orosz et al, 2009). The traffic engineering literature describes these as "phantom jams" in the sense that drivers cannot see any cause of the jam even after they've left the congested region.

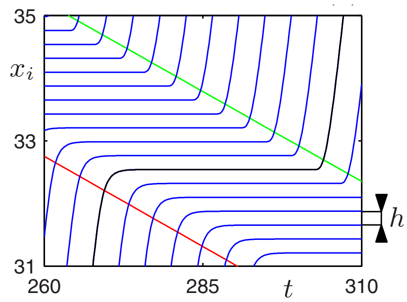

Figure 7: Trajectories of Cars in Traffic Jam

3 The Economic Model

In this paper I build an economic model in which recessions can resemble traffic jams. With the traffic model presented above in mind, in this section I attempt to construct a similar model within an economic environment. In this economy, agents are inter-connected: the output produced by one agent becomes the consumption good of another. In this way, the actions and incentives of agents are very much connected in a way similar to that in the traffic model.

The Model. Time is discrete and indexed by t.

Geography. There are N households indexed by $i \in \{1, \ldots, N\}$. These households live on N islands and each household is composed of a consumer and a producer. While the consumer of household i lives and consumes on island i, the producer of household i lives and produces on island $i+1$. This implies that for any island i, consumer i and producer $i-1$ co-habitate on this island. In particular, the good produced by household $i-1$ is consumed by household i. These households are therefore arranged in a circular network such that household i consumes the output produced by household $i-1$.[9]

Commodity Space. There are $N+1$ consumption goods. First, there are the N different commodities which the N households consume and produce. Consumer i consumes the commodity produced by household $i-1$. Furthermore, these commodities are perishable–they cannot be stored over periods.

There is also a numeraire good, which I call corn. Corn can either be planted as seed corn, consumed, or used to buy the commodities. Corn is consumed by all households. Corn facilitates trade among islands–that is it can be used to purchase goods.

Each household is endowed at time 0 with some amount of corn. Corn can be used for

[9] And household 1 consumes the output of household N.

food or for seed. A farmer can split the corn y_t into consumable sweet corn q_{it} and seed corn a_{it} for next period. If it plants a_{it} seeds of corn this period, it gets $(1+r)\,a_{it}$ seeds next period. To simplify, I set the interest rate $r=0$.

Timing. At the beginning of each period, each household receives revenues from its producer from the previous period. Once each household receives last period's revenues, the goods market on each island takes place. The consumer makes consumption and savings decisions, the producer on that island works and produces the consumption good, and prices adjust so as to clear markets within each island. The household pays the producer for the consumption good in units of corn and the producer on each island plants the corn, to be used by his own consumer at the beginning of next period. This corn gains an interest rate as they are transfered to the following period.

Household Preferences, Budgets, and Technology. The utility of household i is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(c_{it}\right) - \chi n_{it} \right]$$

where β is the household's discount factor and $u\left(c\right)$ is a strictly increasing, concave, one-period utility function satisfying the Inada conditions. Consumption c_{it} is a composite consumption basket given by

$$c_{it} = y_{i-1,t}^{\theta} q_{it}^{1-\theta}$$

composed of its consumption of the output of household $i-1$ at time t, $y_{i-1,t}$, and the numeraire good, which is denoted q_{it}. The household's budget constraint (in terms of the numeraire) is given by

$$p_{i-1,t} y_{i-1,t} + q_{it} + a_{i,t} = \left(1+\rho\right) p_{i,t-1} y_{i,t-1} + a_{it-1} \tag{5}$$

The left hand side is expenditure on both consumption goods and savings in seed corn, where $p_{i-1,t}$ is the price of the good produced by producer $i-1$ at time t and a_{it} are its savings in seed corn. The right hand side is composed of are the revenues the household receives from its producer from last period $p_{i,t-1} y_{i,t-1}$, which is transformed into corn at rate $\left(1+\rho\right)$ as well as corn seed from the previous period, which is transformed into corn one-for-one.

Producer i's production function linear and given by

$$y_{it} = A n_{it}$$

where n_{it} is the labor it employs.

Market clearing. The consumption of any commodity must be equal to the amount produced $y_{i-1,t}$ since there is no storage.

Remarks. Consumption of household i is set equal to production of household $i-1$. This implies that the consumption and investment goods are different in order for islands to produce different amounts. To understand this, suppose the opposite: that production can be used either as consumption or investment. Now consider the following. Island i produces y_i. Island $i+1$ buys this production y_i and uses it either for consumption or investment. If island $i+1$ has its own income and its own saved goods, all of that is spent on consumption and investment. That is, suppose whatever cash-in-hand $i+1$ is $h_{i+1} = (1+r)(y_{i+1} + a_{i+1})$. Then household $i+1$ can spend this cash-on-hand on either either on c or a

$$\begin{aligned} c + a &= (1+r)(y+a) \\ c + (a - (1+r)a) &= y \\ c + x &= y \end{aligned}$$

Household $i+1$ purchases $c+x$ from household i. But this implies that household i must have produced y_i too. So in the end, all households produce the same amount y. This is why I disconnect the consumption and investment goods from one another and therefore introduce another good used for trade–the numeraire (corn).

Next, why do I need a numeraire good. I need some good which can be used to facilitate trade across all households. Rather than complicate matters with money an nominal price determination, I opted for a numeraire good which all households consume.

Finally, another question is about why income comes a period later. Otherwise, all markets clear instantaneously.

4 Equilibrium Characterization

Although I have not introduced any shocks or imperfect information in this economy, I will give a more general definition for equilibrium that allows for household and firm expectations. I define an equilibrium as follows.

Definition 1. *A competitive equilibrium is a collection of allocation and price functions such that*

(i) given current prices and expectations of future prices and income, allocations are optimal for consumers and workers.

(ii) prices clear all markets

I now characterize the equilibrium of this economy. The household's intratemporal condition for consumption over both goods is given by

$$p_{i-1,t} = \frac{u_y(c_{it})}{u_q(c_{it})} = \frac{\theta}{1-\theta}\frac{q_{it}}{y_{i-1,t}} \tag{6}$$

The household's Euler Equation is given by

$$\lambda_{it}\frac{c_{it}}{q_{it}} = \beta\mathbb{E}_{it}\lambda_{i,t+1}\frac{c_{it+1}}{q_{it+1}}$$

where λ_{it} is the Lagrange multiplier on the household's budget constraint.

Next, consider the producer. The producer's optimality condition is given by

$$\chi = \mathbb{E}_{it}\beta\lambda_{t+1}(1+\rho)\,p_{i,t}A$$

Using the Euler equation, this can be reduced to

$$\chi = \lambda_{it}(1+\rho)\,p_{it}A$$

Therefore, the price must satisfy:

$$p_{it} = \frac{1}{1+\rho}\frac{\chi}{A\lambda_{it}}$$

Finally, substituting in for λ_{it} we get that

$$p_{i,t} = \frac{1}{1+\rho}\frac{\chi}{Au'(c_{it})(1-\theta)}\frac{q_{it}}{c_{it}}$$

We conclude that a set of allocations and prices constitute an equilibrium if and only if the following hold

$$p_{i-1,t} = \frac{\theta}{1-\theta}\frac{q_{it}}{y_{i-1,t}} \tag{7}$$

$$u'(c_{it})\frac{c_{it}}{q_{it}} = \mathbb{E}_{it}\beta u'(c_{it+1})\frac{c_{it+1}}{q_{it+1}} \tag{8}$$

$$p_{i,t} = \frac{1}{1+\rho}\frac{\chi}{Au'(c_{it})(1-\theta)}\frac{q_{it}}{c_{it}} \tag{9}$$

along with the resource constraints.

One may reduce these conditions further and state the equilibrium in terms of allocations alone. For simplicity let us assume that $u(c) = \log c$. In this case, the Euler equation (8) reduces to

$$q_{it}^{-1} = \mathbb{E}_{it}\beta q_{it+1}^{-1}$$

And equation () becomes

$$p_{i,t} = \frac{1}{1+\rho}\frac{\chi}{A(1-\theta)}q_{it}$$

Let expenditure be denoted by $z_{it} = p_{i-1,t}y_{i-1,t} + q_{it}$ of household i on consumption in period t. Then, using the optimality condition over consumption goods () it is straightforward to show that

$$z_{it} = p_{i-1,t}y_{i-1,t} + q_{it} = \frac{1}{1-\theta}q_{it}$$

Since consumption of household i is equal to output of household $i-1$, we have that

$$p_{i-1,t}y_{i-1,t} = \theta z_{it} \quad \text{and} \quad q_{it} = (1-\theta) z_{it}$$

this implies that

$$p_{i,t}y_{i,t} = \theta z_{i+1,t}$$

Therefore, the budget constraint can be re-written as

$$z_{it} + a_{i,t} = (1+\rho)\theta z_{i+1,t-1} + a_{it-1}$$

Using the fact that $q_{i,t} = (1-\theta) z_{it}$, one may rewrite the Euler equation as

$$z_{it}^{-1} = \beta\mathbb{E}_{it}z_{it+1}^{-1}$$

We can thus condense the equilibrium characterization to the following

Proposition 2. *Let $z_{i,t} = p_{i-1,t}y_{i-1,t} + q_{i,t}$ denote household i's time t expenditure on the consumption basket. The equilibrium expenditure in this economy is the fixed point to the following two equations: (i) the Euler Equation of each circle household*

$$z_{it}^{-1} = \beta\mathbb{E}_{it}z_{it+1}^{-1} \tag{10}$$

and (ii) the budget constraint of each circle household

$$z_{it} + a_{i,t} = (1+\rho)\theta z_{i+1,t-1} + a_{it-1} \tag{11}$$

(iii) the amount consumed on each good is given by

$$p_{i-1,t}y_{i-1,t} = \theta z_{it} \quad \text{and} \quad q_{it} = (1-\theta)\, z_{it}$$

where

$$p_{i,t} = \frac{1}{1+\rho}\frac{\chi}{A} z_{it}$$

Given equilibrium expenditure one can then easily back out the individual components of consumption q_{it} and y_{it}.Proposition 4 represents the equilibrium as a fixed point in the expenditure of each household z_{it} in terms of each household's Euler Equation and the household's budget constraint. The budget constraint is simply a physical constraint which cannot be violated. The Euler equation, however, describes the optimal behavior or the household in terms of its consumption, or expenditure, path. given it's expectations of future expenditure. This obviously interacts with the budget constraint, as both current and future expenditure must

Therefore this economy reduces to an economy which looks very similar to conventional consumption-savings models. However, the main difference is that the expenditure of one agent becomes the income of another. This is apparent from the budget constraint (11); the expenditure of household $i+1$ at time $t-1$ becomes the income of household i at time t.

Finally, I assume that

$$(1+\rho)\,\theta = 1$$

So that the budget constraint becomes

$$z_{it} + a_{i,t} = z_{i+1,t-1} + a_{it-1}$$

5 Relation to the Traffic Model

I now show how this economic environment is similar in many ways to the traffic model environment presented in Section 2. In this analogy, the expenditure of each circle consumer is similar to the velocity of each car. Thus, the resources any agent spends on consumption in a given period becomes the income for the next agent (the producer of that good) the following period. This increases the latter agent's cash-on-hand in the following period, which he may then choose to spend on consumption, therefore transfering this wealth to the next agent. And so on. Thus, the transferal of resources or wealth from one agent to another is analagous to the idea that whenever a car moves forward it gives space to the car behind it. this increases the headway for the car behind him, in which case that car may move forward.

Here, I will now make these ideas more concrete and show how closely these ideas are aligned. how the economic model outlined above is similar to the traffic model

Position, Velocity, and Acceleration. Let x_{it} denote the value of all expenditure up through period t

$$x_{it} \equiv \sum_{j=0}^{t} z_{i,t-j} + \sum_{k=1}^{i-1} a_{k,-1} \tag{12}$$

where, as before, $z_{it} = p_{i-1,t}y_{i-1,t} + q_{i,t}$ denotes the expenditure on household i's composite consumption basket. Thus, I say that x_{it} denotes the "position" of household i at end of period t. One can think of this as the amount of numeraire the consumer has used. We can think of this position as if agents hold pieces of numeraire. Each unit of numeraire has a number on it, so as a household receives more income, it holds a higher numbered piece of the numeraire.

I define a discounted time-derivative operator as follows

$$\Delta \equiv 1 - L$$

where L is the lag operator. It is straight-forward to show that the velocity of agent i at time t, or the first (discounted) time-derivative of x_{it}, is equal to expenditure this period.

$$v_{it} \equiv \Delta x_{i,t} = x_{it} - x_{i,t-1} = z_{i,t}$$

This is shown in the appendix. Furthermore, the acceleration of household i is simply just the household's change in expenditure: $\Delta z_{it} = z_{it} - z_{it-1}$.

Headway. I now consider the analog of headway, the bumper-to-bumper distance between cars in the traffic model. In the traffic model headway of car i was defined as the difference in position between car i and car $i+1$. In the economic model, I define the headway of household i at time t as a particular difference in position (distance) between that household and the household in front of it. This difference is defined as follows.

$$h_{it} \equiv x_{i+1,t} - x_{i,t}$$

Given this definition along with the sequence of budget constraints in (11), we may obtain the following characterization of headway

Lemma 1. *Headway at the beginning of the period is equal to the household's resources before consuming or investing*

$$h_{i,t-1} = z_{i+1,t-1} + a_{i,t-1}$$

Headway is thereby the household's income and assets at the beginning of the period, before making consumption and investment decisions. This implies that one can rewrite the sequence of budget constraints as follows

$$z_{it} + a_{i,t} = h_{i,t-1} \tag{13}$$

The intuition for this is fairly simple. Suppose household i starts out with assets at time 0. When household 1 buys some goods from household 0, household 1 transfers resources to household 0. Thus, at the beginning of the following period, household 0 can consume using its assets and its income from the previous period.

Boundary Condition. In the traffic model, there was a boundary condition given by $\sum_{i\in I} h_{i,t} = L$. The cars were arranged on a circle of fixed length L. Thus, the aggregate amount of headway remained constant–the length of the circle never shrank nor expanded. In the economic model, the circle is also closed (since household N purchases goods from producer 1) so that a boundary condition must exist in every period. However, aggregate headway can change over time. Headway grows due to interest made on assets, and shrinks as the numeraire leaves the system and is transferred to the mainland worker sector. First, I define the aggregate headway at time t as the sum over all households' headways.

$$H_t = \sum_{i\in I} h_{i,t} = \sum_{i\in I} (z_{i+1,t} + a_{i,t})$$

It is then easy to obtain a law of motion for aggregate headway. Plugging in for $a_{i,t}$ from the budget constraint (13), $a_{i,t} = (1+r)\, h_{i,t-1} - z_{it}$, we obtain

$$H_t = \sum_{i\in I} (z_{i+1,t} + h_{i,t-1} - z_{it})$$

Letting $Z_t = \sum_{i\in I} z_{i,t}$ be aggregate expenditure, this leads to the following characterization of aggregate headway

Lemma 2. *Aggregate headway evolves according to the following law of motion*

$$H_t = H_{t-1} \tag{14}$$

where $Z_t = \sum_{i \in I} z_{i,t}$ *is aggregate expenditure and initial headway given by* $H_{-1} = \sum_{i \in I} h_{i,-1}$.

Thus, the aggregate amount of headway is changing over time, according to the above law of motion. Aggregate headway grows because the amount of wealth held within the circle increases over time due to the fact that the value of bonds increases at the rate of interest.

Transforming equations to continuous time. In discrete time, the equilibrium is described by the following equations

$$\begin{aligned} z_{i,t} &= x_{it} - x_{i,t-1} \\ h_{i,t} &= x_{i+1,t} - x_{i,t} \\ H_t &= H_{t-1} \end{aligned}$$

These equations closely correspond to those in a discrete-time version of the traffic model.[10] Here, instead velocity is interpreted as expenditure, $v_{i,t} = z_{i,t}$, and headway as equivalent to numeraire-on-hand at the beginning of the period, $h_{i,t-1} = z_{i+1,t-1} + a_{i,t-1}$.[11]

I now transform these equations into continuous time such that expenditure is the usual time derivative of position. And hence, taking the limit as the time increment between periods approaches zero, we get the following continuous-time analog of these equations ()

$$\begin{aligned} \hat{z}_i(t) &= \dot{\hat{x}}_i(t) \\ \dot{h}_i(t) &= \hat{x}_{i+1}(t) - \hat{x}_i(t) \\ \dot{H}(t) &= 0 \end{aligned}$$

What remains missing from this system is the policy function.

The next steps in my analysis. I have described the equilibrium as a fixed point of two sets of equations: the set of Euler equations for the circle households, and the set of budget constraints. In the analysis thus far, I have only used the set of budget constraints. There are two equations that must coincide with each other. The only equation I have not used yet is the Euler Equation. The Euler equation must give a policy function as in (15). In order to find this convergence, there are two avenues I pursue.

[10]See for example the discretized traffic version of the optimal velocity model in Tadaki et al. (1997)

[11]In fact, one may consider the hypothetical limit in which $r \to 0$, and $\alpha\theta \to 1$. In this case, there is no discounting of time by the mainland household, and both the specialized good share of the consumption basket and the capital share of output approach one. In this limit, we have that $\Delta = 1 - L$, which implies that $\Delta z_{i,t} = z_{it} - z_{it-1}$, $h_{it} = x_{i+1,t} - x_{i,t}$, and aggregate H_t is constant. Therefore, in this limit, the equations describing the economy converge exactly to those in a discrete-time version of the traffic model.

5.1 Reduced-Form Expenditure Policy Functions

Imposing a Policy Function. The optimal velocity equation in the traffic jam model is given by

$$\dot{v}_i(t) = \alpha(V(h_i(t)) - v_i(t))$$

The goal is to obtain a policy function similar to this in the economic model from first principles. That is, one would ideally want a function governing expenditure behavior that looks like the following

$$\dot{z}_i(t) = G\left(\hat{h}_i(t), z_i(t)\right) \tag{15}$$

with $\partial G/\partial h > 0$ and $\partial G/\partial z < 0$.[12] Thus, I want expenditure to be increasing in cash-on-hand. This is related to the household's marginal propensity to consume. This is similar to the state variables that we often see in many economic problems.

For now, I simply impose a policy function as in (15). In this sense, I just throw away the Euler equation, and exogenously impose a policy function, and I then derive what I need in terms of G in order to obtain traffic jams. One may think of this as a reduced form expression for the behavior of agents. This is what follows in this section I find this simple exercise useful as it gives some guidance as to what the policy function must look like and what properties it must have in order to produce traffic jams.

Equilibrium For now, let's just impose this function (15) exogenously. Then, the equilibrium of this economy is described by the following equations.

Lemma 3. *Imposing a policy function as in (15), an equilibrium of the system is given by*

$$\begin{aligned}
\hat{z}_i(t) &= \dot{\hat{x}}_i(t) \\
\dot{\hat{z}}_i(t) &= G\left(\hat{h}_i(t), \hat{z}_i(t)\right) \\
\hat{h}_i(t) &= \hat{x}_{i+1}(t) - \hat{x}_i(t) \\
\dot{\hat{H}}(t) &= 0
\end{aligned}$$

This system is almost the same as the equations describing the traffic system, with the only difference given by the change in headway.

Uniform Flow Equilibrium. I can now derive what one would consider the uniform flow equilibrium.[13] Let me first define the uniform flow equilibrium. Suppose $\hat{z}_i(t) = \bar{z}$ and $\hat{h}_i(t) = \bar{h} = \bar{H}/N$. This implies $\dot{\hat{z}}(t) = 0$.

[12] This is where I apply the so-called want operator.

[13] Suppose we define the uniform flow equilibrium as follows, $\dot{\hat{z}}_i(t) = 0, \quad \forall i, t$. But, plugging this in, we

Finally, the policy function must also hold. One can linearize around 15 and get that.

$$\dot{\hat{z}}_i(t) = G_h \hat{h}_i(t) - G_z \hat{z}_i(t) \tag{16}$$

Thus, in the uniform flow, it must be the case that

$$0 = G_h \bar{h} - G_z \bar{z}$$

(Note that $\ln(1+r) \simeq r$. This corresponds to the uniform flow-equilibrium in discrete time.)

Proposition 3. *In the uniform flow equilibrium, the transformed expenditure and headway are given by*

$$\hat{z}(t) = \bar{z}, \text{ and } \quad \hat{h}(t) = \bar{h} = \bar{H}/N$$

where

$$\bar{z} = \frac{G_h}{G_z}\bar{h}$$

This describes the uniform flow equilibrium.

Stability. I now consider stability of the uniform flow equilibrium. I obtain the following result

Proposition 4. *The uniform flow equilibrium is stable if and only if*

$$G_h < \frac{1}{2}G_z^2$$

The proof of this is in the Appendix. Thus, for G_h low enough, the uniform flow equilibrium is stable. Otherwise, it is unstable.

5.2 Simulations

I do some simulations with this exogenously imposed consumption function. For simplicity, in these simulations I set $N = 10$, $r = 0$

$$z_{i,t} - z_{i,t-1} = \alpha\left(V\left(h_{i,t-1}\right) - z_{i,t-1}\right)$$

with $\alpha = .8$ and a V function shown in Figure 8

get that $0 = G_h \hat{h}^* - G_z \hat{z}^*$ therefore

$$\hat{z}^* = \frac{G_h}{G_z}\hat{h}^*$$

but this implies that $\hat{h}$ is constant. but this cannot be true since aggregate headway is growing.

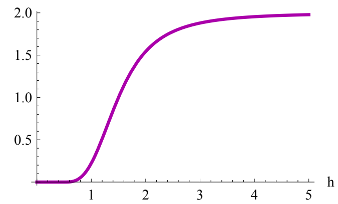

Figure 8: V Function 1

The results of these simulations are presented in Figures 9 and 10. Figure 9 plots aggregate expenditure over time at various levels of aggregate assets. The beginning part of this plot is just allowing the economy to converge. Figure 10 gives the same information, but once the economy has converged. As one can see, for low levels of headway, aggregate expenditure (and hence consumption) falls sharply when aggregate asset holdings becomes low.

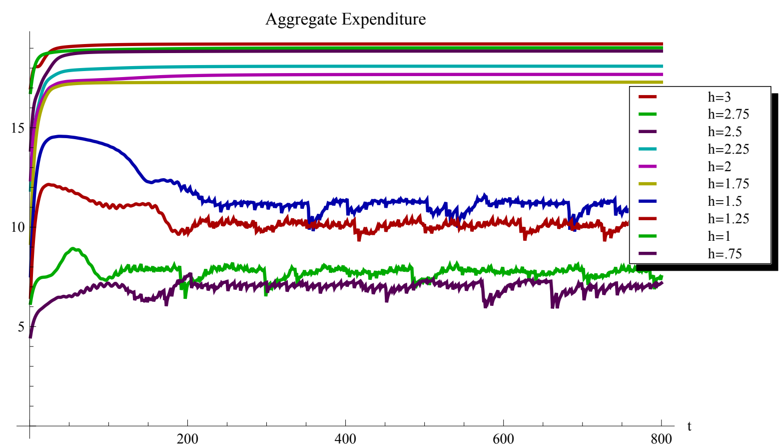

Figure 9: Simulations

Furthermore, I exogenously decrease and increase the amount of aggregate assets in the economy. Given this experiment, Figure 11 plots the time series of aggregate consumption. As one can see, it falls sharply at low levels of aggregate assets, and takes some time to recover before reaching its original level. Figure 12 plots the time series of cross-sectional variation in consumption. When the economy gets into the traffic-jam equilibrium, consumption variance jumps up! Thus, the model endogenously produces consumption variance. and

In summary, from these simulations we see non-linear effects of decreasing cash-on-hand. Furthermore, when average cash-in-hand is low enough, the economic recession looks like traffic jam. Moreover, consumption variance increases.

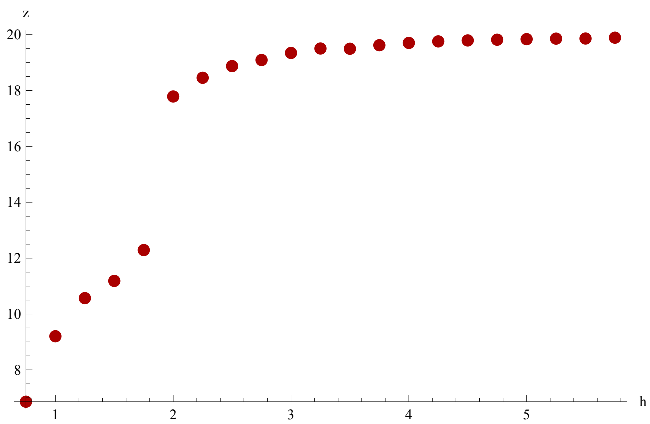

Figure 10: Comparative Statics

Therefore, small perturbations can potentially lead to recessions. These recessions would be ones in which agents cannot identify any large aggregate shock. Furthermore, from the the traffic model presented in Section 2, we see that the theory predicts that traffic jams are more likely to occur under certain conditions–conditions which take us to the other part of the parameter space. Thus, building an economic model may have implications for when recessions are more likely to occur.

Next Steps. The next route obviously then is a question of how to microfound a policy function as in (15) as the result of optimizing behavior of households. The behavior given by agents is the consumer's Euler Equation (10). I look at this more seriously in Section 6.

Agents care only about local interactions. Could optimizing agents follow a similar behavioral rule? If so, perhaps the economy could generate behavior at the micro level that resembles stop-and-go traffic. A decrease in velocity is similar to a decrease in spending. Finally, waiting for headway to increase would be equivalent to waiting for income to increase.

6 Borrowing Constraints Model

I now attempt to provide microfoundations for the type of policy functions considered in the previous section. The goal is to derive a policy function for expenditure that resembles (15), such that it is an increasing function of current headway, but as also a result of optimizing behavior on the behalf of rational consumers. This will depend on the interaction between the household's Euler Equation and its budget constraint.

I consider a variant of the model with idiosyncratic incomes shocks, incomplete markets, and borrowing constraints. This seems to be one of the most natural microfoundations

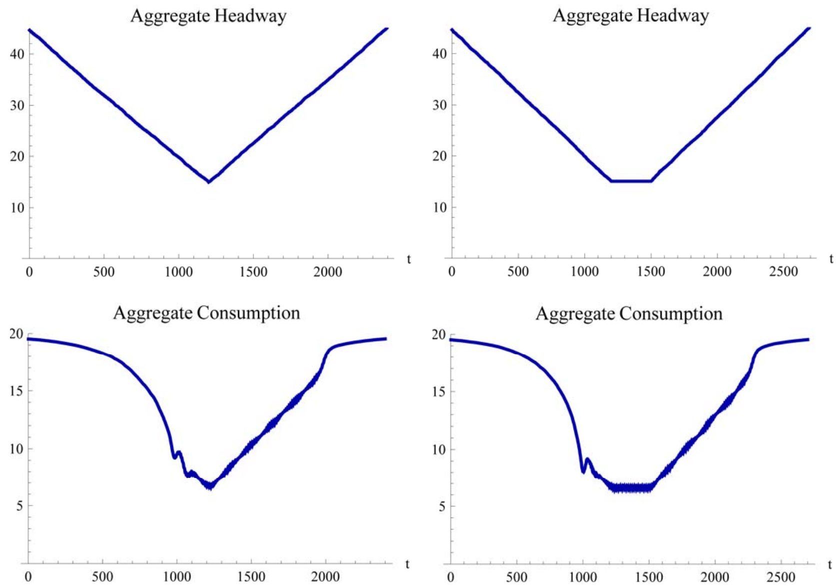

Figure 11: Time Series of Consumption

for an expenditure policy function which has a high marginal propensity to consume when agents are close to their borrowing constraint. I assume that the household faces borrowing constraint a borrowing constraint as follows

$$a_{it} \geq -\phi \tag{17}$$

where ϕ is a known constant. One can also think of this as a simple cash-in-hand constraint if $\phi = 0$. From a large and extensive literature on consumption-savings models with borrowing constraints, we know that this type of simple constraint leads to increasing and concave consumption/expenditure policy functions as well as high marginal propensities to consume when agents are close to their borrowing constraints. Thus, the model here is similar to a consumption savings model with idiosyncratic income (labor) risk, as in Aiyagari, Huggett, Bewley. However, in contrast to these papers, the income risk here is endogenous–the income of one agent is derived from the consumption behavior of another.

With the added borrowing constraint (5), solving the model becomes a bit intractable. In particular, the state space of each agent's problem blows up. Agents must forecast the shocks of all other agents and hence keep track of entire distribution of individual states. Intuitively, imagine each agent's individual state space is composed of his asset holdings,

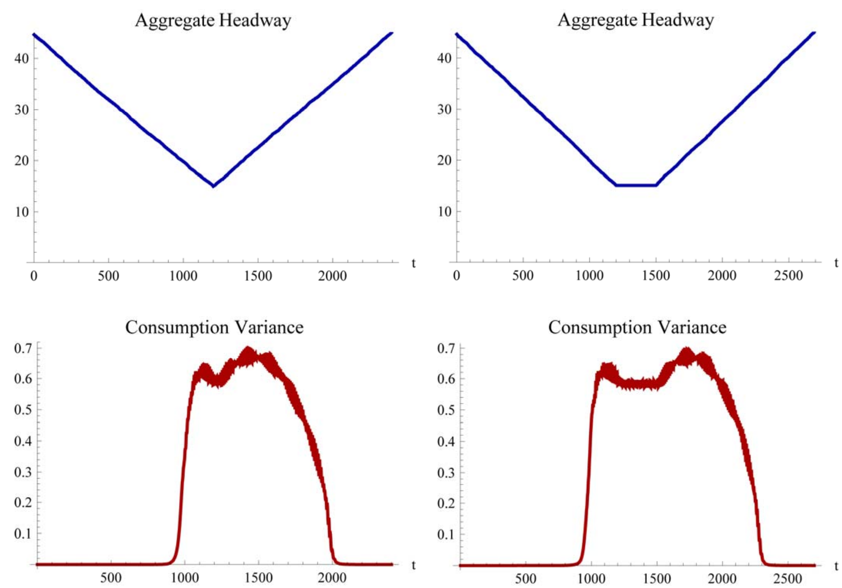

Figure 12: Time Series of Consumption Variance

his income from this producer, and his idiosyncratic income shock. This determines his expenditure. However, in order for him to determine his income next period, he must know the expenditure on the island next to him the current period. But that depend's on agent $i+1$'s current asset holdings, income, and idiosyncratic income shock. Hence, he needs to keep track of that. But in order for him to understand what the income is of agent $i+1$, he must try to understand the state on island $i+2$, and so on... Hence, each agent tries to keep track of all individual states of all islands in the economy. This is clearly an intractable problem, not only for the economist trying to model the economy, but most likely for the agent itself.

Hence, in order to simplify the problem and preserve tractability, I assume that each consumer perceives income $z_{i+1,t}$ as Markov as follows

$$z_{i+1,t+1} = \psi\left(z_{i+1,t}\right)$$

This is clearly a stark assumption. However, it has some underlying economic intuition. That is, suppose agents have constrained information capacity. A growing literature has tried to understand limited information capacity as a constraint on agent's ability to process all information. Sims (2003) models this as a constraint on the conditional entropy, Woodford

(2012) introduces a variant with reference-dependent choice which closely matches experimental evidence on agent's attention, while Gabaix (2011) allows agents to have sparse information sets so that they only keep track of a finite number of state variables. Thus, it seems likely that households cannot keep track of entire state of the world, and instead can only keep track and form expectations over a finite number of moments.

Admittedly, I am not solving the ex-ante problem of what agents would pay attention to with limited information capacity. I am just taking it as given that they only pay attention to their own income, which seems the most relevant for their own consumption choices.

I thus re-define an approximate equilibrium as in Krussell-Smith () as follows.

Definition 2. *A competitive approximate equilibrium is a collection of allocation and price functions such that*

(i) given current prices and expectations of future prices and income, allocations are optimal for households and firms

(ii) prices clear all markets

(iii) household expectations are based on perceived Markov process

$$z'_{i+1} = \psi\left(z_{i+1}\right)$$

where ψ is the best approximation of the true process

Part (iii) is similar to Krusell-Smith. One needs to define "best approximation".

The circle household's consumption-savings problem thus becomes similar to Bewley economy

$$V\left(z_{i+1,-1}, a_i, \omega_i\right) = \max_{z_i, a'_i} \left(1 - \alpha\theta\right) \log z_i + \beta \mathbb{E}_i V\left(z_{i+1}, a'_i, \omega'_i\right)$$

subject to

$$\begin{aligned} z_i + a'_i &= (1+r)\left(\alpha\theta z_{i+1,-1} + a_i + \omega_i\right) \\ a_i &\geq -\phi \end{aligned}$$

and where z_{i+1} evolves according to the law of motion

$$z_{i+1} = \psi\left(z_{i+1,-1}\right)$$

Next, I simulate the economy with borrowing constraints. I obtain policy functions for asset holdings and expenditure given by the following

$$\begin{aligned} a_i' &= d\left(z_{i+1,-1}, a_i, \omega_i\right) \\ z_i &= g\left(z_{i+1,-1}, a_i, \omega_i\right) \end{aligned}$$

where z_i increasing in $z_{i+1,-1}, a_i$, and ω_i.

The parameter values I use for this simple numerical simulationm are as follows. I set $\beta = .9$, $\phi = 0$. The interest rate is set at $r = .02$. I first allow for exogenous beliefs about income

$$z_{i+1}' = \rho z_{i+1} + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$$

numerically approximated with 8 states, $\rho = .2$, $\sigma_\varepsilon = .5$

Consumption is increasing in assets and income. I thus obtain the following expenditure policy functions. The equilibrium expenditure of household i is given by

$$\begin{aligned} a_i' &= D\left(z_{i+1,-1}; a_i; A_{i-1}\right) \\ z_i &= G\left(z_{i+1,-1}; a_i; A_{i-1}\right) \end{aligned}$$

Hence, expenditure is increasing in assets and income; see Figure 13.

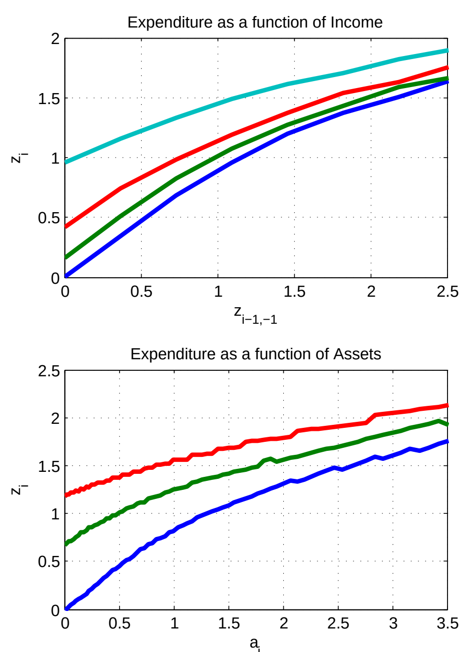

Figure 13: Expenditure Functions

I thus solve for the general equilibrium fixed point as follows

- start at non-stochastic equilibrium.
- compute equilibrium with shocks to ω_{it}
- approximate true process for z_{it} with some Markov process ψ
- use ψ for beliefs in next iteration
- iterate until ψ is "close to" true z process

Comments. I can also follow Kimball and Caroll () and obtain this type of function without a borrowing constraint.

7 Conclusion

I construct a model in which recessions resemble traffic jams. The next steps in this project are clearly two fold. First, one should check the robustness of this in terms of different network structures. Clearly the world is not a circle. At the same time, the world is not a representative household or a representative firm. Third, it would be important to think about efficiency and policy.

Empirical Implications. What are some of the empirical implications of this model? First, more Hand-to-Mouth behavior imply that Recessions more likely. Furthermore, when Agents close to borrowing constraint $\rightarrow$ Recessions more likely. This would potentially be a nice thing to test.

Which recessions could this model potentially apply to? The subprime, The 1907 recession was presumably caused by one trader trying to corner the gold market.

Furthermore, there is evidence... Reinhart Rogoff. Alan Taylor has recently shown that this extends to many recessions.

In the Survey of Consumer Finances, the reason for the household's savings is Liquidity.

Finally, I would like to find data on local interactions and see how to get a flux-like diagram like that in the traffic literature. This would give some empirical evidence for this mechanism.

Appendix

Proof of Equilibrium Characterization in Economic Model Household

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(y_{i-1,t}^{\theta} q_{it}^{1-\theta}\right) - \chi n_{it}\right]$$

subject to

$$p_{i-1,t} y_{i-1,t} + q_{it} + a_{i,t} = (1+\rho)\, p_{i,t-1} A n_{i,t-1} + (1+r)\, a_{it-1} \tag{18}$$

FOCs with respect to $y_{i-1,t}, q_{it}, n_{it}, a_{it}$, respectively, are given by

$$\begin{aligned}
\beta^t u'(c)\, \theta \frac{c_{it}}{y_{i-1,t}} - \beta^t \lambda_{it} p_{i-1,t} &= 0 \\
\beta^t u'(c)\,(1-\theta) \frac{c_{it}}{q_{it}} - \beta^t \lambda_{it} &= 0 \\
-\beta^t \chi + \beta^{t+1} \lambda_{i,t+1} (1+\rho)\, p_{i,t} A &= 0 \\
-\beta^t \lambda_{it} + \beta^{t+1} (1+r)\, \lambda_{i,t+1} &= 0
\end{aligned}$$

euler equation

$$\lambda_{it} = \beta (1+r)\, \lambda_{i,t+1}$$

Proof of Proposition 1 Suppose the vehicle policy function is given more generally by

$$\dot{v}_i(t) = f\left(h_i(t), \dot{h}_i(t), v_i(t)\right) \tag{19}$$

where $v_i(t) = \dot{x}_i(t)$, $h_i(t) = x_{i+1}(t) - x_i(t)$, and $\dot{h}_i(t) = v_{i+1}(t) - v_i(t) = \dot{x}_{i+1}(t) - \dot{x}_i(t)$.

The uniform flow equilibrium is defined as an allocation of headways and velocities for each car in which both are time independent. That is

$$h_i(t) = h^*, v_i(t) = v^*, \dot{h}_i(t) = 0, \dot{v}_i(t) = f(h^*, 0, v^*)$$

To analyze the stability of the uniform flow equilibrium, we linearize (19) about the uniform flow equilibrium. We then have

$$\dot{\tilde{v}}_i(t) = F\tilde{h}_i(t) + G\dot{\tilde{h}}_i(t) - H\tilde{v}_i(t)$$

where

$$F = \partial_h f(h^*, 0, v^*), G = \partial_{\dot{h}} f(h^*, 0, v^*), H = -\partial_v f(h^*, 0, v^*)$$

are all assumed to be positive.

Substituting in $v_i(t) = \dot{x}_i(t)$, $h_i(t) = x_{i+1}(t) - x_i(t)$, and $\dot{h}_i(t) = \dot{x}_{i+1}(t) - \dot{x}_i(t)$ we get that for every i,

$$\ddot{\tilde{x}}_i(t) = F\left(\tilde{x}_{i+1}(t) - \tilde{x}_i(t)\right) + G\left(\dot{\tilde{x}}_{i+1}(t) - \dot{\tilde{x}}_i(t)\right) - H\dot{\tilde{x}}_i(t)$$

Bringing all i on the left side, and $i+1$ on the right side, we get tthe following second order system

$$\ddot{\tilde{x}}_i(t) + (G+H)\dot{\tilde{x}}_i(t) + F\tilde{x}_i(t) = G\dot{\tilde{x}}_{i+1}(t) + F\tilde{x}_{i+1}(t)$$

A standard way to approach the second order system is to define a new variable $\tilde{v}_i(t) = \dot{\tilde{x}}_i(t)$. we can thus rewrite this as

$$\dot{\tilde{v}}_i(t) + (G+H)\tilde{v}_i(t) + F\tilde{x}_i(t) = G\tilde{v}_{i+1}(t) + F\tilde{x}_{i+1}(t)$$

Let

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_N \end{bmatrix}, \tilde{v} = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \vdots \\ \tilde{v}_N \end{bmatrix}, \dot{\tilde{x}} = \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \vdots \\ \dot{\tilde{x}}_N \end{bmatrix}, \dot{\tilde{v}} = \begin{bmatrix} \dot{\tilde{v}}_1 \\ \dot{\tilde{v}}_2 \\ \vdots \\ \dot{\tilde{v}}_N \end{bmatrix}$$

We now have a linear system of $2N$ equations with

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{bmatrix} = M \begin{bmatrix} \tilde{x} \\ \tilde{v} \end{bmatrix}$$

For example, suppose $N = 2$. where M is some matrix that looks like[14]

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{v}}_1 \\ \dot{\tilde{v}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -F & F & -(G+H) & G \\ F & -F & G & -(G+H) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$$

[14] For $N = 3$, then A is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -F & F & 0 & -(G+H) & G & 0 \\ 0 & -F & F & 0 & -(G+H) & G \\ F & 0 & -F & G & 0 & -(G+H) \end{bmatrix}$$

Thus we conjecture a particular solution to the system $\tilde{x}_i = A_i e^{\lambda t}$

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} A_1 e^{\lambda t} \\ A_2 e^{\lambda t} \\ A_1 \lambda e^{\lambda t} \\ A_2 \lambda e^{\lambda t} \end{bmatrix}$$

Therefore, we can plug the trial solution $\tilde{x}_i = A_i e^{\lambda t}$ into equation (), which gives us the following N equations

$$\begin{aligned} A_1 \left(\lambda^2 + (G+H)\lambda + F\right) &= A_2 (G\lambda + F) \\ A_2 \left(\lambda^2 + (G+H)\lambda + F\right) &= A_3 (G\lambda + F) \\ &\vdots \\ A_N \left(\lambda^2 + (G+H)\lambda + F\right) &= A_1 (G\lambda + F) \end{aligned}$$

Iteratively substituting for A_i we have the following equation

$$\left(\lambda^2 + (G+H)\lambda + F\right)^N = (G\lambda + F)^N$$

Taking N-th roots of Equation (), we have the following

$$\lambda^2 + (G+H)\lambda + F = (G\lambda + F) e^{i\theta}$$

where $\theta = \frac{k}{N} 2\pi$ for $k = 1, 2, \ldots, N$ [15]

Now, by substituting

$$\lambda = i\omega \quad \text{for } \omega \in \mathbb{R}^+$$

into equation (),

$$-\omega^2 + (G+H) i\omega + F = (Gi\omega + F)(\cos\theta + i\sin\theta)$$

or

$$-\omega^2 + F + (G+H) i\omega = -G\omega\sin\theta + F\cos\theta + (G\omega\cos\theta + F\sin\theta) i$$

Separating the real and imaginary parts and eliminating ω. The real parts imply

$$-\omega^2 + F = -G\omega\sin\theta + F\cos\theta \tag{20}$$

[15] because note that $e^{i\theta} = \cos\theta + i\sin\theta$, so that $e^{i2\pi} = \cos 2\pi + i\sin 2\pi = 1$

while the imaginary parts imply

$$(G+H)\,\omega = G\omega\cos\theta + F\sin\theta \tag{21}$$

Solving (21) for ω, we get that

$$\omega = \frac{F\sin\theta}{G+H-G\cos\theta}$$

Substituting this into (20) we get that

$$-\left(\frac{F\sin\theta}{G+H-G\cos\theta}\right)^2 + F = -\frac{F\sin\theta}{G+H-G\cos\theta}G\sin\theta + F\cos\theta$$

Therefore, we have an expression only in terms of F, G, H, θ. Rearranging yields

$$F = (1-\cos\theta)\left(\frac{G+H-G\cos\theta}{\sin\theta}\right)^2 + G\,(G+H-G\cos\theta)$$

Let $\alpha = \theta/2$, and using some trigonometric identities, one may determine that stability changes (Hopf bifurcations) occur for

$$F = \frac{1}{2}(2G+H)\left((2G+H)\tan^2\alpha + H\right)$$

where $\alpha = \theta/2$. Thus $\alpha = \frac{k}{N}\pi$ for $k = 1, 2, \ldots, N$.

The stability condition becomes

$$F < \frac{1}{2}(2G+H)\,H \tag{22}$$

We now apply this general stability condition to the optimal velocity model. Here, the acceleration policy function is given by

$$\dot{v}_i(t) = \alpha\left(V(h_i(t)) - v_i(t)\right)$$

Linearizing about the uniform flow equilibrium, we obtain equation () with

$$\begin{aligned} F &= \partial_h f(h^*, 0, v^*) = \alpha V'(h) \\ G &= \partial_{\dot{h}} f(h^*, 0, v^*) = 0 \\ H &= -\partial_v f(h^*, 0, v^*) = \alpha \end{aligned}$$

Plugging these values into (22), the stability condition becomes

$$V'(h) < \frac{1}{2}\alpha$$

QED.

Proof of 2 First, consider the mainland household. The mainland household maximizes utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \tilde{\beta}^t \left[u(\tilde{q}_t) - \chi \tilde{n}_t - \tilde{h}_t \right]$$

subject to its budget constraint.

$$\tilde{q}_t + \tilde{a}_t = w_t \tilde{n}_t + \tilde{h}_t + (1 + r_t)\, \tilde{a}_{t-1}$$

Let $\beta^t \mu_t$ be the Lagrange multiplier on the budget constraint. The FOCs of this problem with respect to $\tilde{c}_t, \tilde{n}_t, \tilde{h}_t, \tilde{b}_t$ are

$$\begin{aligned} u'(\tilde{q}_t) - \mu_t &= 0 \\ -\chi + \mu_t w_t &= 0 \\ -1 + \mu_t &= 0 \\ -\tilde{\beta}^t \mu_t + \mathbb{E}_t (1 + r_t)\, \tilde{\beta}^{t+1} \mu_{t+1} &= 0 \end{aligned}$$

The consumer i's problem is to maximize utility

$$\max \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

where $c_{it} = y_{i-1t}^{\theta} q_{it}^{1-\theta}$ subject to the household's budget constraint.

$$p_{i-1,t} y_{i-1,t} + q_{it} + a_{i,t} = (1 + r_t)\,(\pi_{i,t-1} + a_{it-1})$$

Letting $\beta^t \lambda_{i,t}$ be the Lagrange multiplier on the budget constraint of household i at time t. The FOCs are given by

$$\begin{aligned} u_y(c_{it}) - \lambda_{it} p_{i-1,t} &= 0 \\ u_q(c_{it}) - \lambda_{it} &= 0 \\ -\beta^t \lambda_{i,t} + \mathbb{E}_t (1 + r)\, \beta^{t+1} \lambda_{i,t+1} &= 0 \end{aligned}$$

Proof of Lemma 1 The household budget constraint is given by

$$z_{it} + a_{i,t} = (1+r)\left(\alpha\theta z_{i+1,t-1} + a_{it-1}\right)$$

definition of position

$$x_{it} \equiv \sum_{j=0}^{t} (1+r)^j z_{i,t-j}$$

First, velocity is given by $v_{it} \equiv x_{i,t} - (1+r)\, x_{i,t-1}$

$$\begin{aligned} v_{it} &= \left(z_{i,t} + (1+r)\sum_{j=0}^{\infty}(1+r)^j z_{i+1,t-1-j}\right) - (1+r)\sum_{j=0}^{\infty}(1+r)^j z_{i,t-1-j} \\ &= z_{i,t} \end{aligned}$$

The position of agent $i+1$ at time t is given by

$$x_{i+1,t} = z_{i+1,t} + (1+r)\sum_{j=0}^{\infty}(1+r)^j z_{i+1,t-1-j}$$

Multiplying this by $\alpha\theta$ we have that

$$\alpha\theta x_{i+1,t} = \alpha\theta z_{i+1,t} + \sum_{j=0}^{\infty}(1+r)^j (1+r)\, \alpha\theta z_{i+1,t-1-j}$$

Next, rearranging the budget constraint,

$$z_{it-j} + a_{i,t-j} = (1+r)\,\alpha\theta z_{i+1,t-j-1} + (1+r)\, a_{i,t-j-1}$$

we obtain the following

$$(1+r)\,\alpha\theta z_{i+1,t-1-j} = z_{i,t-j} + a_{i,t-j} - (1+r)\, a_{i,t-j-1}$$

Plugging this into () we get that

$$\alpha\theta x_{i+1,t} = \alpha\theta z_{i+1,t} + \sum_{j=0}^{\infty}(1+r)^j \left(z_{i,t-j} + a_{i,t-j} - (1+r)\, a_{i,t-j-1}\right)$$

Now, if we write out the position of agent i at time t, this is given by

$$x_{i,t} = \sum_{j=0}^{t} (1+r)^j z_{i,t-j}$$

Substituting () and () into our definition of headway,

$$h_{it} \equiv \alpha\theta x_{i+1,t} - x_{i,t}$$

we have that

$$h_{it} = \alpha\theta z_{i+1,t} + \sum_{j=0}^{\infty} (1+r)^j \left(z_{i,t-j} + a_{i,t-j} - (1+r)\, a_{i,t-j-1}\right) - \sum_{j=0}^{t} (1+r)^j z_{i,t-j}$$

Thus,

$$h_{it} = \alpha\theta z_{i+1,t} + \sum_{j=0}^{\infty} (1+r)^j \left(a_{i,t-j} - (1+r)\, a_{i,t-j-1}\right)$$

Expanding the terms in this summation, we have that headway satisfies

$$\begin{aligned} h_{it} &= \alpha\theta z_{i+1,t} + \left(a_{i,t} - (1+r)\, a_{i,t-1}\right) \\ &\quad + (1+r)\left(a_{i,t-1} - (1+r)\, a_{i,t-2}\right) \\ &\quad + (1+r)^2 \left(a_{i,t-2} - (1+r)\, a_{i,t-3}\right) + \cdots \end{aligned}$$

All of the $a_{i,t-j}$ cancel out except for $j = 0$. Thus, we have that

$$h_{it} = \alpha\theta z_{i+1,t} + a_{i,t}$$

Rewriting this for $h_{i,t-1}$ we have that

$$h_{i,t-1} = \alpha\theta z_{i+1,t-1} + a_{i,t-1}$$

Therefore, headway is equal to wealth-on-hand at the beginning of the period. QED.

Proof of Lemma 2 Follows from the main text.

Proof of Lemma 3 Follows from the main text.

Proof of Proposition 3 Follows from the main text.

Proof of Proposition 4 The proof of this follows closely that of Proposition 1. The economic system is described by the following four equations

$$\begin{aligned}\hat{z}_i(t) &= \dot{\hat{x}}_i(t)\\ \dot{\hat{z}}_i(t) &= f(h_i(t), z_i(t))\\ \hat{h}_i(t) &= \alpha\theta\hat{x}_{i+1}(t) - \hat{x}_i(t)\\ \dot{\hat{H}}(t) &= -(1-\alpha\theta)\hat{Z}(t)\end{aligned}$$

Suppose the vehicle policy function is given more generally by

$$\dot{\hat{z}}_i(t) = f\left(h_i(t), \dot{h}_i(t), v_i(t)\right)$$

where $z_i(t) = \dot{x}_i(t)$, $\hat{h}_i(t) = \alpha\theta\hat{x}_{i+1}(t) - \hat{x}_i(t)$, and $\dot{h}_i(t) = \alpha\theta\dot{x}_{i+1}(t) - \dot{x}_i(t)$.

The uniform flow equilibrium is defined as an allocation of headways and velocities for each car in which both are time independent. [need to fix] That is,

$$h_i(t) = h^*, z_i(t) = v^*, \dot{h}_i(t) = 0, \dot{v}_i(t) = f(h^*, 0, v^*)$$

To analyze the stability of the uniform flow equilibrium, we linearize () about the uniform flow equilibrium. We then have

$$\dot{\tilde{z}}_i(t) = F\tilde{h}_i(t) + G\dot{\tilde{h}}_i(t) - H\tilde{z}_i(t)$$

where

$$F = \partial_h f(h^*, 0, v^*), G = \partial_{\dot{h}} f(h^*, 0, v^*), H = -\partial_v f(h^*, 0, v^*)$$

are all assumed to be positive.

Substituting in $\tilde{z}_i(t) = \dot{\tilde{x}}_i(t)$, $\tilde{h}_i(t) = \alpha\theta\tilde{x}_{i+1}(t) - \tilde{x}_i(t)$, and $\dot{\tilde{h}}_i(t) = \alpha\theta\dot{\tilde{x}}_{i+1}(t) - \dot{\tilde{x}}_i(t)$ we get that for every i,

$$\ddot{\tilde{x}}_i(t) = F(\alpha\theta\tilde{x}_{i+1}(t) - \tilde{x}_i(t)) + G\left(\alpha\theta\dot{\tilde{x}}_{i+1}(t) - \dot{\tilde{x}}_i(t)\right) - H\dot{\tilde{x}}_i(t)$$

Bringing all i terms to the left-hand side, and $i+1$ terms to the right-hand side, we get the following second order system

$$\ddot{\tilde{x}}_i(t) + (G+H)\dot{\tilde{x}}_i(t) + F\tilde{x}_i(t) = G\alpha\theta\dot{\tilde{x}}_{i+1}(t) + F\alpha\theta\tilde{x}_{i+1}(t)$$

A standard way to approach the second order system is to define a new variable $\tilde{z}_i(t) = \dot{\tilde{x}}_i(t)$. we can thus rewrite this as

$$\dot{\tilde{z}}_i(t) + (G+H)\tilde{z}_i(t) + F\tilde{x}_i(t) = G\alpha\theta\tilde{z}_{i+1}(t) + F\alpha\theta\tilde{x}_{i+1}(t)$$

Let

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_N \end{bmatrix}, \tilde{z} = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \vdots \\ \tilde{z}_N \end{bmatrix}, \dot{\tilde{x}} = \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \vdots \\ \dot{\tilde{x}}_N \end{bmatrix}, \dot{\tilde{z}} = \begin{bmatrix} \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \\ \vdots \\ \dot{\tilde{z}}_N \end{bmatrix}$$

We now have a linear system of $2N$ equations with

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{bmatrix} = M \begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix}$$

For example, suppose $N = 2$. where M is some matrix that looks like[16]

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{z}}_1 \\ \dot{\tilde{z}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -F & F\alpha\theta & -(G+H) & G\alpha\theta \\ F\alpha\theta & -F & G\alpha\theta & -(G+H) \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix}$$

Thus we conjecture a particular solution to the system $\tilde{x}_i = A_i e^{\lambda t}$

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} A_1 e^{\lambda t} \\ A_2 e^{\lambda t} \\ A_1 \lambda e^{\lambda t} \\ A_2 \lambda e^{\lambda t} \end{bmatrix}$$

Therefore, we can plug the trial solution $\tilde{x}_i = A_i e^{\lambda t}$ into equation (), which gives us the

[16] For $N = 3$, then A is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -F & F & 0 & -(G+H) & G & 0 \\ 0 & -F & F & 0 & -(G+H) & G \\ F & 0 & -F & G & 0 & -(G+H) \end{bmatrix}$$

following N equations

$$\begin{aligned}A_1\left(\lambda^2+(G+H)\lambda+F\right) &= A_2\alpha\theta(G\lambda+F)\\ A_2\left(\lambda^2+(G+H)\lambda+F\right) &= A_3\alpha\theta(G\lambda+F)\\ &\vdots\\ A_N\left(\lambda^2+(G+H)\lambda+F\right) &= A_1\alpha\theta(G\lambda+F)\end{aligned}$$

Iteratively substituting for A_i we have the following equation

$$\left(\lambda^2+(G+H)\lambda+F\right)^N=(\alpha\theta(G\lambda+F))^N$$

Taking N-th roots of Equation (), we have the following

$$\lambda^2+(G+H)\lambda+F=\alpha\theta(G\lambda+F)e^{i\theta}$$

where $\phi=\frac{k}{N}2\pi$ for $k=1,2,\ldots,N$ [17]

Now, by substituting

$$\lambda=i\omega \quad \text{for } \omega\in\mathbb{R}^+$$

into equation (),

$$-\omega^2+(G+H)i\omega+F=\alpha\theta(Gi\omega+F)(\cos\phi+i\sin\phi)$$

or

$$-\omega^2+F+(G+H)i\omega=-\alpha\theta G\omega\sin\phi+\alpha\theta F\cos\phi+(\alpha\theta G\omega\cos\phi+\alpha\theta F\sin\phi)i$$

Separating the real and imaginary parts and eliminating ω. The real parts imply

$$-\omega^2+F=-\alpha\theta G\omega\sin\phi+\alpha\theta F\cos\phi \tag{23}$$

while the imaginary parts imply

$$(G+H)\omega=\alpha\theta G\omega\cos\phi+\alpha\theta F\sin\phi \tag{24}$$

[17] because note that $e^{i\theta}=\cos\theta+i\sin\theta$, so that $e^{i2\pi}=\cos 2\pi+i\sin 2\pi=1$

Solving () for ω, we get that

$$\omega = \frac{\alpha\theta F \sin\phi}{G + H - \alpha\theta G \cos\phi}$$

Substituting this into () we get that

$$-\left(\frac{\alpha\theta F \sin\phi}{G + H - \alpha\theta G \cos\phi}\right)^2 + F = -\frac{\alpha\theta F \sin\phi}{G + H - \alpha\theta G \cos\phi}\alpha\theta G \sin\phi + \alpha\theta F \cos\phi \tag{25}$$

Therefore, we have an expression only in terms of $F, G, H, \alpha, \theta, \phi$. Rearranging yields

$$F = (1 - \alpha\theta\cos\phi)\left(\frac{G + H - \alpha\theta G \cos\phi}{\alpha\theta \sin\phi}\right)^2 + G\left(G + H - \alpha\theta G \cos\phi\right)$$

Let $\beta = \theta/2$, and using some trigonometric identities, one may determine that stability changes (Hopf bifurcations) occur for

$$F = \frac{1}{2}\left(\left(1 + \alpha\theta\right)G + H\right)\left(\left(\left(1 + \alpha\theta\right)G + H\right)\tan^2\beta + H\right)$$

where $\beta = \theta/2$. Thus $\beta = \frac{k}{N}\pi$ for $k = 1, 2, \ldots, N$.

The stability condition becomes

$$F < \frac{1}{1 + \alpha\theta}\left(\left(1 + \alpha\theta\right)G + H\right)H$$

Now, let's apply this to the economic model. Here, the expenditure policy function is given by

$$\dot{v}_i(t) = \alpha\left(V\left(h_i(t)\right) - v_i(t)\right)$$

Linearizing about the uniform flow equilibrium, we obtain equation () with

$$\begin{aligned} F &= \partial_h f\left(h^*, 0, v^*\right) = G_h \\ G &= \partial_{\dot{h}} f\left(h^*, 0, v^*\right) = 0 \\ H &= -\partial_v f\left(h^*, 0, v^*\right) = G_z \end{aligned}$$

Plugging these values into (), the stability condition becomes

$$G_h < \frac{1}{1 + \alpha\theta}G_z^2$$

QED.

Proof of Proposition ?? budget constraints

$$\begin{aligned} z_{it} + a_{i,t} &= (1+r)\left(\alpha\theta z_{i+1,t-1} + a_{it-1}\right) \\ z_{it+1} + a_{i,t+1} &= (1+r)\left(\alpha\theta z_{i+1,t} + a_{it}\right) \end{aligned}$$

imply

$$a_{it} = \frac{1}{1+r}\left(z_{it+1} + a_{i,t+1}\right) - (1-\alpha)\,\theta z_{i+1,t}$$

Iterating the budget constraint forward, we get that..

$$\begin{aligned} z_{i0} + \frac{1}{(1+r)} z_{i1} + \frac{1}{(1+r)^2} z_{i2} + \cdots &= (1+r)\left[\alpha\theta z_{i+1,-1} + a_{i,-1}\right] + \alpha\theta z_{i+1,0} + \frac{1}{1+r}\alpha\theta z_{i+1,1} \\ \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j z_{i,j} &= (1+r)\,h_0 + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j (1-\alpha)\,\theta z_{i+1,j} \end{aligned}$$

therefore

$$\begin{aligned} \frac{1}{1-\frac{1}{1+r}}\bar{z}_i &= \frac{1}{1-\frac{1}{1+r}}\alpha\theta\bar{z}_{i+1} + (1+r)\,h_{i,-1} \\ \bar{z}_i &= \alpha\theta\bar{z}_{i+1} + r h_{i,-1} \\ \bar{z}_i &= \alpha\theta\bar{z}_{i+1} + r\alpha\theta z_{i+1,-1} + r a_{it-1} \\ \bar{z}_i &= (1+r)\,\alpha\theta\bar{z}_{i+1} + r a_{it-1} \end{aligned}$$

Follows from the main text. QED.

Proof of Proposition ?? The the budget constraint is given by

$$\sum_{t=0}^{\infty}\sum_{s^t}\phi_t\left(s^t\right) z_{it}\left(s^t\right) = \phi_0\left(s^t\right)(1+r)\,h_{i,-1} + \sum_{t=1}^{\infty}\sum_{s^t}\phi_t\left(s^t\right)\left((1+r)\,\alpha\theta z_{i+1,t-1} + \omega_{it}\right)$$

let γ_{it} denote income every period. Then

$$\begin{aligned} \gamma_{i,0} &= (1+r)\,h_{i,-1} \\ \gamma_{i,t} &= (1+r)\,\alpha\theta z_{i+1,t-1} + \omega_{it} \end{aligned}$$

Thus, we can write this as

$$\sum_{t=0}^{\infty}\sum_{s^t}\phi_t\left(s^t\right) z_{it}\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t}\phi_t\left(s^t\right)\gamma_{it}$$

or, written in terms of q

$$\sum_{t=0}^{\infty}\sum_{s^t}\phi_t^0\left(s^t\right)\frac{1}{1-\theta}q_{it}\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t}\phi_t^0\left(s^t\right)\gamma_{it}$$

The foc from this problem is

$$\beta^t u_q\left(c_i\left(s^t\right)\right)\pi_t\left(s^t\right)-\mu_i\phi_t^0\left(s^t\right)\frac{1}{1-\theta}=0$$

this implies that

$$\frac{u_q\left(c_i\left(s^t\right)\right)}{\mu_i}=\frac{u_q\left(c_j\left(s^t\right)\right)}{\mu_j}$$

for all pairs (i,j). This implies that consumption only depends on the aggregate

$$\sum_i\gamma_{it}=\sum_i\left((1+r)\,\alpha\theta z_{i+1,t-1}+\omega_{it}\right)=(1+r)\,\alpha\theta Z_t$$

Then z_{it} is constant over time and across histories for all i. Thus the equilibrium satisfies $z_{it}=\bar{z}_i$. Then

$$\beta^t u_q\left(\bar{c}_i\right)\pi_t\left(s^t\right)=\mu_i\phi_t^0\left(s^t\right)\frac{1}{1-\theta}$$

this implies

$$\phi_t^0\left(s^t\right)=\frac{\beta^t u_q\left(\bar{c}_i\right)\pi_t\left(s^t\right)}{\mu_i\frac{1}{1-\theta}}$$

Therefore we take the budget constraint

$$\sum_{t=0}^{\infty}\sum_{s^t}\phi_t^0\left(s^t\right)\left(z_{it}\left(s^t\right)-\gamma_{it}\left(s^t\right)\right)=0$$

plug in for $\phi_t^0\left(s^t\right)$,

$$\sum_{t=0}^{\infty}\sum_{s^t}\frac{\beta^t u_q\left(\bar{c}_i\right)\pi_t\left(s^t\right)}{\mu_i\frac{1}{1-\theta}}\left(z_{it}\left(s^t\right)-\gamma_{it}\left(s^t\right)\right)=0$$

thus

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\left(\bar{z}_i-\gamma_{it}\left(s^t\right)\right)=0$$

thus

$$\begin{aligned}
\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\bar{z}_i &= \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\gamma_{it}\left(s^t\right) \\
\sum_{t=0}^{\infty}\beta^t\bar{z}_i &= (1+r)\,h_{i,-1}+\sum_{t=1}^{\infty}\beta^t(1+r)\,\alpha\theta\bar{z}_{i+1}+\sum_{t=1}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)\omega_{it} \\
\sum_{t=0}^{\infty}\beta^t\bar{z}_i &= (1+r)\,h_{i,-1}+\sum_{t=1}^{\infty}\beta^t(1+r)\,\alpha\theta\bar{z}_{i+1} \\
\frac{1}{1-\beta}\bar{z}_i &= (1+r)\,h_{i,-1}+\frac{\beta}{1-\beta}(1+r)\,\alpha\theta\bar{z}_{i+1}
\end{aligned}$$

As before, let's assume that

$$\beta(1+r)=1$$

hence

$$\bar{z}_i=(1-\beta)(1+r)\,h_{i,-1}+\alpha\theta\bar{z}_{i+1}$$

therefore we get the same thing. QED.

Proof of Proposition ?? Then

$$z_{it}+a_{i,t}=(1+r)\left(\alpha\theta z_{i+1,t-1}+a_{it-1}\right)$$

and

$$a_{it}=\frac{1}{1+r}\left(z_{it+1}+a_{i,t+1}\right)-\alpha\theta z_{i+1,t}$$

Iterating the budget constraint forward, we get that..

$$\begin{aligned}
z_{it}+\frac{1}{(1+r)}z_{it+1}+\frac{1}{(1+r)^2}z_{it+2}+\cdots &= (1+r)\,h_{i,t-1}+\mathbb{E}_{it}\alpha\theta z_{i+1,t}+\frac{1}{1+r}\alpha\theta\mathbb{E}_{it}z_{i+1,t+1}+\cdots \\
\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j z_{i,t+j} &= (1+r)\,h_{i,t-1}+\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j\alpha\theta\mathbb{E}_{it}z_{i+1,t+j}
\end{aligned}$$

thus

$$\frac{1}{1-\frac{1}{1+r}}z_{i,t}=\frac{1}{1-\frac{1}{1+r}}\alpha\theta\mathbb{E}_{it}z_{i+1,t}+(1+r)\,h_{i,t-1}$$

or

$$z_{i,t}=\alpha\theta\mathbb{E}_{it}z_{i+1,t}+rh_{i,t-1}$$

But this is equal to

$$z_{i,t} = \alpha\theta \mathbb{E}_{it} z_{i+1,t} + r\left(\alpha\theta z_{i+1,t-1} + a_{it-1} + \frac{1}{1+r}\omega_{it}\right)$$

my best expectation of tomorrow's income is

$$\mathbb{E}_{it} z_{i+1,t} = z_{i+1,t-1}$$

therefore

$$z_{i,t} = (1+r)\,\alpha\theta z_{i+1,t-1} + \frac{r}{1+r}\left((1+r)\,a_{it-1} + \omega_{it}\right)$$

Proof of Transforming state space Follows from the Main Text.

$$\begin{aligned} z_{i,t} &= (1+r)\left(h_{i,t-1} - a_{it-1} - \frac{1}{1+r}\omega_{it}\right) + \frac{r}{1+r}\left((1+r)\,a_{it-1} + \omega_{it}\right) \\ &= (1+r)\,h_{i,t-1} + (r-(1+r))\left(a_{it-1} + \frac{1}{1+r}\omega_{it}\right) \\ &= (1+r)\,h_{i,t-1} - \left(a_{it-1} + \frac{1}{1+r}\omega_{it}\right) \end{aligned}$$

note that

$$z_{it-1} + a_{it-1} = (1+r)\,h_{i,t-2}$$

thus

$$\begin{aligned} z_{i,t} &= (1+r)\,h_{i,t-1} - \left((1+r)\,h_{i,t-2} - z_{it-1} + \frac{1}{1+r}\omega_{it}\right) \\ z_{i,t} &= (1+r)\,(h_{i,t-1} - h_{i,t-2}) + z_{it-1} - \frac{1}{1+r}\omega_{it} \end{aligned}$$

Therefore

$$\begin{aligned} z_{i,t} - (1+r)\,z_{it-1} &= (1+r)\,(h_{i,t-1} - h_{i,t-2}) - r z_{it-1} - \frac{1}{1+r}\omega_{it} \\ &= h_{i,t-1} - (1+r)\,h_{i,t-2} + r h_{i,t-1} - r z_{it-1} - \frac{1}{1+r}\omega_{it} \end{aligned}$$

Proof of Proposition Stability Permanent Income Follows from the main text.

$$\begin{aligned} 0 &= (h_{t-1} - (1+r)\, h_{t-2}) + r h_{t-1} \\ -r h_{t-1} &= (h_{t-1} - (1+r)\, h_{t-2}) \end{aligned}$$

Therefore,

$$\begin{aligned} H_t - (1+r)\, H_{t-1} &= -(1-\alpha\theta)\, Z_t \\ -r H_{t-1} &= -(1-\alpha\theta)\, Z_t \end{aligned}$$

$$-r h_{t-1} = -(1-\alpha\theta)\, z_t$$

therefore

$$z_t = \bar{z} = \frac{r}{1-\alpha\theta} h_{t-1}$$

intractability of general problem not markov.

The household's general problem is thus given by

$$V_{it}\left(\mathbf{z}_{i+1}; \omega_{i,t}\right) = \max_{c_i, a_i'} (1-\alpha\theta) \log z_i + \theta \mathbb{E}_{i,t} V_{i,t+1}\left(z_{i+1}; \omega_i'\right)$$

subject to

$$\begin{aligned} z_i + a_i &= (1+r)\left(\alpha\theta z_{i+1,t-1} + a_{i,t-1} + \omega_i\right) \\ a_i &\geq -\phi \end{aligned}$$

where

$$\begin{aligned} \mathbf{z}_{i+1} &= \mathbf{z}_{i+1} \cup z_{i,t} \\ z_{i+1,t} &= \psi\left(\mathbf{z}_{i+2}; a_{i,t}\right) \end{aligned}$$

This is a very general formulation of the household's problem. From here it is easy to see how this problem becomes intractable. The main problem here is that agents must keep track of entire state.