Optimal Regulation in the Presence of Reputation Concerns*

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Abstract

We study a market with free entry and exit of firms who can produce high-quality output by making a costly but efficient initial unobservable investment. If no learning about this investment occurs, an extreme "lemons problem" develops, no firm invests, and the market shuts down. Learning introduces reputation incentives such that a fraction of entrants do invest. If the market operates with spot prices, simple regulation can enhance the role of reputation to induce investment, thus mitigating the "lemons problem" and improving welfare.

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1 Introduction

In many market settings, the “lemons problem” (Akerlof (1970)) is an impediment to trade. If buyers are unable to verify the quality of the goods or services provided, and sellers of low-quality goods are free to enter the market, then adverse selection can lead to a complete shut down of that market. If buyers have access to public signals of the quality of sellers’ goods or services upon which a seller’s reputation can be based, then sellers’ concern for their reputation is one mechanism through which the lemons problem is mitigated.[1] But do sellers’ concerns for their reputation in markets subject to a lemons problem lead to allocations that are constrained efficient? Can regulation of markets subject to a lemons problem enhance the role of reputation in improving welfare? If so, what form should this regulation take?

These questions take on added urgency in the aftermath of the 2008 financial crisis. In 1963, former Federal Reserve chairman Alan Greenspan wrote, ”Reputation, in an unregulated economy, is...a major competitive tool.... Left to their own devices, it is alleged, businessmen would attempt to sell unsafe food and drugs, fraudulent securities, and shoddy buildings...[but] it is in the self-interest of every businessman to have a reputation for honest dealings and a quality product.”[2] Forty-five years later, in his remarks before the House of Representatives, he declared, ”Those of us who have looked to the self-interest of lending institutions to protect shareholders’ equity, myself included, are in a state of shocked disbelief.”[3] So, does regulation substitute or complement reputation forces?

In this paper we argue that, as a general matter, simple regulatory interventions in markets subject to a lemons problem can in fact enhance market learning to foster reputation incentives and improve welfare. We do so in a general equilibrium model in which the production of one good is subject to an endogenous lemons problem when traded in spot markets. We consider various assumptions about the information available to the regulator and find that a simple form of entry regulation can improve welfare even without access to direct information about the entering firms’

[1]For an excellent survey of the literature on this subject, see part 4 of the book by Mailath and Samuelson (2006).

[2]”The Assault on Integrity,” *The Objectivist Newsletter*, August 1963. See also this statement from Goldman Sachs’ 2009 Annual Report: “Our assets are our people, capital and reputation. If any of these is ever diminished, the last is the most difficult to restore.”

[3]Edmund L. Andrews, ”Greenspan Concedes Error on Regulation,” *New York Times*, October 23, 2008.

quality. The key is that entry regulation affects the firms' incentives to enter the market with high-quality products, which improves the quality mix of entrants and thus mitigates the severity of the lemons problem.

In our model, consumers have utility over two final goods: a homogeneous good that we term the *numeraire good* and a final good that we term the *experience good*. The experience good is produced by aggregating a continuum of intermediate goods of uncertain quality as inputs. The lemons problem occurs in the market for these intermediate goods.[4] Individual producers of these intermediate goods are long-lived and have zero marginal cost of production at each moment in time up to a capacity constraint. Entering producers of the intermediate good decide whether to make a costly investment of the numeraire good and enter with a high-quality product or to not make the investment and enter with low-quality. A high-quality unit of the intermediate good has a positive marginal product in the production of the experience good, whereas a low-quality unit has a negative marginal product. Producers of intermediate goods exit for exogenous reasons at a fixed rate, but they can also choose to discontinue production and exit if it is optimal for them to do so. A steady-state in this economy has ongoing entry and exit of intermediate good producers.

In the first-best allocation, all entrants make the required initial investment to produce with high-quality, and the level of entry is such that the discounted present value of the marginal high-quality intermediate producer (valued at consumers' marginal utility for the experience good) is equal to the required initial investment of the numeraire good. Under full information, this optimal allocation is also the equilibrium outcome in a market in which the spot market prices for all high- and low-quality intermediate producers are equal to the marginal product of their current output valued at consumers' marginal utility for the experience good.

The lemons problem in this spot market arises when it is not possible to observe if individual producers of intermediate goods have made the investment required to provide high-quality. In this case, it cannot be that all intermediate producers are paid a positive price for their output in equilibrium, or else low-quality producers will earn positive profits from entry. In the complete absence of information about in-

[4]Our assumption that consumers consume the experience good as an aggregate of underlying intermediate goods of uncertain quality simplifies the computation of equilibrium and allows us to construct a straightforward measure of social welfare based on the consumer surplus of a representative household.

termediate producers' quality, this lemons problem leads to a complete market break down and no production of the experience good at all in equilibrium.

The lemons problem is mitigated if producers of the experience good have access to a public signal of the quality of each intermediate goods producer, which serves as the basis for that producer's reputation. In a spot market with such signals, at each moment of time, individual intermediate producers are paid the expected marginal product of their intermediate good valued at consumers' marginal utility for the experience good, where that expectation is based on the individual producer's current reputation. Intermediate good producers' reputations evolve over time according to the stochastic structure of the public signals and the endogenous exit decisions of high and low-quality producers. The reputation level of an entering intermediate goods producer depends on the equilibrium ratio of high and low-quality entrants.

In equilibrium, the ratio of high and low-quality entrants is determined from the firms' free entry conditions, which equate the expected discounted payoffs for high and low-quality entrants to the initial investment cost, or zero, respectively. Likewise, firms are free to exit and will do so if their reputation level falls to a sufficiently low level. These entry and exit conditions, along with the resulting firm dynamics, combine to determine the severity of the adverse selection problem in general equilibrium as a function of the public signal processes governing the reputation dynamics.

We characterize the stationary, competitive spot market equilibrium in this economy with imperfectly informative signals. We then ask whether a regulator can, through the use of taxes and transfers, improve on the spot market equilibrium outcome. The extent to which a regulator can improve spot market equilibrium outcomes depends on the information about market transactions that is available to that regulator.

First we show that a combination of entry fees and subsidies to the production of the final experience good can be used to achieve an allocation arbitrarily close in welfare terms to the informationally unconstrained first best. The entry fee shifts the incentives of firms toward entering with a higher quality, but may also reduce the overall level of entry of both types of firms, lowering overall entry below the first-best level. The production subsidy then enhances entry to the efficient level, and can do so for a quality mix of entering firms that is arbitrarily close to the first best. This simple regulatory scheme is therefore sufficient to drive low-quality intermediate goods producers almost completely out of the market while, at the same time, offering high-quality producers sufficient compensation for their initial investment in quality, such that the economy achieves a nearly efficient level of production of the experience good.

We interpret this finding as indicating that the lemons problem in this environment is a problem of commitment, not one of information. The combination of entry fees and production subsidies shifts the rewards toward high-reputation intermediate producers, thus enhancing rewards for favorable signal outcomes over and above those rewards offered by spot market prices. Similar incentives for quality could be achieved privately if buyers and sellers could commit to long-term contracts involving payments that rewarded favorable signal outcomes and punished unfavorable signal outcomes more steeply than the rewards offered in spot markets. Our central insight is that the lemons problem reduces welfare in spot markets because rewards to signals of quality offered in such markets are not sufficient to support a second-best outcome.

We next consider a regulator that only has information about entry decisions, but cannot directly observe market activity. This regulator can no longer resort to interventions that subsidize or tax market transactions, but is just limited to imposing entry fees.[5]

The imposition of a fixed entry fee leads to a potential trade-off between two opposing effects on equilibrium allocations: as discussed above, it improves the equilibrium quality mix of entrants, but it may also reduce the overall level of entry. The combination may either increase or reduce the overall level of production of the experience good and welfare in the steady-state.

We analyze the impact of a fixed entry fee on welfare in the steady-state for three different processes of stochastic signals: one we term *good news*, one we term *bad news*, and a third we term *Brownian motion*. We solve for entry conditions for high and low-quality firms analytically, making welfare comparisons across different regulation policies tractable. We show how the dynamics of reputation accumulation as governed by these three stochastic signal processes determine how a fixed entry fee impacts equilibrium quality and entry. We show that in the bad news case, there is no trade-off — an increase in the fixed entry fee always increases quality, entry, and steady-state welfare. In the Brownian case, at least initially, there is no trade off —

[5]The regulator could also consider quantitative restrictions on entry. However, the set of allocations that can be achieved using entry quotas is just a subset of the set of allocations that can be achieved using entry costs, and cannot involve an increase in production of the experience good.

starting from an entry fee of zero, an increase in the fixed entry fee always increases quality, entry, and steady-state welfare. In this case, however, once the entry fee gets large, further increases in the fee continue to increase quality but reduce entry and steady-state welfare. In the good news case, there is an immediate trade-off — an increase in the fixed entry fee always increases quality but immediately reduces entry. In this case, we show that, at least initially, starting from an entry fee of zero, an increase in the entry fee has a second-order effect on quality and a third-order effect on entry, and hence steady-state welfare improves. Thus, in all three cases a positive entry fee is optimal.

Literature Review

Our paper connects two mechanisms that mitigate firms' incentives to engage in opportunistic behavior: reputation and regulation. There is a rich literature studying each of these mechanisms in isolation, but to our knowledge they have so far not been systematically connected.

The literature on reputation concerns, surveyed recently in MacLeod (2007), interprets reputation as a valuable asset that the firm may lose if it is found out to act opportunistically (Mailath and Samuelson (2001) and Tadelis (1999, 2002). In these models, firms differ in an unobservable exogenous characteristic. They enter the market with an exogenous reputation level that is updated based on signals about their performance. While the first models considered exit to be exogenous, Hörner (2002), Bar-Isaac (2003), and Daley and Green (2010) introduce endogenous exit of firms, when these firms know their own type.

We contribute to this literature along two important dimensions. First, we fully endogenize the severity of the adverse selection problem, and the resulting trade-off between market size and quality, in general equilibrium. We assume that firms are free to enter or exit the market, and that the unobservable characteristic underlying the lemons problem is the result of an unobserved initial investment decision by otherwise identical firms. The firm's incentive and participation conditions then determine the number of high and low-quality entrants, as well as their respective rates of exit, and the resulting firm dynamics determine the level of high- and low-quality firms (and their respective reputations) in general equilibrium.

Second, as a technical contribution, we show that under natural restrictions on buyer beliefs, the equilibrium is unique as the model approaches continuous time. We also fully characterize value functions in the continuous time limit, when firms know their type and have the option to exit.[6] In contrast with Bar-Isaac (2003), who introduces a perturbation on exit strategies to obtain uniqueness, we show uniqueness without this perturbation, as reputation is updated frequently.

Our paper contributes to the literature on regulation by showing how regulatory interventions can be used to leverage reputational incentives. Leland (1979), extended later by Shaked and Sutton (1981) and Shapiro (1983, 1986), introduce moral hazard and investment decisions in markets with asymmetric information. Lizzeri (1999) and Albano and Lizzeri (2001) analyze the efficiency effects of certification intermediaries. In these environments, entry regulation plays the role of enhancing the information about entrants that is available (”certification”). von Weizsacker (1980) discusses how barriers to entry may increase welfare once we consider economies of scale and differentiated products. Garcia-Fontes and Hopenhayn (2000) show that entry restrictions can improve the average quality of firms in the market. In their case, the welfare benefits arise from heterogeneous preferences about the quality of products among buyers. To contrast our paper with this literature, none of these previous papers considered the role of regulation together with reputational incentives in mitigating information frictions. Furthermore, our study completely abstracts from scale effects and product differentiation or market power as a motive for regulatory interventions, and works with a representative consumer model. These previous papers thus point to different, complementary channels through which regulation may enhance efficiency.

Prescott and Townsend (1984) and Arnott, Greenwald, and Stiglitz (1993) discuss whether government interventions can be Pareto improving in a world of adverse selection and moral hazard even if they cannot directly correct these information imperfections. Klein and Leffler (1981, p. 168) find that ”market prices above the competitive price and the presence of nonsalvageable capital are means of enforcing quality promises.” We contribute to this discussion by showing that the market outcome with spot trade between producers and buyers is not constrained Pareto optimal. This result, however, does not arise from information asymmetry per se, but from the fact that buyers and sellers are unable to commit to dynamic contracts that

[6]Prat and Alos-Ferrer (2010) solve similar value functions but without endogenous exit. Papers of reputation in continuous time are Board and Meyer-ter Vehn (2010) and Faingold and Sannikov (2011); however, they do not consider entry and exit decisions.

generate prices that are different from spot prices. Our paper shows that government interventions are Pareto improving if the private sector cannot reproduce the commitment that a government can replicate with very simple taxes and subsidies. This, of course, does not preclude the possibility of other market-based solutions to the commitment problem through, e.g., longer-term contracts, back-loading of payments, posting of bonds, other contractual clauses, or market-provided intermediation and certification services.

In the following section, we describe the economy and characterize the spot market equilibrium for two extreme benchmarks: full information and no learning. In Section 3 we characterize the spot market equilibrium in steady-state with imperfectly informative signals. In Section 4 we study the role of regulation in improving welfare relative to a spot market economy under two settings: one where the regulator can observe entry and transactions, and another where the regulator can only observe entry. In Section 5 we make some final remarks. The Appendix has proofs.

2 The Model

In this section, we describe the economic environment, characterize the socially optimal allocation, and solve for the spot market equilibrium under two informational benchmarks: full information, in which the quality of the producers of the intermediate goods is fully observable, and no information, in which there are no signals of the quality of the intermediate good producers.

2.1 The Economy

Time is discrete with time periods numbered $t = 0, 1, 2, \ldots$. We denote the length of a time period in calendar time by Δ. For some calculations, we will consider the limit as Δ goes to zero.

At each time t, consumers in this economy derive utility from the consumption of two final goods: one that we term the *experience good* and one that we term the *numeraire good*. Let Y_t denote consumption of the experience good and N_t consumption of the numeraire good at t. Consumers' utility is given by

$$\sum_{t=0}^{\infty} \exp(-r\Delta t) \left[U(Y_t) + N_t\right] \Delta, \tag{1}$$

where $U' > 0$, $U'' < 0$, and r is the discount factor.

At each time t, there is an endowment of Δ units of the numeraire good. This good is not storable. The experience good is produced with a constant returns to scale technology that uses produced intermediate goods as the only inputs.

At each point in time t, there is a stock of "trees" in the economy that yield a flow of the intermediate good as "fruit" at zero marginal cost. In each period, each tree yields a flow of one unit of the intermediate good per unit time for as long as the tree remains active. Each period trees can become inactive for exogenous reasons or endogenously by the owner's decision. Trees that become inactive at t cannot return to production at later dates.

Trees can be one of two types, high-quality (H) or low-quality (L), depending on an initial investment made when the tree enters production (is planted). To plant a high-quality tree at t, an investment of C units of the numeraire good is required at that moment. Low-quality trees can be planted at zero cost at any moment. We refer to the planting of new trees as *entry*.

The quality of the tree yielding Δ units of the intermediate good as fruit per period determines the expected productivity of those units of the intermediate good in use as an input to produce the experience good. One unit of the intermediate good from a high-quality tree contributes $y(1) > 0$ units of output of the experience good at the margin, whereas one unit of fruit from a low-quality tree yields $y(0) < 0$ units of output of the experience good at the margin.[7]

Let ϕ denote the public belief regarding the probability that a given tree is high-quality. We refer to ϕ as the tree's *reputation*. The expected output of the experience good obtained from a unit of the intermediate good from a tree with reputation ϕ is

[7] The assumptions of zero marginal cost of production for the intermediate good and a negative marginal product of low-quality intermediate goods are normalizations that simplify the exposition. Assuming positive production costs and positive marginal product of low-quality intermediate goods delivers the same results but less straightforwardly.

denoted $y(\phi)$ and is given by the affine function

$$y(\phi) = \phi y(1) + (1 - \phi)y(0). \tag{2}$$

The resource constraint for the experience good is then given by

$$Y_t = y(1)m_{Ht} + y(0)m_{Lt}, \tag{3}$$

where m_{Ht} is the measure of active high-quality trees at t and m_{Lt} is the corresponding measure of active low-quality trees.

We denote the measure of new trees entering at t by $m^e_t\Delta \geq 0$. The fraction of those entrants who invest to become high-quality is denoted $\phi^e_t \in [0, 1]$. The corresponding resource constraint for the numeraire good is

$$N_t = 1 - C\phi^e_t m^e_t, \tag{4}$$

where $C\phi^e_t m^e_t$ are the resources invested in planting high-quality trees at t.

A tree that enters at t starts production with reputation ϕ^e_t. Each period that this tree is active, it generates *good* or *bad* signals. We let $\alpha_i(\Delta)$ denote the probability that an active tree of type $i = \{H, L\}$ generates a good signal.

We refer to the removal of active trees from production as *exit*. We denote the probability of continuation of a tree of quality $i = \{L, H\}$ and reputation ϕ_t at t by $\omega_{it}(\phi) \in [0, \exp(-\delta\Delta)]$, where $\delta > 0$ is the exogenous exit rate.

The timing of events within a period is as follows. At the beginning of period t, trees that are incumbent from the previous period start with initial reputation designated by ϕ_t. These trees choose rates at which to continue $\omega_{it}(\phi)$ and new trees enter. Buyers form *interim beliefs* $\phi^c_t(\phi)$ ($\phi^c : [0, 1] \rightarrow [0, 1]$) about the quality of those incumbent trees that started the period with reputation ϕ and that continue. Likewise, buyers form beliefs ϕ^e_t about the quality of entering trees. Trade occurs at these interim beliefs. Signals are then realized, leading to updating of reputations to ϕ_{t+1} to start period $t + 1$ for all trees that were active in period t.

The evolution of reputation for a tree is governed by Bayes' rule where applicable. A buyer who contemplates purchasing the fruit from a tree that began period t with reputation ϕ and that continues to operate in period t has interim beliefs about the quality of the tree consistent with trees' strategies for continuation

$$\phi_t^c(\phi) = \frac{\phi\omega_{Ht}(\phi)}{\phi\omega_{Ht}(\phi) + (1-\phi)\omega_{Lt}(\phi)}, \tag{5}$$

where this expression is well defined.

Likewise, the updating of reputations given signals after trade takes place is also governed by Bayes' rule. After a tree that started in period t with reputation ϕ_t operates and generates a signal, it starts period $t+1$ with reputation ϕ_{t+1} given by either $\phi^g(\phi^c(\phi_t))$ or $\phi^b(\phi^c(\phi_t))$ depending on whether a good or a bad signal is realized. The functions ϕ^g and ϕ^b are defined by

$$\phi^g(\phi^c) = \frac{\phi^c\alpha_H(\Delta)}{\phi^c\alpha_H(\Delta) + (1-\phi^c)\alpha_L(\Delta)} \tag{6}$$

and

$$\phi^b(\phi^c) = \frac{\phi^c(1-\alpha_H(\Delta))}{\phi^c(1-\alpha_H(\Delta)) + (1-\phi^c)(1-\alpha_L(\Delta))}. \tag{7}$$

At the beginning of each period t, there is a measure of reputations across high-quality trees $\nu_{Ht}(\phi)$ and across low-quality trees $\nu_{Lt}(\phi)$. Production is determined by the extent of trade. Hence, for an allocation to be feasible, we must have

$$m_{Ht} = \exp(-\delta\Delta)\phi_t^e m_t^e \Delta + \int_\phi \omega_{Ht}(\phi)d\nu_{Ht}(\phi), \tag{8}$$

$$m_{Lt} = \exp(-\delta\Delta)(1-\phi_t^e) m_t^e \Delta + \int_\phi \omega_{Lt}(\phi)d\nu_{Lt}(\phi). \tag{9}$$

The evolution of the measures of reputations across high- and low-quality trees $\nu_{it}(\phi)$ from one period to the next is determined in the standard way. First, firms continuation strategies $\omega_{it}(\phi)$ reduce the measure of high- and low-quality trees by reputation through exit. Second, the reputations of those trees that continue are updated according to buyers' interim beliefs $\phi_t^c(\phi)$, and a measure $\exp(-\delta\Delta)m_t^e\Delta$ of trees enters with fraction ϕ_t^e of those trees being high-quality and $(1-\phi_t^e)$ low-quality. Finally, trade occurs, good or bad public signals are observed, and trees' reputations are updated again according to Bayes' rule in (6) and (7).

An *allocation* in this environment is a sequence of consumption of the experience and numeraire goods for the representative household $\{Y_t, N_t\}$, rates of entry of trees and initial reputations for entrants $\{m_t^e, \phi_t^e\}$, buyers' interim beliefs $\{\phi_t^c(\phi)\}$, and for $i = \{L, H\}$, continuation strategies $\{\omega_{it}(\phi)\}$, reputational distributions $\{\nu_{it}(\phi)\}$, and corresponding measures of active high- and low-quality trees $\{m_{it}\}$.

An allocation is *feasible* if it satisfies the final good resource constraints (3) and (4), the constraints on the evolution of reputation measures implied by continuation strategies, buyers' interim beliefs, the rates of entry and initial reputations for entrants, and the signals stated above, and the stocks of high- and low-quality trees satisfy (8)-(9).

2.2 Signal Structures and Reputation

In what follows, we consider five signal structures on which reputation can be based, which we term *full information*, *no information*, *good news*, *bad news*, and *Brownian motion*. We define these signal structures here.

Under full information, there is an immediate, perfect signal of trees' quality so that the reputation of a high-quality tree jumps to $\phi = 1$ upon entry, whereas that for a low-quality tree jumps to $\phi = 0$ upon entry.

Under no information, there are no signals so that the reputation of a tree entered into production with reputation ϕ_t^e evolves over time only according to buyers' interim beliefs.

In the good news case, if the tree is of high-quality, a signal that reveals that quality arrives at rate $\lambda > 0$ per unit of time. No such signal can arrive if the tree is low-quality. This corresponds to $\alpha_L(\Delta) = 0$ and $\alpha_H(\Delta) = \lambda\Delta$. Note that in this case, $\phi^g(0)$ is not defined by Bayes' rule. We impose that $\phi^g(0) = \lim_{\phi\to 0} \phi^g(\phi) = 1$.

In the bad news case, the assumption is reversed: if the tree is of low-quality, a signal that reveals that quality arrives at rate $\lambda > 0$ per unit of time. No such signal can arrive if the tree is high-quality. This corresponds to $(1-\alpha_H(\Delta)) = 0$ and $(1-\alpha_L(\Delta)) = \lambda\Delta$. Note that in this case, $\phi^b(1)$ is not defined by Bayes' rule. We impose that $\phi^b(1) = \lim_{\phi\to 1} \phi^b(\phi) = 0$.

Finally, to approximate a Brownian motion in discrete time, we choose $\alpha_H(\Delta)$ and

$\alpha_L(\Delta)$ so that for all ϕ

$$\log\left(\frac{\phi^g(\phi)}{1-\phi^g(\phi)}\right) - \log\left(\frac{\phi}{1-\phi}\right) = \zeta\sqrt{\Delta}$$

and

$$\log\left(\frac{\phi^b(\phi)}{1-\phi^b(\phi)}\right) - \log\left(\frac{\phi}{1-\phi}\right) = -\zeta\sqrt{\Delta}.$$

This is achieved if

$$\alpha_H(\Delta) = \left(\frac{\exp(\zeta\sqrt{\Delta})}{1+\exp(\zeta\sqrt{\Delta})}\right) > \frac{1}{2} \tag{10}$$

and $\alpha_L(\Delta) = 1 - \alpha_H(\Delta)$, where ζ is the signal-to-noise ratio.

The signals in the good news, bad news, and Brownian motion cases are public signals of the quality of each tree that are observed by all potential buyers. These signals might be interpreted as ratings in some widely published guide derived from either specialized testing or noisy surveys of past customers' experiences with the intermediate good obtained from each tree. Under this interpretation, individual past buyers of the intermediate good from a particular tree have more precise information about that tree's quality from their past consumption experience, but this experience is not fully revealed to other buyers by a survey. We assume that this private information does not affect demand for a given tree's fruit because buyers do not buy repeatedly from the same tree.

Alternatively, one might interpret the signals as reflecting a noisy outcome of production of the experience good with the intermediate output supplied by a particular tree. For example, if the good signal is a positive contribution of 1 to the production of the experience good and the bad signal is a negative contribution of -1, the expected contribution from a high-quality tree is $y(1) = 2\alpha_H(\Delta) - 1$ and the expected contribution from a low-quality tree is $y(0) = 2\alpha_L(\Delta) - 1$. Under this interpretation, the outcome of each individual buyer's experience is public information.

2.3 A Spot Market Equilibrium

We now consider the equilibrium allocation in a market in which the owners of trees sell the intermediate goods obtained as fruit from their trees to producers of the experience good. We assume producers of the experience good face competition both in buying their inputs and in selling their output. Hence, at each time t, experience good producers buy fruit at a spot market price $p_t(\phi)$ that depends on the reputation of the tree. This spot market price is equal to the expected value of the marginal product of the intermediate good when used in production of the experience good, with expectations based on the reputation of the tree. This expected value of the marginal product has two components: the relative price of the experience good with respect to the numeraire good and the expected marginal product of the intermediate good from a tree with a given reputation, $y(\phi)$ from equation (2).

We assume the experience and numeraire final goods are also transacted at spot prices in each period t. We denote this relative price by P_t. In equilibrium, given our assumption about the competition that experience good producers face, this price of the experience good relative to the numeraire good is given by the marginal utility of the experience good:

$$P_t = U'(Y_t). \tag{11}$$

Thus, the *spot market price* at t in units of the numeraire good, for a unit of the intermediate good from a tree that is believed to be of high-quality with probability ϕ, is given by

$$p_t(\phi) = y(\phi)P_t. \tag{12}$$

In what follows, we focus on *steady-state spot market equilibrium*, a spot market equilibrium in which all prices and quantities are constant over time. To keep the notation simple, we suppress the time subscript. If $p(\phi)$ is the steady-state spot market price for intermediate goods based on reputation and Y is the steady-state production of the experience good, we find it useful to directly use prices normalized by the marginal utility of the experience good. From the previous two equations, these normalized prices are just the expected marginal products of the intermediate good, $y(\phi)$.

Given that trees produce Δ units of the intermediate good at zero marginal cost, $y(\phi)\Delta$ corresponds to the flow of normalized profits from an active tree with reputation ϕ at t.

In the next lemma, we show that in a steady-state spot market equilibrium, given buyers' interim beliefs $\phi^c(\phi)$, the discounted expected value of the profits earned by a tree of quality $i \in \{H, L\}$ and reputation ϕ can be characterized by a Bellman equation in which the normalized profits $y(\phi)\Delta$ are the current reward. We denote the fixed point of this Bellman equation by $V_i(\phi)$ and refer to it as the *normalized value function* of a tree of quality i and reputation ϕ. Because trees find it optimal to continue when they expect positive profits and exit if they expect negative profits from continuation, the actual value functions, denoted $W^i(\phi)$, are simply given by the normalized value functions scaled by the price of the marginal utility of the experience good $U'(Y)$ in steady state, i.e., $W_i(\phi) = V_i(\phi)U'(Y)$.

Lemma 1 *Normalized value functions of intermediate good producers.*

Take buyers' interim beliefs based on continuation $\phi^c(\phi)$ as given. The value of a tree with quality $i \in \{L, H\}$ and reputation ϕ is given as the unique solution $V_i(\phi)$ to the Bellman equation

$$V_i(\phi) = \max_{\omega \in [0, \exp(-\delta\Delta)]} \omega V_i^c(\phi^c(\phi)), \tag{13}$$

where

$$V_i^c(\phi^c) = y(\phi^c)\Delta + \exp(-r\Delta)\left(\alpha_i(\Delta)V_i(\phi^g(\phi^c)) + (1-\alpha_i(\Delta))V_i(\phi^b(\phi^c)\right) \tag{14}$$

with $\phi^g(\phi^c)$ and $\phi^b(\phi^c)$ defined by (6) and (7).

The proof of this lemma is given in the Appendix.

This Bellman equation (13) also defines the set of optimal continuation strategies for a tree of quality i given buyers' interim beliefs. Specifically, a continuation strategy $\omega_i(\phi)$ is a *best response* to buyers' interim beliefs $\phi^c(\phi)$ only if $\omega_i(\phi) = \exp(-\delta\Delta)$ when $V_i(\phi) > 0$ and $V_i(\phi) \geq 0$ for all ϕ. Note that this second requirement implies that $\omega_i(\phi) = 0$ whenever $V_i^c(\phi^c(\phi)) < 0$.

In a steady-state spot market equilibrium, we also require that there be non-positive profits associated with entry for both high- and low-quality trees; that is,

$$V_H(\phi^e)U'(Y) - C \leq 0, \tag{15}$$

with equality if $\phi^e m^e \Delta > 0$, and

$$V_L(\phi^e)U'(Y) \leq 0, \tag{16}$$

with equality if $(1-\phi^e)m^e\Delta > 0$.

We summarize this discussion with the following definition of a *steady-state spot market equilibrium.*

Definition 1 *Steady-state spot market equilibrium*

A steady-state spot market equilibrium consists of a feasible allocation in which all variables are constant over time $\{Y, N, m^e, \phi^e, \phi^c(\phi), \omega_i(\phi), \nu_i(\phi), m_i\}$ *and normalized value functions* $\{V_i(\phi)\}$ *defined as in (13) such that*

(i) *The continuation strategies* $\omega_i(\phi)$ *are a best response to buyers' interim beliefs* ϕ^c.

(ii) *Buyers' interim beliefs* $\phi^c(\phi)$ *are consistent with the continuation strategies* $\omega_i(\phi)$ *as in (5) where Bayes' rule is defined, and*

(iii) *The zero profits on entry conditions (15) and (16) are satisfied.*

In the next two subsections, we solve for the steady-state spot market equilibrium under two extreme informational benchmarks: full information and no information. We show that under full information, the socially optimal allocation can be implemented as a spot market equilibrium, whereas under no information, there is no production of the experience good in a steady-state spot market equilibrium.

2.4 Full Information Benchmark

It is straightforward to characterize the socially optimal allocation in the full information case. We have that the measure of reputation across trees has mass on $\phi = 0$ and on $\phi = 1$, with no trees with intermediate reputations. The evolution of the stocks of trees (8) and (9) in steady-state is given by

$$m_H = \exp(-\delta\Delta)\phi^e m^e \Delta + \omega_H(1) m_H \tag{17}$$

and

$$m_L = \exp(-\delta\Delta)(1 - \phi^e) m^e \Delta + \omega_L(0) m_L. \tag{18}$$

Clearly, since the output of a tree known to be low-quality is expected to subtract from production of the experience good ($y(0) < 0$), it is optimal to set $\omega_L(0) = 0$ and $\phi^e = 1$. Likewise, since an existing tree known to be of high-quality can contribute

$y(1)$ to production of the experience good at zero cost as long as it continues in production, it is optimal to set $\omega_H(1) = \exp(-\delta\Delta)$, its maximum value. These results then characterize the optimal continuation decisions.

Now consider the optimal level of entry of high-quality trees. The marginal social cost, in terms of utility, of creating a new tree at t with probability $\phi^e = 1$ of being high-quality is given by C, whereas the marginal benefit is given by

$$\exp(-\delta\Delta)\sum_{\tau=t}^{\infty}\exp(-(r+\delta)\Delta\tau)y(1)U'(\bar{Y})\Delta,$$

where $\bar{Y}$ denotes the full information production of the experience good in steady-state. Then

$$\frac{\Delta\exp(-\delta\Delta)}{1-\exp(-(r+\delta)\Delta)}y(1)U'(\bar{Y}) = C. \tag{19}$$

As $\Delta \to 0$, this equation converges to

$$y(1)U'(\bar{Y}) = C(r+\delta).$$

There is an optimal stock of high-quality trees in steady state determined by equation (3), $\bar{m}_H = \bar{Y}/y(1)$.

The optimal dynamic choice of entry $m_t^e\Delta$ is the following. If $y(1)m_{H0}$ is less than this optimal level $\bar{Y}$, the regulator creates an atom of new high-quality trees at $t = 0$ to attain the optimal stock $\bar{m}_H$ of high-quality trees immediately (since utility is quasi-linear). If $y(1)m_{H0}$ exceeds this optimal level, the regulator creates no new trees until the stock of existing high-quality trees has depreciated down to this level at rate $\exp(-\delta\Delta)$. Once this optimal stock of high-quality trees is attained, the regulator chooses a flow of entry of new trees $m^e = \frac{1-\exp(-\delta\Delta)}{\Delta}\bar{m}_H$ to maintain the stock at a constant level. If $\Delta \to 0$ the flow of new trees necessary to maintain the stock at a constant level is then $m^e = \delta\bar{m}_H$.

The value function associated with a high-quality tree in the socially optimal allocation in steady-state is

$$V_H(1) = \exp(-\delta\Delta)\sum_{\tau=t}^{\infty}\exp(-(r+\delta)\Delta\tau)U'(\bar{Y})y(1)\Delta > 0,$$

whereas that associated with operating a low-quality tree is

$$V_L(0) = 0.$$

This last result follows from the assumption that $y(0) < 0$, so it is always optimal to remove a low-quality tree from production as rapidly as possible. Note also that in the transition to steady state, $V_H(1) = C$ whenever there is positive entry and $V_H(1) < C$ when there is no entry.

Clearly, the spot market prices ($p(1) = y(1)U'(\bar{Y})$ and $p(0) = y(0)U'(\bar{Y})$) implement the value functions above and hence the optimal allocation, characterized by entry of high-quality trees, $\phi^e = 1$, and high production of the experience good, $\bar{Y}$.

2.5 No Information Benchmark

In contrast to the full information case, in the extreme case of non-observable investment and no signals from which to learn, the adverse selection problem associated with free entry of low-quality trees is so severe that there is no production of the experience good in steady state.

This result follows from the observation that it is impossible to offer high-quality producers of the intermediate good a positive price for their good without attracting unbounded entry of low-quality trees. With no dependence of the public signal on the quality of the tree, reputation for high- and low-quality trees will not change over time if both types of trees have the same continuation rates $\omega_i(\phi) > 0$. Likewise, both types of trees will have the same exit rates if reputation does not evolve because, if reputation does not evolve, then they both expect the same profits, that is, $V_H(\phi) = V_L(\phi)$. Of course, this equality of value functions means that it is impossible to satisfy the entry condition for high-quality trees (15) as an equality (with positive entry of high-quality trees) without violating the entry condition (16) for low-quality trees. As a result, there can be no positive production of the experience good once the initial stock of high-quality trees dies out. Thus, the steady-state equilibrium allocation with no information has no entry of trees and $Y = 0$.

3 Reputation with Imperfectly Informative Signals

In this section, we solve for a steady-state spot market equilibrium when public signals about each intermediate goods producing tree are revealed over time following the three signal structures defined above (good news, bad news, and Brownian motion). In each case, the steady-state spot market equilibrium can be solved for in a two-step recursive manner. In the first step, we solve for the buyers' interim beliefs $\phi^c(\phi)$, trees' normalized value functions $V_i(\phi),$ and continuation strategies $\omega_i(\phi)$ consistent with conditions (i) and (ii) in the definition of a steady-state spot market equilibrium. In the second step, we solve for the reputation of entering trees ϕ^e and the level of output of the experience good Y consistent with the entry conditions for high- and low-quality trees (15) and (16) expressed as equalities. The resource constraints that define a feasible allocation then give the remainder of the equilibrium quantities.

We impose two "reasonable" restrictions on buyers' interim beliefs to rule out multiplicity of equilibrium generated by off-equilibrium beliefs.

Assumption 1 *Monotonic Updating*

We say that buyers' beliefs show monotonic updating if $\phi^c(\phi)$ is non-decreasing in ϕ for all $\phi \in [0, 1]$.

Using the Bellman equations developed in Lemma 1, this restriction implies that value functions $V_i(\phi)$ are weakly increasing and $V_H(\phi) \geq V_L(\phi)$.

Assumption 2 *High quality trees are more likely to continue than low-quality trees.*

We say that buyers' beliefs regarding trees' continuation strategies are consistent with the hypothesis that high-quality trees are more likely to continue than low-quality trees if $\phi^c(\phi) \geq \phi$ for all ϕ.

Note that this assumption is stronger that a restriction on that $\omega_H(\phi) \geq \omega_L(\phi)$ for all ϕ. It further imposes restriction on interim beliefs when both continuation strategies are equal to zero.

Now we show that the steady-state spot market equilibrium that we focus on here is in fact the unique steady-state spot market equilibrium allocation as the model converges to continuous time (this is, as $\Delta \to 0$) given the three information structures and these two restrictions on buyers' interim beliefs. That result, and the structure of the *steady-state spot market equilibrium with imperfectly informative signals*, is described in the following proposition and proved in the Appendix.

Proposition 1 *Steady-state spot market equilibrium under assumptions 1 and 2 on buyers' beliefs.*

The steady-state spot market equilibrium implemented by normalized spot prices $y(\phi)$ is uniquely characterized, as $\Delta \to 0$, by the following four results:

(i) There is entry of some low-quality trees, ($\phi^e < 1$),

(ii) Reputations of all active trees in steady-state remain in an interval $[\bar{\phi}, 1]$ with $\bar{\phi} > 0$. High-quality trees always strive to remain active, i.e., $\omega_H(\phi) = \exp(-\delta\Delta)$ for $\phi > 0$. Low-quality trees also strive to remain active if $\phi > \bar{\phi}$, and otherwise they randomize continuation with a probability $\omega_L(\phi) \in (0, \exp(-\delta\Delta)]$ such that buyers' interim beliefs $\phi^c(\phi) = \bar{\phi}$.

(iii) The steady-state equilibrium entry reputation equals the exit threshold: $\phi^e = \bar{\phi}$.

(iv) The steady-state equilibrium level of production of the experience good Y satisfies

$$V_H(\phi^e)U'(Y) = C. \tag{20}$$

The structure of this equilibrium allows for a two-step equilibrium construction procedure:

Equilibrium construction step 1: The first step of our equilibrium construction is illustrated in Figures 1 and 2. The equilibrium buyers' interim beliefs $\phi^c(\phi)$, trees' continuation strategies $\omega_i(\phi)$, and trees' normalized value functions $V_i(\phi)$ are all indexed by a reputation level $\bar{\phi} \in (0, 1)$ that we refer to as the *exit threshold* such that

(a) buyers' interim beliefs are given by $\phi^c(\phi) = \phi$ for $\phi \geq \bar{\phi}$, $\phi^c(\phi) = \bar{\phi}$ for $\phi \in (0, \phi)$, and $\phi^c(0) = 0$,

(b) trees' continuation strategies are given by $\omega_H(\phi) = \exp(-\delta\Delta)$ for all $\phi > 0$ and $\omega_H(0) = 0$ for high-quality trees, whereas, for low-quality trees $\omega_L(\phi) = \exp(-\delta\Delta)$ for all $\phi \geq \bar{\phi}$, and $\omega_L(\phi)$ solves

$$\bar{\phi} = \frac{\phi \exp(-\delta\Delta)}{\phi \exp(-\delta\Delta) + (1-\phi)\omega_L(\phi)}$$

for all $\phi \in (0, \bar{\phi})$, and $\omega_L(0) = 0$, and, finally,

(c) the normalized value function for high-quality trees $V_H(\phi)$ is strictly positive for $\phi > 0$ and equal to zero for $\phi = 0$, whereas, for low-quality trees $V_L(\phi) = 0$ for $\phi \leq \bar{\phi}$ and $V_L(\phi) > 0$ otherwise.

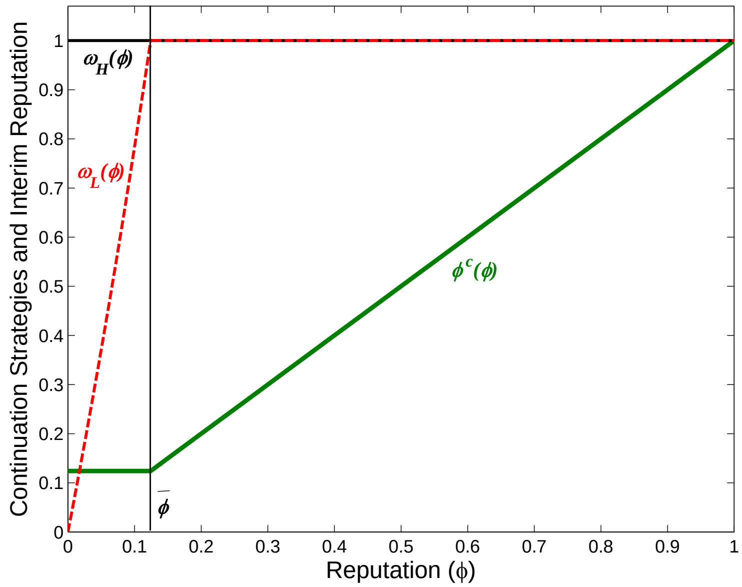
In each of our three cases of imperfectly informative signals, this first step reduces to a fixed point problem of finding an exit threshold $\bar{\phi}$ such that the associated buyers' interim beliefs defined in (a) imply normalized value functions such that $\bar{\phi}$ is the greatest lower bound on the set of ϕ such that $V_L(\phi) > 0$ as in (c), thus ensuring that the continuation strategies specified in (b) are a best response to the buyers' interim beliefs.[8] In the Appendix we provide closed-form solutions for the value functions and exit thresholds for all three of our signal structures in the continuous time limit as $\Delta \to 0$. Figures 1 and 2 are constructed based on Brownian motion and $\Delta \to 0$. Figures 3 and 4 show the value functions for the bad and good news cases, respectively, again as $\Delta \to 0$.

Equilibrium construction step 2: Given a solution to step 1, the reputation at entry is chosen as $\phi^e = \bar{\phi}$. This clearly implies that the zero profit at entry condition for low-quality trees (16) is satisfied. The aggregate production of the experience good Y is chosen such that the normalized value function for high-quality firms, when scaled by the marginal utility of the experience good $U'(Y)$, satisfies the condition that high-quality trees earn zero profits at entry, i.e., (15) is an equality. The remaining elements of the equilibrium allocation can then be solved from the conditions defining feasibility.

Of course, the normalized value of high-quality trees at $\phi^e = \bar{\phi}$, given by $V_H(\phi^e) > 0$, is determined by the specifics of the signal structure. Hence, the equilibrium level of production of the experience good Y, and the corresponding marginal utility of

[8] At $\phi = 0$, we have assumed that buyers' interim beliefs are absorbing at $\phi^c(0) = 0$ and that both high- and low-quality trees exit. Bayes' rule is not defined at this point.

Figure 1: Continuation Decisions and Interim Reputation



that experience good $U'(Y)$ needed to satisfy the zero profit condition on entry for high-quality trees (15), are pinned down by the signal structure as well.

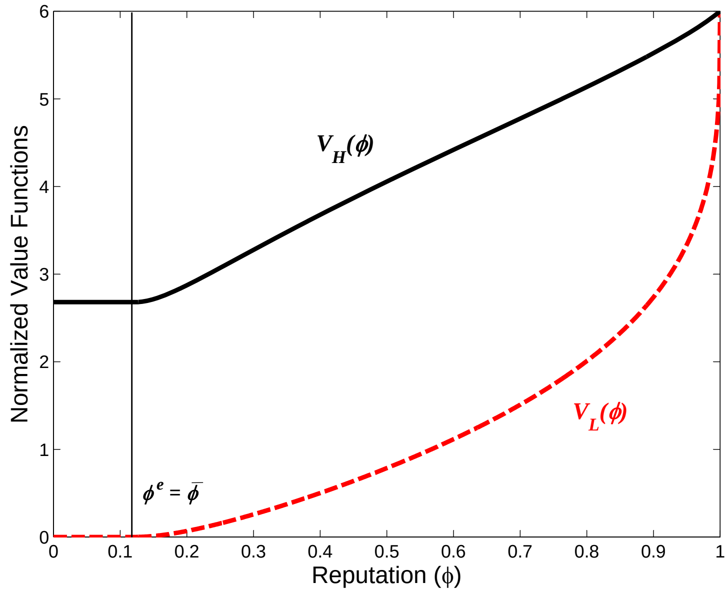
This procedure for computing a steady-state spot market equilibrium with imperfectly informative signals gives us the result that the lemons problem that arises with imperfectly informative signals leads to a reduction in the output of the experience good relative to the full information steady state. We state this result in the next proposition.

Proposition 2 *Comparison of equilibrium outcome with full information benchmark.*

The steady-state level of the experience good output when signals about trees' quality are not perfectly informative is lower than that in the full information benchmark. That is, $Y < \bar{Y}$.

Proof Using (19) from the full information benchmark, the first-best level of experience good production is given by $V_H(1)U'(\bar{Y}) = C$. With imperfectly informative signals, we have that the fraction of high-quality trees that enter is equal to the lowest level of reputation sustained by the market; that is, $\phi^e = \bar{\phi} < 1$. From (15), the output of the experience good is given by $V_H(\bar{\phi})U'(Y) = C$. Since $V_H(\bar{\phi}) < V_H(1)$, then $Y < \bar{Y}$. Q.E.D.

Figure 2: Value Functions: Brownian motion



As we see from this proof, although reputation mitigates the lemons problem and allows for some positive production of the experience good (relative to the no-information benchmark), the need for high-quality trees to endure lower profits after entry as they accumulate a good reputation constrains efficient production. Hence, in a steady-state spot market equilibrium with imperfectly informative signals, the time that high-quality trees spend accumulating a reputation to distinguish themselves from low-quality trees leads to a social cost relative to the full information benchmark. The next section focuses on the potential of regulation to mitigate this social cost.

In the next proposition, we show that regardless of the information structure, the steady-state spot market equilibrium converges to the benchmark without information as the precision of signals goes to zero and converges to the benchmark with perfect information as the precision of signals goes to infinity. Hence, as the effectiveness of learning improves, the equilibrium ranges from complete market shutdown to the unconstrained first best.

Proposition 3 *The benchmarks are the equilibria at the informational limits as* $\Delta \to 0$.

In the three information structures considered (bad news, good news, and Brownian motion), the spot market equilibrium converges to $Y = 0$ *as the precision of signals goes to zero and to the unconstrained first best* $Y = \bar{Y}$, *as the precision of signals goes to infinity.*

Figure 3: Value Functions: Bad News

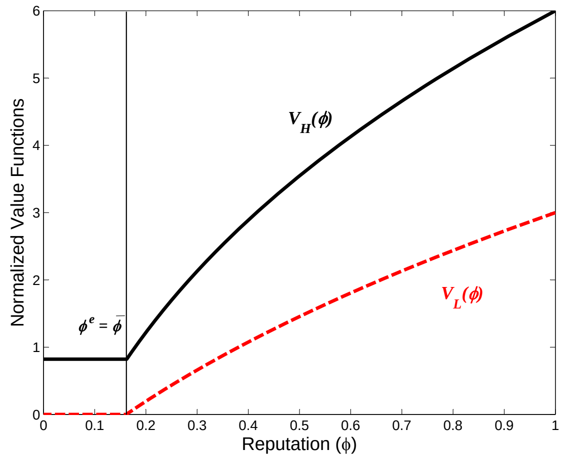
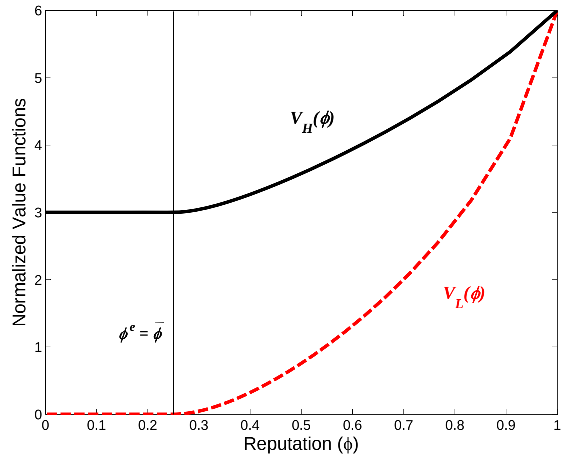


Figure 4: Value Functions: Good News



We prove this proposition in the Appendix by direct calculation using the analytical solutions for the value functions in the continuous time limit.

4 Regulation with Imperfectly Informative Signals

We now study the role of regulation in improving welfare relative to a market economy with reputation based on imperfectly informative signals. We assume the regulator has access to two simple instruments: subsidies s per unit of the experience good purchased (such that the effective price obtained by experience good producers is $P = sU'(Y)$ per unit) and taxes F on entrants into intermediate goods production. We assume the net proceedings of these taxes and subsidies are rebated lump sum to consumers or obtained lump sum from consumers.

We begin this section with an extension of our procedure for constructing the steady-state spot market equilibrium with imperfectly informative signals to the case of equilibrium with the subsidies s and taxes F. We then consider the extent to which a regulator can improve on welfare in the spot market equilibrium with these two policy instruments and how these possibilities for welfare improvement depend on the information available to the regulator.

Consider the construction of a steady-state equilibrium with regulation. Note that since a subsidy s to the sale of the experience good simply alters the relative price of the experience good to $P = sU'(Y)$ and that a tax F on entry of intermediate good producers simply alters the conditions (15) and (16) that high- and low-quality trees earn zero profits on entry, then the first step in our construction of equilibrium summarized as finding an exit threshold $\bar{\phi}$ and corresponding interim beliefs for buyers $\phi^c(\phi)$, continuation strategies for trees $\omega_i(\phi)$, and normalized value functions $V_i(\phi)$ that satisfy conditions (a), (b), and (c) is unchanged. Hence, the regulatory instruments considered here have no impact on buyers' beliefs and trees' continuation strategies.

The second step of equilibrium construction, however, is altered. Specifically, the zero profit on entry conditions for high- and low-quality trees (15) and (16) now become

$$V_H(\phi^e)sU'(Y) = C + F \tag{21}$$

and

$$V_L(\phi^e)sU'(Y) = F. \tag{22}$$

Thus, the second step of our equilibrium construction now becomes one of solving the two zero profit on entry conditions (21) and (22) for the endogenous reputation of entrants ϕ^e and output of the experience good Y.

In what follows, it is useful to write these two zero profit conditions for entry (21) and (22) equivalently as follows. First taking the ratio of (22) to (21), we have

$$\frac{V_L(\phi^e)}{V_H(\phi^e)} = \frac{F}{C + F}. \tag{23}$$

Second, taking the difference between (21) and (22), we have

$$\left(V_H(\phi^e) - V_L(\phi^e)\right) sU'(Y) = C. \tag{24}$$

As these equations make clear, the reputation of entrants $\phi^e \geq \bar{\phi}$ depends only on F. Subsidies to the sale of the experience good do not differentially affect the entry decisions of high- and low-quality trees. In contrast, total production of the experience good Y depends on ϕ^e (hence on F) and on s.

Note also that the relationship between entry taxes F and the reputation of entrants ϕ^e depends on how the ratio of the value functions V_L and V_H varies with ϕ. In particular, we clearly have that the right-hand side of (23) is strictly increasing from 0

to 1 as F increases from 0 to ∞. We also have, by construction, that the ratio of the value functions on the left-hand side of (23) is equal to zero at $\phi^e = \bar{\phi}$ and equal to 1 at $\phi^e = 1$. In the Appendix we prove that for the three signal structures we consider, in the continuous time limit, the ratio of the value functions $V_L(\phi^e)/V_H(\phi^e)$ is strictly increasing in ϕ^e from 0 at $\phi^e = \bar{\phi}$ to 1 at $\phi^e = 1$, so that we have that the equation (23) implicitly defines the reputation of entrants as a function of the tax on entry denoted $\phi^e(F)$. Moreover, this result implies that it is feasible for a regulator to implement any initial reputation of entrants $\phi^e \in [\bar{\phi}, 1)$ desired by an appropriate choice of entry costs F. Thus, in terms of solving for the equilibrium outcome as a function of the regulatory instruments F and s, we have that (23) gives the reputation of entrants ϕ^e as a function of the entry cost F and then (24) gives the equilibrium level of production of the experience good Y as a function of $\phi^e(F)$ and s.

4.1 Regulation When Transactions Are Observable

We assume that a regulator who observes transactions in the market can choose both the subsidy to the sale of the experience good s and the tax on entrants F. Our main result here is that a regulator with access to these two policy instruments can implement a steady-state equilibrium outcome with welfare for the representative household that is arbitrarily close to welfare in the full information first-best outcome. Specifically, let $\bar{Y}$ and $\bar{N}$ denote the full information optimal steady-state levels of consumption of the experience and numeraire good. We then have the following proposition.

Proposition 4 *Optimal regulation with policies based on transactions.*

When a regulator observes transactions, it is possible to find a combination of F and s that implements a steady-state allocation with $Y = \bar{Y}$ and $N = \bar{N} - \epsilon$ for any $\epsilon > 0$.

The regulator can set F large enough to make ϕ^e arbitrarily close to 1. Then, it is possible to find a subsidy to target the optimal $\bar{Y}$

$$s = \frac{C}{U'(\bar{Y})\left[V_H(\phi^e) - V_L(\phi^e)\right]}.$$

The corresponding value of $\phi^e m^e \Delta$ needed to produce $\bar{Y}$ is slightly higher than in the full information first best because there is a small fraction of low-quality active trees in steady state detracting from the output of the experience good. This additional expenditure of the numeraire good required to pay for the extra entry of high-quality trees can be made arbitrarily small by setting ϕ^e arbitrarily close to 1. Details of this proof are in the Appendix.

We interpret this proposition as indicating that the lemons problem in this economy is one of commitment rather than one of information. The lemons problem arises because the competitive market prices based on the spot gains to trade between a buyer and a seller do not offer sufficient rewards to reputation to ensure high-quality. There is a welfare gain to be achieved here if buyers are able to commit to pay prices that reward good reputation or punish poor reputation over and above the incentives provided by spot market prices.

In some environments, it may be possible to achieve such commitment through long-term contracts between buyers and sellers. If a contract between a buyer and seller with prices based on reputation can be enforced, then the two parties can, with an appropriate choice of parameters F and s, design an incentive contract guaranteeing that most sellers entering into the contract are high-quality. Here we interpret the relationships between the buyers and sellers of intermediate goods as one-shot or short-lived, and hence long-term contracts are not feasible. In this case, regulation is a substitute for missing private capabilities to commit, and can replicate the commitment allocation.

Remark on budget balance: We have not assumed budget balance in which taxes on entry by new intermediate good producers exactly compensate subsidies to experience good producers. Budget balance would impose restrictions on how closely one can approximate the unconstrained first best. The government revenues from entry taxes are $Fm^e\Delta$ per period, whereas, the government expenditures on subsidies are $(s-1)U'(Y)Y\Delta$ per period. Revenues have an upper bound (when $\phi^e \to 1$ and $Y = \bar{Y}$) of $V_L(1)\frac{1-\exp(-\delta\Delta)}{\Delta}\frac{\bar{Y}}{y(1)}\Delta$, which is finite. Contrarily, required subsidies explode to infinity as $\phi^e \to 1$ from equation (24). This implies that budget balance restricts the possibilities for regulation to approximate the first-best allocation.

Remark on non-Markov transfers: One can also see immediately that our assumption that subsidies are based on transactions rather than the full history of signals for each tree is restrictive. The standard result that a reputation of $\phi = 1$ is an absorbing state implies that $V_L(1) = V_H(1)$, so it is impossible to have only high-quality trees entering. We can get arbitrarily close to having only high-quality trees enter, but not all the way there.

In contrast, if we allowed the regulator to make transfers based on the full history of signals of quality associated with each tree, then, for a wide range of stochastic signal structures, the regulator could implement an allocation with exactly $\phi^e = 1$. This result follows if the distribution of signal histories for high- and low-quality trees differs sufficiently such that over time, arbitrarily precise statistical tests of tree quality can be performed given long enough realized signal histories. A transfer scheme that back-loads payments to trees and conditions them on this statistical test of signal histories can then reward the investment of a high-quality tree and (with an entry cost $F > 0$) and, at the same time, deter entry by low-quality trees by leaving them with strictly negative expected profits upon entry. This is not possible with transfers that are Markov in reputation because buyers ignore further signals of quality once $\phi = 1$. Still, the cost in terms of welfare of having such a simple subsidy scheme is negligible compared to the possibility of having subsidies as a complex function of the whole history of signals.

4.2 Regulation When Transactions Are Not Observable

We now assume that the regulator does not observe, and hence cannot subsidize, transactions of the experience good. Instead, we assume that the regulator takes as given that active trees are paid the spot market prices $p(\phi) = y(\phi)U'(Y)$ and that the regulator can only use fixed regulatory entry costs F (rebated lump sum to consumers) to influence steady-state welfare. We evaluate steady-state welfare by $U(Y) + N$, where Y and N are the steady-state equilibrium levels of consumption of the experience and numeraire goods.

The computation of equilibrium when transactions are not observable is very similar to the case in which transactions are observable, except that now the subsidy s to the sale of the experience good is constrained to be $s = 1$. Specifically, again, the first step in our construction of equilibrium summarized as finding an exit threshold $\bar{\phi}$ and corresponding interim beliefs for buyers $\phi^c(\phi)$, continuation strategies for trees

$\omega_i(\phi)$, and normalized value functions $V_i(\phi)$ that satisfy conditions (a), (b), and (c) is unchanged. It is only the second step, in which we compute the entry reputation ϕ^e and the scale of production of the experience good Y necessary to satisfy the zero profit at entry conditions for high- and low-quality trees, that is altered. Here, these entry conditions are given as follows.

The steady-state equilibrium entry reputation ϕ^e satisfies

$$\frac{V_L(\phi^e)}{V_H(\phi^e)} = \frac{F}{C+F}. \tag{25}$$

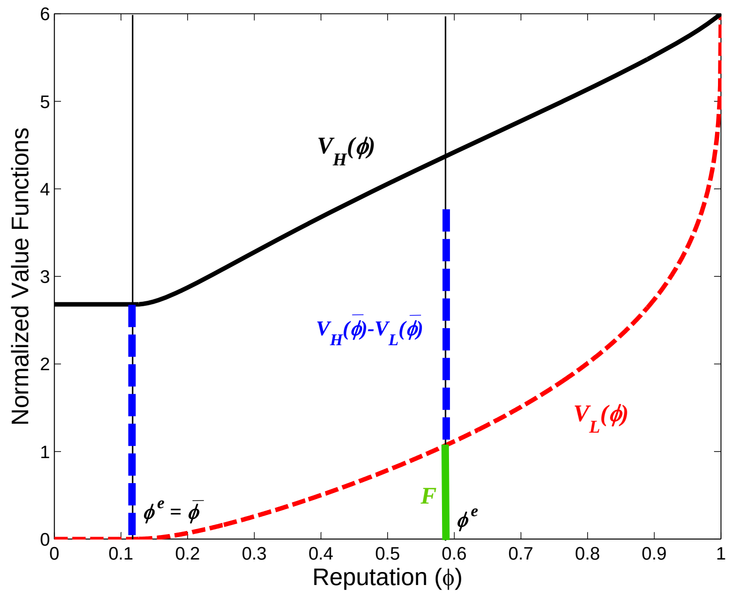
The steady-state equilibrium level of production of the experience good Y satisfies

$$\left(V_H(\phi^e) - V_L(\phi^e)\right) U'(Y) = C. \tag{26}$$

These equations are the same as (23) and (24) except that the subsidy rate s is set to one. Again, fixed entry costs F increase the reputation of entrants ϕ^e and thus the average quality of intermediate goods producers in the market. Now, however, there is no subsidy s that can be used to target the equilibrium level of production of the experience good Y. Instead, the steady-state scale of production of the experience good Y is determined by the difference of the value functions $V_H(\phi^e) - V_L(\phi^e)$. This is a choice of F that pins down ϕ^e and thus also pins down Y. Thus, a regulator potentially faces a conflict in choosing F between relaxing the lemons problem, increasing average quality at entry ϕ^e, and discouraging production of the experience good Y.

We illustrate these computations in Figure 5. The value functions shown in this figure and the associated continuation strategies and interim beliefs are the same as those shown in Figures 1 and 2 because an entry cost does not affect the normalized value functions and continuation strategies for firms that have already entered. The entry cost F impacts the reputation of entrants and the equilibrium level of production of the experience good through equations (25) and (26). As we have argued, equation (25) implies that the entry reputation ϕ^e is an increasing function of entry costs F because the ratio of the value functions in the left-hand side of the equation is increasing in ϕ^e for the three signal structures. From equation (26) we see that the impact of entry costs on the equilibrium level of production of experience goods depends on how the difference of the value functions depends on the entry reputation ϕ^e. As this entry reputation rises, this difference can either rise or fall.

Figure 5: Entry Reputation with Entry Fees



Consider now how changes in the entry reputation ϕ^e and equilibrium level of production of experience goods Y induced by an entry cost F impact equilibrium steady-state welfare.

By choosing an entry cost $F \geq 0$, a social planner can implement any steady-state allocation (Y, N, ϕ^e, m^e) that satisfies equation (26), $\phi^e \geq \bar{\phi}$ and

$$Y = \left(\tilde{V}_H(\phi^e)\phi^e + \tilde{V}_L(\phi^e)(1 - \phi^e)\right) m^e \tag{27}$$

$$N = 1 - C\phi^e m^e. \tag{28}$$

Here, $\tilde{V}_i(\phi^e)$ is defined as the value function for trees of quality $i \in \{L, H\}$, imposing $r = 0$ and the steady-state equilibrium continuation strategies and interim beliefs. Equation (27) measures steady-state output because the average across high- and low-quality entrants of the expected discounted value of profits with an interest rate of zero (output since marginal cost is zero) is equal to the integral across the cross section of profits in the steady state. Note that the computation of the value functions takes the impact of endogenous exit on the cross section of output into account.

We can now calculate the impact of entry cost F on equilibrium steady-state welfare by decomposing this impact into the part coming from the change in entry reputation ϕ^e and the part coming from the change in the equilibrium consumption of the experience good Y.

Consider first the part coming from the change in entry reputation ϕ^e, holding Y fixed. Solving equation (27) for m^e, imposing equation (26), gives that

$$N = 1 - C\frac{\phi^e}{\tilde{V}_H(\phi^e)\phi^e + \tilde{V}_L(\phi^e)(1-\phi^e)}Y. \tag{29}$$

Clearly, if we hold Y fixed, the equilibrium steady-state welfare increases with ϕ^e only if the fraction $\frac{\phi^e}{\tilde{V}_H(\phi^e)\phi^e + \tilde{V}_L(\phi^e)(1-\phi^e)}$ decreases with ϕ^e. An increase in ϕ^e potentially has ambiguous impact on equilibrium steady-state welfare because an increase in the quality of entrants has two opposing effects on the average productivity of active trees. On the one hand, each entering cohort of trees has a higher average quality. On the other hand, as the entry reputation ϕ^e rises above the exit reputation $\bar{\phi}$, those low-quality trees that do enter potentially remain active for a longer period of time. We show in the Appendix that for our three information structures, this fraction decreases with ϕ^e (at least initially), and hence an increase in ϕ^e, holding Y fixed, increases equilibrium steady-state welfare (again, at least initially).

Now consider the impact on equilibrium steady-state welfare of increasing output of the experience good Y, holding fixed the entry reputation ϕ^e. In this case, welfare can be expressed by

$$U(Y) + 1 - C\frac{\phi^e}{\tilde{V}_H(\phi^e)\phi^e + \tilde{V}_L(\phi^e)(1-\phi^e)}Y. \tag{30}$$

Differentiating this expression with respect to Y and imposing the constraint (26) implies that the partial derivative of social welfare with respect to Y is given by

$$U'(Y)\left[1 - \frac{\phi^e(V_H(\phi^e) - V_L(\phi^e))}{\tilde{V}_H(\phi^e)\phi^e + \tilde{V}_L(\phi^e)(1-\phi^e)}\right]dY. \tag{31}$$

Because the functions $\tilde{V}_i(\phi^e)$ are computed in the same way as $V_i(\phi^e)$, except that r is set to zero, it is straightforward to show that $V_H(\phi^e) - V_L(\phi^e) \leq \tilde{V}_H(\phi^e) - \tilde{V}_L(\phi^e)$ and $\tilde{V}_L(\phi^e) \geq 0$. These inequalities imply that this partial derivative is non-negative.

We use these results in the next proposition to show that the fixed cost F that maximizes the welfare of a representative household in steady state under our three signal structures is strictly positive.

Proposition 5 *Optimal regulation with policies not based on transactions, only entry.*

The optimal level of entry costs F is always positive, in all three information cases.

The proof is in the Appendix. Here we provide the intuition for this result. Note first that we have already shown that the equilibrium value of the reputation of entrants ϕ^e is an increasing function of the regulatory entry cost F under all three information structures. In the Appendix, from value functions for the three signal structures in continuous time we show the difference $V_H(\phi^e) - V_L(\phi^e)$ is increasing under bad news for $\phi^e \geq \bar{\phi}$ (as illustrated in Figure 3), is decreasing under good news for $\phi^e > \bar{\phi}$ (as illustrated in Figure 4), and is increasing for a range of low-entry reputation $\phi^e \in (\bar{\phi}, \phi^*)$, and decreasing for a range of high-entry reputation $\phi^e \in (\phi^*, 1)$ under Brownian motion (as illustrated in Figure 2). Furthermore, when signals follow Brownian motion and good news, the difference does not change marginally at $\phi^e = \bar{\phi}$, since $V_H'(\bar{\phi}) = V_L'(\bar{\phi}) = 0$ in those cases.

Putting these results together, we have that in the case of bad news the regulator does not face a direct conflict between the objectives of increasing quality at entry ϕ^e and increasing production of the experience good Y. Both the ratio $V_L(\phi)/V_H(\phi)$ and the difference $V_H(\phi) - V_L(\phi)$ are increasing in ϕ for $\phi > \bar{\phi}$, so a regulator who increases F increases ϕ^e and Y simultaneously. In this case, the regulator wants to increase F to drive ϕ^e arbitrarily close to 1. Since Y increases as ϕ^e increases, in the bad news case, a policy of increasing the entry cost F to drive the average quality of entrants ϕ^e toward one is always welfare improving.

When the signal structure follows a Brownian motion, the regulator faces a direct conflict between the objectives of increasing quality at entry ϕ^e and increasing production of the experience good Y, but only after the reputation at entry has achieved a level ϕ^* strictly above the exit threshold $\bar{\phi}$. Since $\bar{\phi} < \phi^*$, then initially the ratio $V_L(\phi)/V_H(\phi)$ and the difference $V_H(\phi) - V_L(\phi)$ are increasing in ϕ, hence inducing an increase in both ϕ^e and Y, guaranteeing a higher welfare from a higher F, at least initially.

In the case of good news, the regulator does face a direct conflict between the objectives of increasing quality ϕ^e and increasing production of the experience good Y. This follows from the result that the ratio $V_L(\phi)/V_H(\phi)$ is increasing and the difference $V_H(\phi) - V_L(\phi)$ is decreasing in ϕ for $\phi > \bar{\phi}$. Thus, a regulator who increases $F > 0$ increases ϕ^e but reduces Y simultaneously. Initially, at $\phi^e = \bar{\phi}$, the partial derivative of equilibrium steady-state welfare with respect to ϕ^e and Y is zero in both cases. Hence, there is no first-order impact of an increase in entry fee F on equilibrium steady-state welfare starting from an entry fee of zero. We show in the Appendix, however, that the second derivative of welfare with respect to an increase in entry fee is strictly positive at $F = 0$, and hence even in this case the optimal entry fee is strictly positive.

Remark on the welfare impact of regulation: The magnitude of the impact of regulation on welfare is non-monotonic in the precision of signals. When the precision of the signals goes to zero, entry costs do not increase Y much, since the difference between value functions is negligible. However, since the production of the experience good is very small, the marginal welfare gain can still be important. At the other extreme, when the precision of the signal goes to infinity, there is not much room for improvement on the market outcome to be achieved through regulation, since this outcome is already close to the unconstrained first best. This suggests that regulatory policies are more effective in improving the outcome of a market with spotprices when the precision of signals is intermediate.

Remarks on other regulatory tools: Naturally, a regulator can use other regulatory tools, in addition to entry costs, to increase welfare if he or she has access to intermediate levels of information. For example, a regulator can also offer operational subsidies. Loosely, on the one hand these subsidies are expected to compensate mostly high-quality trees, since in expectation they live longer. On the other hand, these subsidies delay exit of low-quality trees, since they prefer to wait longer before exit to receive the subsidy. Depending on this trade-off, operational subsidies may be helpful in increasing welfare even further.

It may also be possible to consider policies that subsidize variables more likely to be experienced by high-quality trees, such as age, or that punish variables more likely to be experienced by low-quality trees, such as exit. Since these two variables are only imperfect signals of reputation, however, these tools are likely not as effective as subsidies to the experience good that are transferred to intermediate goods producers reflecting their reputation with precision. More generally, the regulator can use a wide array of policy combinations to impact welfare. Our model offers a simple framework for analyzing the impact of such policies.

5 Conclusions

We have provided a tractable model for analyzing the interaction of regulation and reputation on spot markets with a lemons problem and with imperfect signals about sellers' quality. We have argued that the lemons problem in this environment is a problem of commitment and not a problem of information. The lemons problem can essentially be eliminated if buyers can commit to offer sellers incentives strong enough to invest in high-quality so as to improve their reputation. When a regulator can design taxes and subsidies contingent on sellers' reputation, a simple taxing scheme may provide the commitment required to mitigate the lemons problem.

Even if a regulator does not have the ability to tax or subsidize sellers contingent on their reputation, that regulator still has the ability to improve welfare by mitigating the lemons problem in a spot market equilibrium by imposing a positive fixed entry cost that is then rebated lump sum to households. Although the regulator potentially faces a trade-off between increasing the average quality of entering sellers and restricting the overall volume of production, we show under our three signal structures that this trade-off is resolved in favor of increasing quality, at least for small entry costs.

Entry costs are typically criticized for reducing production and market size. The main logic is clearly exposed in Hopenhayn (1992): higher entry costs must be compensated by higher aggregate prices, hence by less total output. This argument has been widely used by the economic literature — from supporting trade liberalization to explaining total factor productivity differences across countries — and by international organisms in proposing policy reforms to underdeveloped countries. Still, as shown by Djankov et al. (2002), there is heavy regulation of entry for start-up firms around the world, under the main justification of discouraging the entry of low-quality firms. In this paper, we provide a unifying framework to study the trade-off that entry costs create between production and quality. Interestingly, we show there is a range of entry costs that increase quality without reducing total output — sometimes also increasing total output — and we characterize the optimal level of entry costs that maximize welfare by enhancing market-provided reputation incentives.

From a technical viewpoint, we contribute in providing analytical solutions in continuous time for a model of reputation with free entry and exit of firms that know their type, since they know their own initial investments that determine their type. The explicit analytical solution allows a complete welfare comparison across different regulation policies. We also endogenize the initial reputation assigned to entrants in a market, since the lemons problem is generated by an endogenous decision in general equilibrium of ex ante identical firms.

An important next step in understanding optimal regulation in the presence of reputation concerns is considering moral hazard problems at each moment. We have assumed that quality is fixed as the result of a one-time investment decision. A large literature examines outcomes when sellers must maintain ongoing investments to preserve quality.[9] We anticipate that our first main result will extend to this setting: the problem of moral hazard arises because buyers cannot commit to pay sellers prices contingent on reputation that are high enough to preserve the incentives to invest in quality. We conjecture, then, that a regulator with sufficient flexibility to design transfers contingent on reputation would be able to mitigate both the lemons problem and the moral hazard problem associated with investments to maintain quality. We are not able to derive these results formally, as the required transfer schemes are likely to be non-linear in reputation and thus outside the scope of what we can solve at this time.

Another natural extension is to study mechanisms and institutions the market can endogenously create to reduce commitment problems and align learning and reputation compensations to improve welfare. Possible institutions are vertical integration between experience good producers and intermediate good producers that relax informational problems and financial intermediaries or horizontal integration of intermediate goods producers that commit to cross-subsidize members with different reputation (in the spirit of Biglaiser and Friedman (1994)).

Similarly, an alternative channel that markets can use to replicate positive entry costs is burning money at the moment of entry as a signal of investment. Multiple equilibria introduce this possibility, all of them sustained by an implausible degree of coordination among producers of the intermediate good. Based on this required degree of coordination, but beyond the scope of this paper, we conjecture that the only robust equilibrium, from an evolutionary perspective, is the one we characterize without money burning. Furthermore, money burning is an inefficient way to replace entry

[9] See, for example, Marvel and McCafferty (1984), Maksimovic and Titman (1991), and, more recently, Board and Meyer-ter Vehn (2010).

fees, unless that money goes back to the economy, as we assume the regulator does by making lump-sum transfers of the entry fees to households.

Finally, it is important to mention that most of the literature that studies the effects of costly certification to enter into a market focuses on the informational element of certificates as screening of the initial investment (see Lizzeri (1999) and Albano and Lizzeri (2001)). Our case is more extreme and suggests that even if certification does not provide any additional information about the quality of new firms, it may still be welfare improving, simply because it is costly to entrants.

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A Appendix. "For Online Publication"

A.1 Proof Lemma 1

We use standard recursive techniques to prove this lemma. Let B represent the set of bounded real functions with domain $[0,1]$ with the sup norm. We can specify the perpetuity value of $y(1)\Delta$ as an upper bound on functions in this space and zero as a lower bound on functions in this space. This space is a complete normed metric space. Use the Bellman equation (13) to define operators $T_i : B \to B$ for $i \in \{H, L\}$ in the standard manner. Since functions $f \in B$ are bounded between the perpetuity value of $y(1)\Delta$ and 0, and $T_i(f)$ satisfies the same bounds, then it is also in B.

We now prove that T_i satisfy Blackwell's sufficient conditions for a contraction (monotonicity and discounting), which guarantees there is a unique solution to the recursive equations (13) above. We let V_i denote the unique fixed point of T_i.

a. Monotonicity: For any $f, h \in B(0,1)$, with $f(\phi) \geq h(\phi)$ for all $\phi \in (0,1)$ then $T_i(f)(\phi) \geq T_i(h)(\phi)$ for all $\phi \in (0,1)$. This is immediate.

b. Discounting: There is some constant $\varrho \in (0,1)$ such that $(T_i)(f)(\phi) + \varrho a \geq (T_i)(f + a)(\phi)$, for all constants $a \geq 0$. Direct computation for all $a \geq 0$ gives

$$\begin{aligned} T_i(f+a)(\phi) &= \max_{\omega} \omega \left[y(\phi^c)\Delta + \exp(-r\Delta) \left(\alpha_i(\Delta) f(\phi^g(\phi^c) + a) + (1 - \alpha_i(\Delta)) f(\phi^b(\phi^c) + a) \right) \right] \\ &\leq T_i(f)(\phi) + \exp(-r\Delta) a, \end{aligned}$$

which gives the result since $\exp(-r\Delta) \in (0,1)$. Note here in the last inequality we are allowing the tree to continue (choose $\omega = \exp(-\delta\Delta)$) for the $a > 0$ separately from the choice of ω in $T_i(f)$, and that is what generates the inequality.

A.2 Proof Proposition 1

In what follows we characterize equilibria under the two "reasonable" restrictions on buyers' interim beliefs ϕ^c defined in Assumptions 1 and 2, and then we show that the equilibrium characterized in Proposition 1 is unique when the model converges to continuous time ($\Delta \to 0$).

Without restrictions on buyers' interim beliefs, there is a generic multiplicity of equilibrium that arises from off-equilibrium beliefs. If buyers expect both types of trees to exit, then Bayes' rule does not discipline interim beliefs about trees that continue, and thus these beliefs can be specified in a manner that encourages trees to indeed exit. For example, one can generate any number of equilibria by setting beliefs $\phi^c(\phi) = 0$ for arbitrary subsets of $[0,1]$. Such beliefs, together with value functions $V_i(\phi) = 0$

and continuation strategies $\omega_i(\phi) = 0$ on those same subsets of $[0, 1]$ satisfy our definition of equilibrium. Our multiplicity here arises from the standard problem that if buyers interpret continuation as a sure signal that a tree is low-quality, then all trees prefer to exit, and Bayes' rule cannot be used to restrict buyers' interpretation of out of equilibrium continuation of a tree.

Using the Bellman equations developed in Lemma 1, Assumption 1 implies that value functions $V_i(\phi)$ are weakly increasing (since operators T_i map weakly increasing functions to weakly increasing functions under this restriction) and $V_H(\phi) \geq V_L(\phi)$ (since $T_H(V_L)(\phi) \geq T_L(V_L)(\phi) = V_L(\phi)$, because $\alpha_H(\Delta) > \alpha_L(\Delta)$ and $V_L(\phi)$ is weakly increasing in ϕ).

This first assumption, however, is not enough to rule out multiplicity of equilibrium generated by off-equilibrium beliefs, simply because the beliefs $\phi^c(\phi) = 0$ for all ϕ satisfy monotonic updating, as do beliefs that set $\phi^c(\phi) = 0$ over regions $[0, k]$. This is the reason we introduce Assumption 2. If Bayes' rule can be used, then $\phi^c(\phi) \geq \phi$ iff $\omega^H(\phi, \Delta) \geq \omega^L(\phi, \Delta)$. This assumption itself is stronger in that we require it of beliefs even when Bayes' rule cannot be used.

The combination of these two assumptions implies that there exists a level of reputation $\bar{\phi} < 1$ such that both $V_H(\phi)$ and $V_L(\phi)$ are strictly greater than zero for all $\phi > \bar{\phi}$. This result follows from the observation that $V_i(\phi) \geq \max(0, y(\phi)\Delta)$ when Assumption 2 is satisfied and we have $y(\phi) > 0$ for sufficiently high ϕ (a tree can always operate for one period, earning $y(\phi^c(\phi))\Delta$ and then exit). Assumption 2 together with Assumption 1 (weakly increasing value functions) implies that the region for which $V_L(\phi) > 0$ is an interval that we denote by $(\bar{\phi}, 1]$ and the region for which $V_H(\phi) > 0$ is an interval that we denote by $(\underline{\phi}, 1]$, with $\underline{\phi} \leq \bar{\phi}$.

We next prove that equilibrium beliefs, value functions, and exit strategies in all equilibria with beliefs that satisfy Assumptions 1 and 2 have the following form.

Lemma 2 *Characterization of equilibrium beliefs, value functions, and exit strategies*

Under assumptions 1 and 2 on buyers' beliefs, the equilibrium beliefs, value functions, and continuation strategies have the following form. There exists a value of $\bar{\phi} \in (0, 1)$, which is pinned down uniquely independently of the specification of beliefs $\phi^c(\phi)$ for $\phi < \bar{\phi}$, such that $V_L(\phi) = 0$ for $\phi \leq \bar{\phi}$ and $V_L(\phi) > 0$ for $\phi > \bar{\phi}$. There exists a value of $\underline{\phi} \in [0, \bar{\phi})$ such that $V_H(\phi) = 0$ for $\phi \leq \underline{\phi}$, $V_H(\phi)$ is constant and strictly positive on $(\underline{\phi}, \bar{\phi}]$ and $V_H(\phi)$ is strictly increasing for $\phi > \bar{\phi}$. The associated interim beliefs are $\phi^c(\phi) = \phi$ for $\phi \geq \bar{\phi}$, $\phi^c(\phi) = \bar{\phi}$ for $\phi \in (\underline{\phi}, \bar{\phi})$, and $\phi^c(\phi)$ is any function bounded above by $\underline{\phi}$ and that satisfies assumptions 1 and 2 on $\phi \in [0, \underline{\phi}]$. The continuation strategies have the form $\omega^H(\phi, \Delta) = \exp(-\delta\Delta)$ for all $\phi > \underline{\phi}$, $\omega^H(\phi, \Delta) = 0$ for all $\phi \leq \underline{\phi}$, $\omega^L(\phi, \Delta) = \exp(-\delta\Delta)$ for $\phi > \bar{\phi}$, $\omega^L(\phi, \Delta)$ is set greater than zero so that $\phi^c(\phi) = \bar{\phi}$ for $\phi \in (\underline{\phi}, \bar{\phi}]$ and $\omega^L(\phi, \Delta) = 0$ for all $\phi \leq \underline{\phi}$.

Proof We have already shown that given Assumption 2, there exist ϕ sufficiently large such that $V_i(\phi) > 0$. Given Assumption 1, the value functions V_i are non-decreasing with $V_H(\phi) \geq V_L(\phi)$ so that the regions for which V_i are strictly positive are intervals $(\underline{\phi}, 1]$ and $(\bar{\phi}, 1]$ for high- and low-quality trees respectively, with $\underline{\phi} \leq \bar{\phi}$. The equilibrium requirement that $\omega^i(\phi, \Delta) = \exp(-\delta\Delta)$ whenever $V_i(\phi) > 0$ implies that these must be the continuation strategies for $\phi > \bar{\phi}$ and Bayes' rule then implies that $\phi^c(\phi) = \phi$ for $\phi > \bar{\phi}$. Assumptions 1 and 2 then imply that $\phi^c(\bar{\phi}) = \bar{\phi}$ as well.

Observe next that $V_H(\bar{\phi}) > 0$. To see this, observe that $\phi^g(\bar{\phi}) > \bar{\phi}$ and $\phi^b(\bar{\phi}) < \bar{\phi}$ which implies that $V_L(\phi^g(\bar{\phi})) > 0$ and $V_L(\phi^b(\bar{\phi})) = 0$. This implies that $T_H(V_L)(\bar{\phi}) > T_L(V_L)(\bar{\phi}) \geq 0$, which gives the result.

Note that it is not possible to have an equilibrium with $\bar{\phi} = 0$ as, in this case, $\phi^c(\phi) = \phi$ for all $\phi \in [0, 1]$ and we have already assumed parameters such that $V_i^c(\phi) < 0$ for sufficiently low ϕ under these interim beliefs. Also note that the value of $\bar{\phi}$ is the pinned down uniquely independently of the specification of beliefs $\phi^c(\phi)$ for $\phi < \bar{\phi}$. This last result follows from the fact that $V_L(\phi) = 0$ for all $\phi < \bar{\phi}$. To be specific, note that the most optimistic beliefs that satisfy Assumptions 1 and 2 and the result that $\phi^c(\phi) = \phi$ for all $\phi \geq \bar{\phi}$ have $\phi^c(\phi) = \bar{\phi}$ for all $\phi \in (0, \bar{\phi})$. Likewise, the most pessimistic beliefs that satisfy Assumptions 1 and 2 have $\phi^c(\phi) = \phi$ for all ϕ. It is straightforward to verify that if V_L is the fixed point of the operator T_L with beliefs $\phi^c(\phi) = \phi$ for all $\phi \geq \bar{\phi}$ and $\phi^c(\phi) = \bar{\phi}$ for all $\phi \in (0, \bar{\phi})$ and satisfies $V_L(\phi) = 0$ for all $\phi \leq \bar{\phi}$, then that same V_L is the fixed point of the operator T_L with beliefs $\phi^c(\phi) = \phi$ for all ϕ. Hence, $\bar{\phi}$ is independent of the specification of beliefs for $\phi < \bar{\phi}$.

For $\phi \in (\underline{\phi}, \bar{\phi})$ we have $V_H(\phi) > 0$ and hence $\omega^H(\phi, \Delta) = \exp(-\delta\Delta)$. Monotonicity of updating thus requires that $\omega^L(\phi, \Delta) > 0$ for ϕ in this region. The definition of $\bar{\phi}$ implies that $V_L(\phi) = 0$ in this region. Hence, we must have $V_L^c(\phi) = 0$ in this region. The strict monotonicity of $y(\phi)$ and the weak monotonicity of $V_L(\phi)$ implies that we must have $\phi^c(\phi)$ constant for ϕ in this region. The only constant that satisfies assumption 2 is $\phi^c(\phi) = \bar{\phi}$, which pins down the continuation strategy $\omega^L(\phi, \Delta)$ in this region.

What remains is to characterize equilibrium behavior for $\phi \leq \underline{\phi}$. In this region, we have $V_i(\phi) = 0$. Note that since $V_H^c(\bar{\phi}) > 0$, we must have $\phi^c(\phi) < \bar{\phi}$ in this region. The strict monotonicity of $y(\phi)$ and weak monotonicity of $V_L(\phi)$ then implies that $V_L^c(\phi^c(\phi)) < 0$ in this region and thus $\omega^L(\phi, \Delta) = 0$ in this region. To ensure that $V_H^c(\bar{\phi}) < 0$, we must then have $\phi \leq \underline{\phi}$ in this region as well. Q.E.D.

This end the proof of Lemma 2. Now we argue that there is a value of Δ low enough such that there is a unique equilibrium in which $\underline{\phi} \to 0$. By doing so, we will have proved part (ii) of Proposition 1, as this result will imply that $\phi^c(\phi^b(\bar{\phi})) = \bar{\phi}$.

First, note that Assumption 2 imposes $\phi^c \geq \phi$, then the most pessimistic beliefs consistent with such assumption are $\phi^c = \phi$. In this case, the value of $\underline{\phi}$ is also pinned down uniquely independently of the specification of beliefs $\phi^c(\phi)$ for $\phi < \underline{\phi}$. As in the case of $\bar{\phi}$, this result follows from the fact that $V_H(\phi) = 0$ for all $\phi < \underline{\phi}$. By Assumption 1 $V_H(\phi) \geq V_L(\phi)$. Since $\alpha_H(\Delta) > \alpha_L(\Delta)$, then $\underline{\phi} < \bar{\phi}$ for all $\Delta > 0$. Furthermore, $\underline{\phi} < \bar{\phi}$

also holds strictly in the continuous time limit, as $\Delta \to 0$, (we show this later using analytical value functions in continuous time, but intuitively this comes from a higher drift in reputation for high-quality firms also in the limit). This implies there exists a Δ small enough such that, $\underline{\phi} < \phi^b(\bar{\phi}) < \bar{\phi}$. Assuming $\phi^c(\phi^b(\bar{\phi})) = \phi^b(\bar{\phi})$, $V_H(\phi^b(\bar{\phi})) > 0$ and $V_L(\phi^b(\bar{\phi})) < 0$. However, this imply $\omega^H(\phi^b(\bar{\phi})) = 1$ and $\omega^L(\phi^b(\bar{\phi})) = 0$, which are inconsistent with pessimistic beliefs $\phi^c(\phi^b(\bar{\phi})) = \phi^b(\bar{\phi})$. This implies that for values of Δ low enough the equilibrium interim beliefs and continuation strategies described in point (ii) of Proposition 1 is unique.

To complete the proof of Proposition 1, note that points (i) and (iii) follow from the free entry requirement that $V_L(\phi^e) = 0$. Point (iv) of Proposition 1 is implied by subtracting the free entry requirement for low-quality trees from that for high-quality trees.

A.3 Value Functions in Continuous Time and Their Properties

Here we obtain analytical solutions for the value functions $V_i(\phi)$, under bad news, good news, and Brownian motion, for general payment functions $q(\phi)$. We also show the properties described in the text hold when the function $q(\phi)$ is linear in ϕ, as we assume is the case with spot prices where $q(\phi) = y(\phi)$. In this section, for notational simplicity we denote

$$y(\phi) = a_1\phi - a_0,$$

where $a_1 = y(1) - y(0) > 0$ and $a_0 = -y(0) > 0$. For simplicity we also define $\hat{r} = r + \delta$.

A.3.1 Bad News

In this case $dS_t \in \{0, 1\}$, which means there is either a signal or no signal at each t. The bad news case is defined by $Pr(dS_t = 1|H) = 0$ and $Pr(dS_t = 1|L) = \lambda dt$, which means there is a positive Poisson arrival only for low-quality trees. When a signal arrives, the tree is revealed to be of low quality and hence the public belief about its quality drops to $\phi = 0$. With this reputation, the tree would never be able to sell its output at a non-negative price. Thus, following this event, it is optimal for the tree to cease production and exit as quickly as possible.

It is convenient to use a transformed variable $l = (1 - \phi)/\phi : [0, 1] \to (\infty, 0]$ to summarize the reputation level of a tree. The evolution of l is determined by

$$\frac{dl_t}{dt} = \left[\frac{Pr(dS_t|L) - Pr(dS_t|H)}{Pr(dS_t|H)}\right] l_t.$$

When bad news arrives (i.e., $dS_t = 1$)

$$\frac{dl_t}{dt} = \left[\frac{\lambda dt - 0}{0}\right] l_t = \infty,$$

and reputation jumps immediately to $l = \infty$. Since $\phi = \frac{1}{1+l}$, this means reputation drops immediately to $\phi = 0$.

While there are no news (i.e., $dS_t = 0$), reputation increases. From the Poisson distribution, the probability that a high-quality tree does not generate news for an interval of time of length t is $e^{-\lambda t}$. Then, after a time interval of length t of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_t = 0|L)}{Pr(S_t = 0|H)}\right] l_0 = \left[\frac{e^{-\lambda t}}{1}\right] l_0 = e^{-\lambda t} l_0,$$

where $l_0 = \frac{1-\phi^e}{\phi^e}$. This means l_t is decreasing (reputation is increasing) over time at a rate $\lambda \in [0, \infty)$. While there is no news, the evolution of reputation for trees with high and low-quality is the same. After bad news, a tree exits and obtains zero thereafter. Then, the value functions for both types only differ in their discount factor.

Lemma 3 *Value functions for general profit functions and bad news*

A value function for a low-quality tree with reputation l, for a general $q(l)$, is

$$\hat{V}_L(l) = \int_{\tau=0}^{\infty} e^{-(\hat{r}+\lambda)\tau} q(e^{-\lambda\tau} l) d\tau,$$

and the value function for a high-quality tree with reputation l is

$$\hat{V}_H(l) = \int_{\tau=0}^{\infty} e^{-\hat{r}\tau} q(e^{-\lambda\tau} l) d\tau.$$

Solving explicitly the integrals for the case of linear payoffs and no marginal costs, $q(\phi) = y(\phi) = a_1\phi - a_0$ (hence $q(l) = \frac{a_1}{1+l} - a_0$),

$$\hat{V}_L(l) = \frac{1}{\hat{r}+\lambda}\left[a_1 \Upsilon_{m_{\hat{r}+\lambda}}(-l) - a_0\right], \tag{32}$$

$$\hat{V}_H(l) = \frac{1}{\hat{r}}\left[a_1 \Upsilon_{m_{\hat{r}}}(-l) - a_0\right], \tag{33}$$

where $\Upsilon_m(-l) = {}_2F_1(1, m; m+1, -l)$ is an hypergeometric function, and

$$m_{\hat{r}} = \frac{\hat{r}}{\lambda} > 0 \qquad \texttt{and} \qquad m_{\hat{r}+\lambda} = \frac{\hat{r}+\lambda}{\lambda} = 1 + m_{\hat{r}}.$$

The hypergeometric function has well-defined properties when $m > 0$. In particular, it is monotonically increasing in ϕ (from 0 to 1) and monotonically decreasing in m.

$$\Upsilon_m\left(-\frac{1-\phi}{\phi}\right) : [0,1] \to [0,1] \qquad \texttt{and} \qquad \frac{\partial \Upsilon_m(\cdot)}{\partial m} < 0.$$

Now we denote $V_i(\phi) = \hat{V}_i(l)$ for all ϕ and $i \in \{L, H\}$. Since $\lim_{\phi \to 0} V_L(\phi) = -\frac{a_0}{\hat{r}+\lambda} < 0$ with no exit, there is a $\phi = \bar{\phi}$ such that $V_L(\bar{\phi}) = 0$. Hence $\bar{\phi}$ is the highest reputation at which low-quality trees are indifferent between exiting or not. As discussed above, exiting strategies imply that in equilibrium, no tree has a reputation below $\bar{\phi}$. Value functions in the range $[\bar{\phi}, 1]$ are

$$\begin{aligned} V_L(\phi) &: [\bar{\phi}, 1] \to [0, \frac{a_1 - a_0}{\hat{r}+\lambda}] \\ V_H(\phi) &: [\bar{\phi}, 1] \to [V_H(\bar{\phi}), \frac{a_1 - a_0}{\hat{r}}], \end{aligned}$$

where $V_H(\bar{\phi}) = \frac{1}{\hat{r}}\left[a_1 \Upsilon_{m_{\hat{r}}}\left(-\frac{1-\bar{\phi}}{\bar{\phi}}\right) - a_0\right] > 0$ (since $m_{\hat{r}} < m_{\hat{r}+\lambda}$).

Lemma 4 *Derivatives with linear payoffs*

For all $\phi \in [\bar{\phi}, 1]$ and all $\hat{r} \geq 0$

$$V_L'(\phi) > 0 \qquad \textit{and} \qquad V_H'(\phi) > 0$$

First recall that derivatives of hypergeometric functions are

$$\frac{\partial \Upsilon_{m_{\hat{r}}}(-l)}{\partial l} = \Upsilon'_{m_{\hat{r}}}(-l) = m_{\hat{r}} l \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-l)\right] < 0.$$

Then

$$\hat{V}_L'(l) = \frac{a_1 l}{\lambda}\left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}+\lambda}}(-l)\right], \tag{34}$$

$$\hat{V}_H'(l) = \frac{a_1 l}{\lambda}\left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-l)\right], \tag{35}$$

which are negative for all $[0, \bar{l}]$. In particular, at $\bar{l}$, $\hat{V}_L(\bar{l}) = 0$, then $\Upsilon_{m_{\hat{r}+\lambda}}(-\bar{l}) = \frac{a_0}{a_1}$.

The lemma follows from

$$V_i'(\phi) = V_i'(l)\frac{\partial l}{\partial \phi} = -\frac{V_i'(l)}{\phi^2}.$$

Applying the properties of hypergeometric functions, the ratio $V_L(\phi)/V_H(\phi)$ is monotonically increasing, from 0 to $\frac{\hat{r}}{\hat{r}+\lambda}$, and the difference $V_H(\phi) - V_L(\phi)$ is also monotonically increasing, from $V_H(\bar{\phi})$ to $\frac{\lambda}{\hat{r}+\lambda} V_H(1)$.

A.3.2 Good News

In this case $Pr(dS_t = 1|H) = \lambda dt$ and $Pr(dS_t = 1|L) = 0$. When a signal arrives, the tree is revealed to be of high-quality, and hence the public belief ϕ regarding this tree jumps up to $\phi = 1$. After good news, the tree maintains a reputation of $\phi = 1$ until it exits exogenously.

Again, we use the variable $l = (1 - \phi)/\phi$. When good news arrives (i.e., $dS_t = 1$)

$$\frac{dl}{dt} = \left[\frac{0 - \lambda dt}{\lambda dt} \right] l_t = -l_t,$$

and reputation jumps immediately to $l = 0$, or $\phi = 1$.

While there is no news (i.e., $dS_t = 0$), reputation decreases. After a time interval of length t of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_t = 0|L)}{Pr(S_t = 0|H)} \right] l_0 = \left[\frac{1}{e^{-\lambda t}} \right] l_0 = e^{\lambda t} l_0,$$

which means l_t is increasing (reputation is decreasing) over time at a rate λ.

Denoting $q(l(1))$ the payoffs for a tree with $\phi = 1$, the value function for a tree that has experienced good news is,

$$V(l(1)) = \frac{q(l(1))}{\hat{r}}.$$

There is a key difference between good news and bad news. Under bad news, reputation only increases, which means endogenous exit never occurs unless a bad signal is revealed. Under good news, reputation continuously decreases and low-quality trees that hit $\bar{\phi}$ will exit at a rate λ such that reputation never drifts below $\bar{\phi}$.

Lemma 5 *Value functions for general profit functions and good news*

The value function for a low-quality tree with reputation l is

$$\hat{V}_L(l) = \int_{\tau=0}^{T(l)} e^{-\hat{r}\tau} q(e^{\lambda \tau} l) d\tau. \tag{36}$$

The value function for a high-quality tree with reputation l is

$$\hat{V}_H(l) = \int_{\tau=0}^{T(l)} e^{-(\hat{r}+\lambda)\tau}\left[q(e^{\lambda\tau}l) + \lambda\frac{q(l(1))}{\hat{r}}\right]d\tau + \int_{\tau=T(l)}^{\infty} e^{-(\hat{r}+\lambda)(\tau-T(l))}\lambda\frac{q(l(1))}{\hat{r}}d\tau, \quad (37)$$

where $T(l)$ is the time required for l to increase up to $\bar{l} = \frac{1-\bar{\phi}}{\bar{\phi}}$.

$$T(l) = \frac{log(\bar{l}/l)}{\lambda} > 0. \quad (38)$$

In the case of linear payoffs and no marginal costs, the reputation at which low-quality trees are willing to exit is given by $q(\bar{l}) = \frac{a_1}{1+\bar{l}} - a_0 = 0$. In this case, $\bar{l}$ is given by the reputation below which profits are negative. Then $\bar{l} = \frac{a_1-a_0}{a_0}$ and $T(l)$ is given following equation (38). The value functions are

$$\begin{aligned}\hat{V}_L(l) &= \frac{1}{\hat{r}}\left[a_1\left(1-\Upsilon_{m_{\hat{r}}}\left(-\frac{1}{l}\right)\right)-a_0\right] \\ &\quad -\frac{e^{-\hat{r}T(l)}}{\hat{r}}\left[a_1\left(1-\Upsilon_{m_{\hat{r}}}\left(-\frac{a_0}{a_1-a_0}\right)\right)-a_0\right], \end{aligned} \quad (39)$$

$$\begin{aligned}\hat{V}_H(l) &= \frac{1}{\hat{r}+\lambda}\left[a_1\left(1-\Upsilon_{m_{\hat{r}+\lambda}}\left(-\frac{1}{l}\right)\right)-a_0+\lambda\frac{a_1-a_0}{\hat{r}}\right] \\ &\quad -\frac{e^{-(\hat{r}+\lambda)T(l)}}{\hat{r}+\lambda}\left[a_1\left(1-\Upsilon_{m_{\hat{r}+\lambda}}\left(-\frac{a_0}{a_1-a_0}\right)\right)-a_0\right]. \end{aligned} \quad (40)$$

Now we denote $V_i(\phi) = \hat{V}_i(l)$ for all ϕ and $i \in \{L, H\}$. Since $T(l(1)) = \infty$, using the previously discussed properties of the hypergeometric functions,

$$\begin{aligned} V_L(\phi) &: [\bar{\phi}, 1] \to [0, \frac{a_1-a_0}{\hat{r}}], \\ V_H(\phi) &: [\bar{\phi}, 1] \to [\frac{\lambda}{\hat{r}+\lambda}\frac{a_1-a_0}{\hat{r}}, \frac{a_1-a_0}{\hat{r}}]. \end{aligned}$$

Lemma 6 *Derivatives with linear payoffs*

For all $\hat{r} \geq 0$,

$$\begin{aligned} &V_L'(\phi) > 0 \quad \textit{and} \quad V_H'(\phi) > 0 \qquad \textit{for all} \ \ \phi \in (\bar{\phi}, 1] \\ &V_L'(\bar{\phi}) = V_H'(\bar{\phi}) = 0 \\ &V_L''(\bar{\phi}) = V_H''(\bar{\phi}) > 0. \end{aligned}$$

First recall that derivatives of hypergeometric functions are

$$\frac{\partial \Upsilon_{m_{\hat{r}}}(-1/l)}{\partial l} = \Upsilon'_{m_{\hat{r}}}(-\frac{1}{l}) = m_{\hat{r}} l \left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-1/l)\right]\left(-\frac{1}{l^2}\right)$$

and define

$$X_{m_{\hat{r}}} = (a_1 - a_0) - a_1 \Upsilon_{m_{\hat{r}}}\left(-\frac{1}{\bar{l}}\right).$$

Taking derivatives

$$V'_L(l) = -\frac{a_1}{\hat{r}}\Upsilon'_{m_{\hat{r}}}(-\frac{1}{l}) + \frac{\hat{r}T'(l)e^{-\hat{r}T(l)}}{\hat{r}}X_{m_{\hat{r}}}$$

and plugging in the expressions above for $\Upsilon'_{m_{\hat{r}}}(-\frac{1}{l})$ and $T'(l)$, we have that

$$V'_L(l) = \frac{a_1}{\lambda l}\left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}}}(-\frac{1}{l})\right] - \frac{e^{-\hat{r}T(l)}}{\lambda l}\left[a_1\left(1 - \Upsilon_{m_{\hat{r}}}(-\frac{1}{\bar{l}})\right) - a_0\right]$$

and similarly for high-quality firms

$$V'_H(l) = \frac{a_1}{\lambda l}\left[\frac{l}{1+l} - \Upsilon_{m_{\hat{r}+\lambda}}(-\frac{1}{l})\right] - \frac{e^{-(\hat{r}+\lambda)T(l)}}{\lambda l}\left[a_1\left(1 - \Upsilon_{m_{\hat{r}+\lambda}}(-\frac{1}{\bar{l}})\right) - a_0\right].$$

These derivatives are negative for $l \in [0, \bar{l})$ and exactly zero at $\bar{l} = \frac{a_1 - a_0}{a_0}$ (this is the point $\bar{\phi}$ at which profits are zero, $a_1\bar{\phi} - a_0 = 0$), regardless of the $\hat{r}$ used. Since

$$V'_i(\phi) = V'_i(l)\frac{\partial l}{\partial \phi} = -\frac{V'_i(l)}{\phi^2}.$$

Then the first derivatives of value function are positive for $\phi \in (\bar{\phi}, 1]$ and

$$V'_L(\bar{\phi}) = V'_H(\bar{\phi}) = 0 \qquad \text{for all } \hat{r}. \tag{41}$$

Now we take second derivatives, for low-quality firms

$$V''_L(l) = -\frac{a_1}{\lambda(1+l)^2}\left[1 - \frac{\hat{r}(1+l)}{\lambda l}\right] + \frac{\lambda - \hat{r}}{\lambda^2 l^2}\left[a_1 \Upsilon_{m_{\hat{r}}}(-\frac{1}{l}) + e^{-\hat{r}T(l)}X_{m_{\hat{r}}}\right]$$

and similarly for high-quality firms

$$V''_H(l) = -\frac{a_1}{\lambda(1+l)^2}\left[1 - \frac{(\hat{r}+\lambda)(1+l)}{\lambda l}\right] - \frac{\hat{r}}{\lambda^2 l^2}\left[a_1 \Upsilon_{m_{\hat{r}+\lambda}}(-\frac{1}{l}) + e^{-(\hat{r}+\lambda)T(l)}X_{m_{\hat{r}+\lambda}}\right].$$

Evaluating these expressions at $\bar{l}$ for low-quality firms, for all $\hat{r}$

$$V_L''(\bar{l}) = \frac{\lambda a_0^3}{\lambda^2 a_1(a_1 - a_0)}$$

and similarly for high-quality firms

$$V_L''(\bar{l}) = V_H''(\bar{l}) = \frac{a_0}{\lambda \bar{l}(1+\bar{l})} > 0.$$

Again, using transformed derivatives to compute derivatives with respect to ϕ

$$V_i''(\phi) = \frac{V_i''(l)}{\phi^4} + \frac{2V_i'(l)}{\phi^3}.$$

This implies that, at $\bar{\phi}$,

$$V_L''(\bar{\phi}) = V_H''(\bar{\phi}) > 0 \qquad \text{for all } \hat{r}. \tag{42}$$

Finally, applying the properties of hypergeometric functions, the ratio $V_L(\phi)/V_H(\phi)$ is monotonically increasing, from 0 to 1, and the difference $V_H(\phi) - V_L(\phi)$ is monotonically decreasing, from $\frac{\lambda}{\hat{r}+\lambda}\frac{a_1-a_0}{\hat{r}}$ to 0.

A.3.3 Brownian Motion

Assume now the signal process follows a Brownian motion

$$dS_t = \mu_i dt + \sigma dZ_t,$$

where $i = \{L, H\}$, drifts depend on the tree's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

The following proposition shows that reputation, for both high- and low-quality trees, also follows a Brownian motion process when based purely on signals. As discussed in Proposition 1, given the equilibrium exit rates, this is also the updating rule for all $\phi > \bar{\phi}$, whereas the updating for all $\phi \leq \bar{\phi}$ follows an immediate jump up to $\bar{\phi}$.

Lemma 7 *Reputation process based on Brownian motion signals.*

The reputation process high-quality trees expect is a submartingale

$$d\phi_t^H = \frac{\lambda^2(\phi_t)}{\phi_t} dt + \lambda(\phi_t) dZ_t, \tag{43}$$

and the reputation process low-quality trees expect is a supermartingale

$$d\phi_t^L = -\frac{\lambda^2(\phi_t)}{(1-\phi_t)}dt + \lambda(\phi_t)dZ_t, \tag{44}$$

where $\lambda(\phi_t) = \phi_t(1-\phi_t)\zeta$ *and* $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ *is the signal-to-noise ratio.*

Proof The activities of the two types of trees induce two different probability measures over the paths of the signal S_t. Fix a prior ϕ^e and assume exogenous exit. Then reputation evolves following the equation:

$$\phi_t = \frac{\phi^e Pr(S_t|H)}{\phi^e Pr(S_t|H) + (1-\phi^e)Pr(S_t|L)}$$

or

$$\phi_t = \frac{\phi^e \xi_t}{\phi^e \xi_t + (1-\phi^e)}, \tag{45}$$

where ξ_t is the ratio between the likelihood that a path $S_s : s \in [0,t]$ arises from type H and the likelihood that it arises from type L. As in Faingold and Sannikov (2011), from Girsanov's Theorem, this ratio follows a Brownian motion characterized by $\mu_\xi = 0$ and $\sigma_\xi = \xi_t\zeta$,

$$d\xi_t = \xi_t \zeta dZ_s^L, \tag{46}$$

where $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ and $dZ_s^L = \frac{dS_t - \mu_L dt}{\sigma}$ is a Brownian motion under the probability measure generated by type L.[10]

By Ito's formula,

$$d\phi = [\phi' \mu_\xi + \frac{1}{2}\phi'' \sigma_\xi^2]dt + \phi' \sigma_\xi dZ_s^L$$

$$d\phi_t = -\frac{1}{2}\frac{2\phi^{e2}(1-\phi^e)}{(\phi^e\xi_t + (1-\phi^e))^3}\xi_t^2\zeta^2 dt + \frac{\phi^e(1-\phi^e)}{(\phi^e\xi_t + (1-\phi^e))^2}\xi_t\zeta dZ_s^L,$$

and from equation (45) we can express it in terms of ϕ_t rather than ϕ^e

$$d\phi_t = -\phi_t^2(1-\phi_t)\zeta^2 dt + \phi_t(1-\phi_t)\zeta dZ_s^L$$

$$d\phi_t = \phi_t(1-\phi_t)\zeta[dZ_s^L - \phi_t\zeta dt]$$

replacing by the definition of dZ_s^L,

$$d\phi_t = \lambda(\phi_t)dZ_t^\phi, \tag{47}$$

[10] It is also possible to solve the problem defining $\xi_t = \frac{Pr(S_t|L)}{Pr(S_t|H)}$ such that $\phi_t = \frac{\phi^e}{\phi^e + (1-\phi^e)\xi_t}$, where $d\xi_t = \xi_t\zeta dZ_s^H$.

where $dZ_t^{\phi} = \frac{1}{\sigma}[dS_t - (\phi_t \mu_H + (1-\phi_t)\mu_L)dt]$ and

$$\lambda(\phi_t) = \phi_t(1-\phi_t)\frac{\mu_H - \mu_L}{\sigma}. \tag{48}$$

Conversely, suppose that ϕ_t is a process that solves equation (47). Define ξ_t using equation (45),

$$d\xi_t = -\frac{1-\phi^e}{\phi^e}\frac{\phi_t}{1-\phi_t}$$

By applying Ito's formula again, ξ_t satisfies equation (46). This implies ξ_t is the ratio between the likelihood that a path $S_s : s \in [0,t]$ arises from type H and the likelihood it arises from type L. Hence, ϕ_t is determined by Bayes' rule.

Finally, consider that different types will have different paths, that in expectation will move their reputation. Replacing dS_t^i in dZ_t^{ϕ} in equation (47) for the two different types of trees, deliver equations (43) and (44). Q.E.D.

Four clear properties arise from inspecting equations (43) and (44). First, high-quality trees expect a positive drift in their evolution of reputation, whereas low-quality trees expect a negative drift. Second, when reputation ϕ_t is either 0 or 1, drifts and volatilities are zero, which means at those points reputation do not change, for both high- and low-quality trees. Third, reputation varies more at intermediate levels of ϕ_t, and volatilities are larger. Finally, the drift for high-quality trees is higher than for low-quality trees for bad reputations and lower for good reputations, since ϕ_t is in the denominator of the drift for high-quality trees, whereas $(1-\phi_t)$ is in the denominator of the drift for low-quality trees.

Lemma 8 *The ordinary differential equations that characterize the value functions for high and low-quality trees are*

$$\hat{r}\rho V_L(\phi) = \rho q(\phi) - \phi^2 (1-\phi) V_L'(\phi) + \frac{1}{2}\phi^2 (1-\phi)^2 V_L''(\phi), \tag{49}$$

$$\hat{r}\rho V_H(\phi) = \rho q(\phi) + \phi (1-\phi)^2 V_H'(\phi) + \frac{1}{2}\phi^2 (1-\phi)^2 V_H''(\phi), \tag{50}$$

where

$$\rho = \frac{\sigma^2}{(\mu_H - \mu_L)^2}. \tag{51}$$

Proof First, we prove the following lemma.

Lemma 9 *Define Ψ the space of progressively measurable processes ψ_t for all $t \geq 0$ with $E[\int_0^T \psi_t^2 dt] < \infty$ for all $0 < T < \infty$. A bounded process W_t^i for all $t \geq 0$ is the continuation* *value for type $i = \{H, L\}$ if and only if, for some process ψ_t^i in Ψ, we have*

$$dW_t^i = \hat{r}[W_t^i - q(\phi_t)]dt + \psi_t^i dZ_t. \tag{52}$$

Proof The flow continuation value W_t^i for type i is the expected payoff at time t,

$$W_t^i = \hat{r}E_t^i \left[\int_t^{\infty} e^{-\hat{r}(\tau - t)} q(\phi_\tau) d\tau \right].$$

Denote U_t^i the discounted sum of payoffs for type i conditional on the public information available at time t,

$$U_t^i = \hat{r}E_t^i \left[\int_0^{\infty} e^{-\hat{r}\tau} q(\phi) d\tau \right] = \int_0^t e^{-\hat{r}\tau} \hat{r} q(\phi_\tau) d\tau + W_t^i. \tag{53}$$

Since U_t^i is a martingale, by the Martingale Representation Theorem, there exists a process ψ_t^i in Ψ such that

$$dU_t^i = e^{-\hat{r}t} \psi_t^i dZ_t. \tag{54}$$

Differentiating (53) with respect to time

$$dU_t^i = e^{-\hat{r}t} \hat{r} q(\phi_t) dt - \hat{r} e^{-\hat{r}t} W_t^i dt + e^{-\hat{r}t} dW_t^i. \tag{55}$$

Combining (54) and (55), we can obtain (52). Q.E.D.

In a Markovian equilibrium, we know $W_t^i = V_i(\phi_t)$. Since this continuation value depends on the reputation, which follows a Brownian motion, using Ito's Lemma,

$$dV_i(\phi) = \left[\mu_{i,\phi} V_i'(\phi) + \frac{1}{2} \sigma_\phi^2 V_i''(\phi) \right] dt + \sigma_\phi V_i'(\phi) dZ, \tag{56}$$

where $\mu_{H,\phi} = \frac{\lambda^2(\phi)}{\phi}$, $\mu_{L,\phi} = -\frac{\lambda^2(\phi)}{(1-\phi)}$ and $\sigma_\phi = \lambda(\phi)$ from Proposition 7.

Matching drifts of equations (52) and (56) for each type i yields the linear second-order differential equation that characterizes continuation values $V_H(\phi)$ and $V_L(\phi)$,

$$\frac{1}{2}\lambda^2(\phi) V_L''(\phi) - \frac{\lambda^2(\phi)}{(1-\phi)} V_L'(\phi) - \hat{r} V_L(\phi) + q(\phi) = 0 \tag{57}$$

and

$$\frac{1}{2}\lambda^2(\phi) V_H''(\phi) + \frac{\lambda^2(\phi)}{\phi} V_H'(\phi) - \hat{r} V_H(\phi) + q(\phi) = 0. \tag{58}$$

Using the definition for $\lambda(\phi)$ from equation (48) we obtain the second-order differential equations in the proposition. Q.E.D.

Solving these ordinary differential equations (ODEs), we can obtain the value functions for high- and low-quality trees. Imposing that these value functions are non-negative introduces endogenous exit, which regulates the reputation process.

Lemma 10 *Value functions for general profit functions and Brownian motion*

The value function of low-quality trees with reputation l is

$$\hat{V}_L(l) = K\left\{\int_0^1 \theta^{\gamma-1} q(\theta l)\, d\theta - \int_{\chi/l}^1 \theta^{-\gamma} q(\theta l)\, d\theta\right\}. \tag{59}$$

The value function of high-quality trees with reputation l is

$$\hat{V}_H(l) = K\left\{\int_0^1 \theta^{\gamma-2} q(\theta l)\, d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l)\, d\theta + \frac{q(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\}, \tag{60}$$

where $\theta = l'/l$,

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2\hat{r}\rho} \qquad \text{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2\hat{r}\rho}}.$$

Proof We first solve the previous ODEs imposing the boundary conditions at $\phi = 1$ and $\phi = \bar{\phi}$. After simplifying the expressions we take derivatives, which will be useful later to characterize the properties of these value functions.

Changing variables to $l = (1-\phi)/\phi$ and defining $\hat{V}(l) = V(\phi)$, the ODEs above can be written as

$$\begin{aligned}\hat{r}\rho\hat{V}_L(l) &= \rho q(l) + l\hat{V}_L'(l) + \frac{1}{2}l^2\hat{V}_L''(l)\\ \hat{r}\rho\hat{V}_H(l) &= \rho q(l) + \frac{1}{2}l^2\hat{V}_H''(l).\end{aligned}$$

1. Solving the ODEs

1.a) Solving for $\hat{V}_L(l)$: We conjecture a solution of the form

$$\hat{V}_L(l) = K\left[l^{-\gamma}\int_{\chi_1}^l l'^{\gamma}\frac{q(l')}{l'}dl' - l^{\gamma-1}\int_{\chi_2}^l l'^{1-\gamma}\frac{q(l')}{l'}dl'\right]$$

for some parameters γ and K. With this, we have

$$\hat{V}_L'(l) = K\left[(-\gamma)\, l^{-\gamma-1}\int_{\chi_1}^l l'^{\gamma}\frac{q(l')}{l'}dl' - (\gamma-1)\, l^{\gamma-2}\int_{\chi_2}^l l'^{1-\gamma}\frac{q(l')}{l'}dl'\right]$$

$$\hat{V}_L''(l) = K\left[(-\gamma)(-\gamma-1)\, l^{-\gamma-2}\int_{\chi_1}^{l} l'^{\gamma}\frac{q(l')}{l'}dl' - (\gamma-1)(\gamma-2)\, l^{\gamma-3}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{q(l')}{l'}dl'\right] + K(1-2\gamma)\frac{q(l)}{l^2}$$

$$l\hat{V}_L'(l) + \frac{1}{2}l^2\hat{V}_L''(l) = \frac{\gamma(\gamma-1)}{2}\hat{V}_L(l) + K\left(\frac{1-2\gamma}{2}\right)q(l),$$

which solves the ODE when $2\hat{r}\rho = \gamma(\gamma-1)$ and $K(1-2\gamma) = -2\rho$, or

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2\hat{r}\rho} \qquad \texttt{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2\hat{r}\rho}}.$$

Recall $\gamma(\rho) : [0, \infty] \to [1, \infty]$ and $K(\rho) > 0$. The parameters χ_1 and χ_2 will be determined later from boundary conditions.

1.b) Solving for $\hat{V}_H(l)$**:** Define: $\Delta_H(l) = q(0) - \hat{V}_H(l)$, $\bar{q}(l) = q(0) - q(l)$. Notice $\bar{q}(l)$ is increasing in l.

Rewriting the ODE for the high type as

$$\rho\Delta_H(l) = \rho\bar{q}(l) + \frac{1}{2}l^2\Delta_H''(l).$$

Proceeding as above, we conjecture a solution of the form

$$\Delta_H(l) = K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{q}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{q}(l')}{l'}dl'\right]$$

for the same parameters γ and K defined previously. With this, we have

$$\Delta_H'(l) = K\left[(1-\gamma)l^{-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{q}(l')}{l'}dl' + \gamma l^{\gamma-1}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{q}(l')}{l'}dl'\right]$$

$$\Delta_H''(l) = K\left[-\gamma(1-\gamma)\, l^{-\gamma-1}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{q}(l')}{l'}dl' + \gamma(\gamma-1)\, l^{\gamma-2}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{q}(l')}{l'}dl'\right] + K(1-2\gamma)\frac{\bar{q}(l)}{l^2}$$

$$\frac{1}{2}l^2\Delta_H''(l) = \frac{\gamma(\gamma-1)}{2}\Delta_H(l) + K\left(\frac{1-2\gamma}{2}\right)q(l)$$

that fulfill the ODE by construction with the parameters γ and K defined above. The parameters ψ_1 and ψ_2 will be determined later also from boundary conditions.

2. Dealing with the boundary conditions at $l = 0$*.*

Notice that we need $\lim_{l\to 0} \hat{V}_L(l) = \lim_{l\to 0} q(l) = q(0)$, and $\lim_{l\to 0} \Delta_H(l) = \lim_{l\to 0} \bar{q}(l) = \lim_{l\to 0} q(l) - q(0) = 0$. The two limiting properties hold if and only if $\chi_1 = 0$ and $\psi_1 = 0$ (we then relabel $\chi_2 = \chi$ and $\psi_2 = \psi$).

We will proceed with the proof for the high type. The proof for the low type is related. Using Lipschitz continuity of $\bar{q}(l)$, assuming $\bar{q}(l) \le \Lambda l$, and $\psi_2 \le \infty$:

$$
\begin{aligned}
\Delta_H(l) &= K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{q}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{q}(l')}{l'}dl'\right]\\
&\le \Lambda K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}dl'\right]\\
&= \Lambda K\left[l^{1-\gamma}\left(\frac{l^{\gamma}}{\gamma} - \frac{\psi_1^{\gamma}}{\gamma}\right) + l^{\gamma}\left(\frac{\psi_2^{1-\gamma}}{1-\gamma} - \frac{l^{1-\gamma}}{1-\gamma}\right)\right]\\
&= \Lambda K\left[l\left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right)\right]\\
&= \Lambda l
\end{aligned}
$$

if and only if $\psi_1 = 0$ and assuming $\psi_2 = \infty$. Hence, $\lim_{l\to 0} \Delta_H(l) = 0$ if and only if $\psi_1 = 0$. A similar analysis delivers $\lim_{l\to 0} \hat{V}_L(l) = q(0)$ if and only if $\chi_1 = 0$.

3. Simplifying the Value Functions.

Changing variables inside the integrals: $\theta = l'/l$, so $l d\theta = dl'$ and the limits of integration. We start from obtaining $V_H(l)$.

$$
\Delta_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\bar{q}(\theta l)\,d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}\bar{q}(\theta l)\,d\theta\right\}
$$

Since $\bar{q}(\theta l) = q(0) - q(\theta l)$ and $\hat{V}_H(l) = q(0) - \Delta_H(l)$

$$
\begin{aligned}
\hat{V}_H(l) &= q(0)\left(1 - K\int_0^1 \theta^{\gamma-2}d\theta - K\int_1^{\psi/l}\theta^{-\gamma-1}d\theta\right)\\
&\quad +K\left\{\int_0^1 \theta^{\gamma-2}q(\theta l)\,d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}q(\theta l)\,d\theta\right\}\\
\hat{V}_H(l) &= q(0)\left[\frac{K}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right] + K\left\{\int_0^1 \theta^{\gamma-2}q(\theta l)\,d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}q(\theta l)\,d\theta\right\}.
\end{aligned}
$$

Hence,

$$\hat{V}_H(l) = K\left\{\int_0^1 \theta^{\gamma-2} q(\theta l)\, d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l)\, d\theta + \frac{q(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\}. \tag{61}$$

Similarly, the low type's value function can be written as

$$\hat{V}_L(l) = K\left\{\int_0^1 \theta^{\gamma-1} q(\theta l)\, d\theta - \int_{\chi/l}^1 \theta^{-\gamma} q(\theta l)\, d\theta\right\}. \tag{62}$$

In reduced form,

$$\hat{V}_L(l) = K[B_L(l) - A_L(l)] \tag{63}$$

$$\hat{V}_H(l) = K[B_H(l) - A_H(l)], \tag{64}$$

where

$$B_L(l) = \int_0^1 \theta^{\gamma-1} q(\theta l)\, d\theta \qquad \texttt{and} \qquad A_L(l) = \int_{\chi/l}^1 \theta^{-\gamma} q(\theta l)\, d\theta$$

$$B_H(l) = \int_0^1 \theta^{\gamma-2} q(\theta l)\, d\theta \qquad \texttt{and} \qquad A_H(l) = \int_{\psi/l}^1 \theta^{-\gamma-1} q(\theta l)\, d\theta - \frac{q(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}.$$

4. Taking Derivatives.

Taking derivatives of $\hat{V}_L(l)$ components and multiplying by l,

$$l\frac{\partial A_L(l)}{\partial l} = \int_{\chi/l}^1 \theta^{-\gamma} q'(\theta l)\, \theta l d\theta - \left(\frac{\chi}{l}\right)^{-\gamma} q(\chi)\left(-\frac{\chi}{l^2}\right) l.$$

Integrating the first term by parts,

$$\begin{aligned}\int_{\chi/l}^1 \theta^{1-\gamma} q'(\theta l)\, l d\theta &= \theta^{1-\gamma} q(\theta l)|_{\chi/l}^1 - \int_{\chi/l}^1 (1-\gamma)\theta^{-\gamma} q(\theta l) d\theta \\ &= q(l) - \left(\frac{\chi}{l}\right)^{1-\gamma} q(\chi) - (1-\gamma)\int_{\chi/l}^1 \theta^{-\gamma} q(\theta l) d\theta.\end{aligned}$$

Then,

$$l\frac{\partial A_L(l)}{\partial l} = q(l) - (1-\gamma)\int_{\chi/l}^1 \theta^{-\gamma} q(\theta l) d\theta = q(l) - (1-\gamma) A_L(l).$$

Similarly,

$$
\begin{aligned}
l\frac{\partial A_H(l)}{\partial l} &= q(l)+\gamma\int_{\psi/l}^{1}\theta^{-\gamma-1}q(\theta l)d\theta-\frac{q(0)}{\gamma}(-\gamma)\left(\frac{\psi}{l}\right)^{-\gamma-1}\left(-\frac{\psi}{l^2}\right)l\\
&= q(l)+\gamma\int_{\psi/l}^{1}\theta^{-\gamma-1}q(\theta l)d\theta-\gamma\frac{q(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}=q(l)+\gamma A_H(l)
\end{aligned}
$$

$$
l\frac{\partial B_L(l)}{\partial l}=q(l)-\gamma\int_0^1\theta^{\gamma-1}q(\theta l)d\theta=q(l)-\gamma B_L(l)
$$

$$
l\frac{\partial B_H(l)}{\partial l}=q(l)-(\gamma-1)\int_0^1\theta^{\gamma-2}q(\theta l)d\theta=q(l)-(\gamma-1)B_H(l).
$$

The derivatives can then be simplified as follows:

$$
l\hat{V}_L'(l) = K[-\gamma B_L(l)+(1-\gamma)A_L(l)] \tag{65}
$$

$$
l\hat{V}_H'(l) = K[(1-\gamma)B_H(l)-\gamma A_H(l)]. \tag{66}
$$

5. Smooth Pasting Conditions.

Boundary conditions (value matching and smooth pasting for low and high types) must be satisfied at $\bar{l}$. These conditions jointly determine $\bar{l}$, χ and ψ:[11]

$$
\hat{V}_L\left(\bar{l}\right)=\hat{V}_L'\left(\bar{l}\right)=\hat{V}_H'\left(\bar{l}\right)=0.
$$

Using the formal expressions of the value functions and derivatives,

$$
\begin{aligned}
\hat{V}_L(\bar{l}) &= 0\Rightarrow\int_{\chi/\bar{l}}^{1}\theta^{-\gamma}q\left(\theta\bar{l}\right)d\theta=\int_0^1\theta^{\gamma-1}q\left(\theta\bar{l}\right)d\theta,\\
\bar{l}\hat{V}_L'(\bar{l}) &= 0\Rightarrow(1-\gamma)\int_{\chi/\bar{l}}^{1}\theta^{-\gamma}q\left(\theta\bar{l}\right)d\theta=\gamma\int_0^1\theta^{\gamma-1}q\left(\theta\bar{l}\right)d\theta.
\end{aligned}
$$

Combining the two conditions, we find the equation that pins down $\bar{l}$:

$$
\int_0^1\theta^{\gamma-1}q\left(\theta\bar{l}\right)d\theta=0 \tag{67}
$$

[11]Value matching and smooth pasting conditions for low-quality trees arise from optimal exiting decisions, and the smooth pasting condition for high-quality trees arises from the belief process that is reflecting at $\bar{\phi}$.

and the equation that pins down χ:

$$\int_{\chi/\bar{l}}^{1} \theta^{-\gamma} q\left(\theta \bar{l}\right) d\theta = 0 \qquad \Rightarrow \qquad \chi = \bar{l}. \tag{68}$$

Finally, the condition that pins down ψ is

$$(1-\gamma)\int_0^1 \theta^{\gamma-2} q(\theta\bar{l})d\theta = \gamma\left[\int_{\psi/\bar{l}}^1 \theta^{-\gamma-1} q(\theta\bar{l})d\theta - \frac{q(0)}{\gamma}\left(\frac{\psi}{\bar{l}}\right)^{-\gamma}\right]. \tag{69}$$

These expressions completely characterized value functions and the reputation at which low-quality trees exit. Q.E.D.

Here we can make a digression. In proving Proposition 1, we show that $\underline{\phi} < \bar{\phi}$. Here we can see this property also holds in continuous time. The condition that pins down $\bar{\phi}$ is the one shown in equation (67). Similarly, under the pessimistic beliefs $\phi^c = \phi$, $\underline{\phi} = \frac{1}{1+\underline{l}}$ can be pinned down from the conditions

$$\begin{aligned} \hat{V}_H(\underline{l}) &= 0 \Rightarrow A_H(\underline{l}) = B_H(\underline{l}) \\ \underline{l}\hat{V}_H'(\underline{l}) &= 0 \Rightarrow \gamma A_H(\underline{l}) = (1-\gamma)B_H(\underline{l}). \end{aligned}$$

This implies $A_H(\underline{l}) = 0$, since the condition that pins down $\bar{l}$ is $A_L(\bar{l}) = 0$. Then,

$$\int_0^1 \theta^{\gamma-2} q\left(\theta \underline{l}\right) d\theta = \int_0^1 \theta\theta^{\gamma-2} q\left(\theta\bar{l}\right) d\theta = 0.$$

Since $\theta \in [0,1]$, then $q\left(\theta\underline{l}\right) < q\left(\theta\bar{l}\right)$ for all θ. Hence, $\underline{\phi} < \bar{\phi}$ strictly.

Lemma 11 *Ratio and Differences of Value Functions for linear payoffs*

The ratio of value functions $V_L(\phi)/V_H(\phi)$ is monotonically increasing in ϕ. The difference between value functions $V_H(\phi) - V_L(\phi)$ is increasing for low reputation levels and decreasing for high reputation levels.

Proof First we prove the ratio of value functions $V_L(\phi)/V_H(\phi)$ is monotonically increasing in ϕ. Then we prove the difference between value functions $V_H(\phi) - V_L(\phi)$ is increasing for low reputation levels and decreasing for high reputation levels.

1. Increasing Ratio $V_L(\phi)/V_H(\phi)$

The ratio $\frac{V_L(\phi^e)}{V_H(\phi^e)}$ is an increasing function of ϕ^e if and only if $\frac{V_L(l_0)}{V_H(l_0)}$ is a decreasing function of l_0. This is because $l_0 = \frac{1-\phi^e}{\phi^e}$, then $V_i'(\phi^e) = V_i'(l_0)\frac{\partial l_0}{\partial \phi^e} = -\frac{V_i'(l_0)}{\phi^{e^2}}$ and $\frac{\partial(V_L(\phi^e)/V_H(\phi^e))}{\partial\phi^e} = -\frac{1}{\phi^{e^2}}\frac{\partial(V_L(l_0)/V_H(l_0))}{\partial l_0}$.

First, we define the domain and image of the function. The lowest possible reputation in the market is $\bar{l}$, where $\hat{V}_L(\bar{l}) = 0$ and $\hat{V}_H(\bar{l}) > 0$. We also know that $\hat{V}_L(1) = \hat{V}_H(1) > 0$. Finally, $0 < \hat{V}_L(l) < \hat{V}_H(l)$ for all other $l_0 \in [0, \bar{l})$. This implies $\frac{\hat{V}_L(l_0)}{\hat{V}_H(l_0)}$ is a mapping from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

We show the ratio $\frac{\hat{V}_L(l)}{\hat{V}_H(l)}$ is monotonically decreasing in $l \in [0, \bar{l}]$. This is the case if

$$\frac{l\hat{V}_L'(l)}{\hat{V}_L(l)} < \frac{l\hat{V}_H'(l)}{\hat{V}_H(l)}.$$

Using the simplified expressions for the value functions, after some algebra, dropping the argument l, this condition implies

$$B_H \left[\left(1 - \gamma \frac{A_H}{B_H}\right)(B_L - A_L) + (2\gamma - 1)A_L \right] > A_H \left[\gamma(B_L - A_L) + (2\gamma - 1)A_L\right]. \quad (70)$$

We show the left-hand side of (70) is positive and the right-hand side of (70) is negative for all $l \in [0, \bar{l}]$; hence, the condition is always satisfied and the ratio of value functions decreasing in that range.

1.a. $B_H(l) > 0$ *for all* $l \in [0, \bar{l}]$

First, we develop the integrals $B_L(l)$ and $B_H(l)$.

Recall the profit function is linear in ϕ, ($y(\phi) = a_1\phi - a_0$) and $\phi = \frac{1}{1+l}$,

$$q(\theta l) = \frac{a_1}{1 + \theta l} - a_0$$

and consider the general solution to the following integral (see Abramowitz and Stegun (1972)),

$$\int \theta^m \left(\frac{a_1}{1 + \theta l} - a_0 \right) d\theta = a_1 \theta^{m+1} \Phi(-\theta l, 1, m + 1) - \frac{\theta^{m+1}}{m + 1} a_0,$$

where $\Phi(-\theta l, 1, m + 1)$ is a Hurwitz Lerch zeta-function.

Applying this result to B_L,

$$B_L(l) = \int_0^1 \theta^{\gamma-1} \left(\frac{a_1}{1 + \theta l} - a_0 \right) d\theta = \left[a_1 \theta^\gamma \Phi(-\theta l, 1, \gamma) - \frac{\theta^\gamma}{\gamma} a_0 \right]_0^1$$

$$B_L(l) = a_1 \Phi(-l, 1, \gamma) - \frac{a_0}{\gamma}$$

and similarly,

$$B_H = a_1\Phi(-l,1,\gamma-1) - \frac{a_0}{\gamma-1}.$$

Our strategy is to prove first $B_L(l) > 0$ for all $l \in [0,\bar{l}]$ and then to prove $B_H(l) > B_L(l)$ for all $l \in [0,\bar{l}]$.

Important properties of Herwitz Lerch zeta functions for the parameters we are considering ($\gamma \geq 1$) are (see Laurincikas and Garunkstis (2003)):

- $\Phi(0,1,\gamma) = \frac{1}{\gamma}$
- $\frac{\partial\Phi(-l,1,\gamma)}{\partial l} = \frac{1}{l}\left[\frac{1}{l+1} - \gamma\Phi(-l,1,\gamma)\right] < 0$
- $(\gamma-1)\Phi(-l,1,\gamma-1) > \gamma\Phi(-l,1,\gamma)$.

By construction, $B_L(\bar{l}) = 0$, hence $\Phi(\bar{l},1,\gamma) = \frac{a_0}{\gamma a_1}$. Given the properties above,

$$B_L(l) : [0,\bar{l}] \to [\frac{a_1 - a_0}{\gamma}, 0].$$

Furthermore, $B_L(l)$ is monotonically decreasing in the range

$B_H(l) > B_L(l)$ for all $l \in [0,\bar{l}]$ if

$$\gamma(\gamma-1)[\Phi(-l,1,\gamma-1) - \Phi(-l,1,\gamma)] > \frac{a_0}{a_1}.$$

Considering the third property above,

$$(\gamma-1)\Phi(-l,1,\gamma-1) > \Phi(-l,1,\gamma) + (\gamma-1)\Phi(-l,1,\gamma) > \frac{a_0}{\gamma a_1} + (\gamma-1)\Phi(-l,1,\gamma),$$

and hence, $B_H(l) > 0$ for all $l \in [0,\bar{l}]$.

1.b. $A_H(l) < 0$ *for all* $l \in [0,\bar{l}]$

We develop the integral $A_L(l)$ and $A_H(l)$ following the steps above:

$$A_L(l) = \int_{\chi/l}^{1} \theta^{-\gamma}\left(\frac{a_1}{1+\theta l} - a_0\right)d\theta = \left[a_1\theta^{1-\gamma}\Phi(-\theta l,1,1-\gamma) - \frac{\theta^{1-\gamma}}{1-\gamma}a_0\right]_{\chi/l}^{1}$$

$$A_L(l) = a_1\left[\Phi(-l,1,1-\gamma) - (\chi/l)^{1-\gamma}\Phi(-\chi,1,1-\gamma)\right] + \frac{a_0}{\gamma-1}\left(1-(\chi/l)^{1-\gamma}\right)$$

and,

$$A_H(l) = a_1 \left[\Phi(-l,1,-\gamma) - (\psi/l)^{-\gamma}\Phi(-\psi,1,-\gamma)\right] + \frac{a_0}{\gamma} - \frac{a_1}{\gamma}(\psi/l)^{-\gamma}.$$

Consider $A_H(0) = A_H(\psi) = -\frac{a_1-a_0}{\gamma} < 0$. We show that, if the function grows, the maximum is still negative. That is, we prove that $A_H(\hat{l}) < 0$, where $\hat{l} = argmax A_H(l)$ (hence $\frac{\partial A_H(l)}{\partial l}|_{l=\hat{l}} = 0$):

$$\frac{\partial A_H(l)}{\partial l} = \frac{a_1}{l}\left[\left(\frac{1}{1+l} + \gamma\Phi(-l,1,-\gamma)\right) - \gamma(l/\psi)^{\gamma}\Phi(-\psi,1,-\gamma)\right] - \frac{a_1}{l}(l/\psi)^{\gamma}.$$

The condition satisfied at $l\frac{\partial A_H(l)}{\partial l} = 0$ is

$$\left[\Phi(-l,1,-\gamma) - (\psi/l)^{-\gamma}\Phi(-\psi,1,-\gamma)\right] = \frac{1}{\gamma}(l/\psi)^{\gamma} - \frac{1}{1+l}.$$

Evaluating $A_H(\hat{l})$ considering that condition

$$a_1\left[\frac{1}{\gamma}(l/\psi)^{\gamma} - \frac{1}{1+l}\right] + \frac{a_0}{\gamma-1}\left(1-(\chi/l)^{1-\gamma}\right) < 0$$

since

$$\gamma a_1 \frac{1}{1+l} > a_0.$$

Hence, $A_H(l) < 0$ for all $l \in [0,\bar{l}]$.

Finally, just for completeness, $A_L(0) = -\frac{a_1-a_0}{\gamma-1} < 0$, $A_L(\chi) = 0$ because $\chi/l = 1$ and $A_L(\bar{l}) = 0$ by construction. It can be further shown that $A_L(l) < 0$ for all $l \in (0,\chi)$ and $A_L(l) > 0$ for all $l \in (\chi,\bar{l})$.

1.c. $\gamma(B_L(l) - A_L(l)) + (2\gamma-1)A_L(l) > 0$ *for all* $l \in [0,\bar{l}]$

Recall $\gamma(B_L - A_L) + (2\gamma-1)A_L = \gamma B_L + (\gamma-1)A_L = -\frac{l\hat{V}_L'(l)}{K}$.

By construction $\gamma B_L + (\gamma-1)A_L = 0$ at $l = 0$ and $l = \bar{l}$.

For $l \in (\chi,\bar{l})$, since $A_L(l) \geq 0$ and $B_L(l) > 0$, $\gamma B_L + (\gamma-1)A_L > 0$. In particular, at $\bar{l}$, $A_L(\chi) = 0$ and $\gamma B_L(\chi) > 0$.

As shown above, for $l \in [0,\chi]$, $B_L(l) > 0$ monotonically increasing and $A_L(l) < 0$ monotonically increasing. This implies $\gamma B_L + (\gamma-1)A_L$ goes monotonically from 0 at $l = 0$ to $\gamma B_L(\bar{l}) > 0$ and hence positive in the whole range.

1.d. $\left[\left(1-\gamma\frac{A_H(l)}{B_H(l)}\right)(B_L(l) - A_L(l)) + (2\gamma-1)A_L(l)\right] > 0$ *for all* $l \in [0,\bar{l}]$

First, recall $(\gamma-1)B_H+\gamma A_H = -\frac{l\hat{V}_H'(l)}{K}$. Hence, as in the point above, $(\gamma-1)B_H+\gamma A_H = 0$ at $l=0$ and $l=\bar{l}$ by construction, which we can rewrite as $1-\gamma\frac{A_H(0)}{B_H(0)} = 1-\gamma\frac{A_H(\bar{l})}{B_H(\bar{l})} = \gamma$. Hence at these two extreme points, the term in the left-hand side is zero, the same as the one in the right-hand side.

More generally $(\gamma-1)B_H+\gamma A_H > 0$ (and then $1 < 1-\gamma\frac{A_H(l)}{B_H(l)} < \gamma$). Since $A_L(\chi)=0$, $\left(1-\gamma\frac{A_H(l)}{B_H(l)}\right) B_L(l) > 0$. This part of the proof follows from the same monotonicity arguments above.

2. Non-monotonic Difference $V_H(\phi) - V_L(\phi)$

First, $\hat{V}_L'(\bar{\phi}) = \hat{V}_H'(\bar{\phi}) = 0$ by construction and $\hat{V}_L'(1) = \hat{V}_H'(1) = 0$, from the expressions above. Second $\hat{V}_L'(\phi)$ and $\hat{V}_H'(\phi)$ are positive for all $\phi \in (\bar{\phi}, 1)$. Third, these derivatives are single peaked and the reputation that maximizes $\hat{V}_H'(\phi)$ is lower than the reputation that maximizes $\hat{V}_L'(\phi)$. Finally, $\hat{V}_H''(\bar{\phi}) > \hat{V}_L''(\bar{\phi})$ and $\hat{V}_H''(1) < \hat{V}_L''(1)$, which means the two derivatives cross only one time, at ϕ^*. These properties arise from inspection of the derivatives of linear profits value functions and from properties of the hypergeometric functions that characterize them.

Q.E.D.

A.4 Proof Proposition 3

We split the proof in two parts.

1) As the precision of signals go to zero.

In this case, to prove the steady-state spot market equilibrium converges to the benchmark without information ($Y \to 0$), it is enough to prove that $V_H(\phi) \to V_L(\phi)$ for all ϕ. This is because, from equation (26), $Y \to 0$ as $V_H(\phi^e) \to V_L(\phi^e)$.

In the bad and good news cases, the precision of signals is zero when $\lambda = 0$, hence there are no news about the true type of the firm. It is trivial to see, from Propositions 3 and 5, that $V_L(\phi) = V_H(\phi) = q(\phi)/\hat{r}$ for all ϕ when $\lambda = 0$.

In the Brownian motion case, the precision of signals is zero when the signal to noise ratio $\frac{\mu_H-\mu_L}{\sigma} = 0$, and then $\rho = \infty$. From the ODEs (49) and (50), $V_H(\phi) = V_L(\phi) = q(\phi)/\hat{r}$.

2) As the precision of signals go to infinity.

In this case, to prove that the steady-state spot market equilibrium converges to the unconstrained first best benchmark with perfect information ($Y \to \bar{Y}$), it is enough to prove that $V_H(\phi) \to V_H(1)$ and $V_L(\phi) \to 0$ for all $\phi > 0$, as precision goes to infinity.

This is because, from Proposition 1, $V_H(\phi^e)U'(Y) = C$. As precision goes to infinity, low-quality firms exit fast, and the reputation at entry does not matter to determine the average quality of firms in steady-state (that is, $\frac{m(1)}{m(0)+m(1)} \to 1$ regardless of $\bar{\phi} > 0$).

In the bad and good news cases, the precision of signals is infinity when $\lambda = \infty$, hence news about the true type of the firm arrive immediately. In this case, low-quality firms spend almost no time with a reputation different than 0. From Propositions 3 and 5, it is straightforward to check that $V_H(\phi) = q(1)/r$ and $V_L(\phi) = 0$ for $\lambda = \infty$ and all $\phi > 0$. Even when $\bar{\phi} < 1$, since all low-quality firms almost instantaneously leave the market when $\lambda \to \infty$, effectively $\frac{m(1)}{m(0)+m(1)} \to 1$ in the market.

In the Brownian motion case, the precision of signals is infinite when the signal-to-noise ratio $\frac{\mu_H - \mu_L}{\sigma} = \infty$. Then $\rho = 0$ and $\gamma = 1$. From evaluating equation (50) at l with $\rho = 0$, $V''_H(l) = 0$ for all l. Combining this result with equations (69) and the definition of $V'_H(l)$ in the Appendix, $V'_H(\bar{l}) = 0$. This implies that $V_H(\bar{\phi}) = V_H(1)$, and then the production of the experience good is $\bar{Y}$. Furthermore, even when $\bar{\phi} < 1$, since all low-quality firms almost instantaneously leave the market, effectively $\frac{m(1)}{m(0)+m(1)} \to 1$ in the market.

A.5 Proof Proposition 4

Specifically, under full information, the steady-state measure of high-quality trees is $\bar{m}_H = \bar{Y}/y(1)$, so the rate of entry of high-quality trees is $\phi^e \bar{m}^e = \frac{(1-\exp(-\delta\Delta))}{\Delta} \frac{\bar{Y}}{y(1)}$ and the associated consumption of the numeraire good is $\bar{N} = 1 - \frac{(1-\exp(-\delta\Delta))}{\Delta} \frac{\bar{Y}}{y(1)} C$. In continuous time, $\bar{N} = 1 - \delta \frac{\bar{Y}}{y(1)} C$.

In the equilibrium with regulation, a fraction $(1 - \phi^e)$ of entering trees are low-quality in steady state, since we proved $\phi^e < 1$ in equilibrium. Hence, there is a fraction of all active trees that is low-quality, where this fraction is positive. Denote the equilibrium steady-state ratio of low- to high-quality trees by $\hat{m}_L/\hat{m}_H$, also positive. From the resource constraint for the experience good (3) in steady state, to produce output $\bar{Y}$ there must be a stock of $\hat{m}_H$ high-quality trees given by

$$\hat{m}_H = \frac{\bar{Y}}{y(1) + y(0)\hat{m}_L/\hat{m}_H} > \bar{m}_H,$$

and a steady-state entry rate of high-quality trees of $\phi^e m^e = \frac{(1-\exp(-\delta\Delta))}{\Delta} \hat{m}_H$ is required to maintain that production. As a result of this required elevated rate of entry of high-quality trees, consumption of numeraire good is $N = 1 - \frac{(1-\exp(-\delta\Delta))}{\Delta} \hat{m}_H C < \bar{N}$. The gap between N and $\bar{N}$ can be made arbitrarily small by choosing F to set ϕ^e as close to one as is required to drive $\hat{m}_L/\hat{m}_H$ sufficiently close to zero and hence drive $\hat{m}_H$ sufficiently close to $\bar{m}_H$.

A.6 Proof Proposition 5

By choosing an entry cost $F \geq 0$, a social planner can implement any steady-state allocation (Y, N, ϕ^e, m^e) that satisfies the following constraints:

$$Y = \left(\tilde{V}_H(\phi^e)\phi^e + \tilde{V}_L(\phi^e)(1-\phi^e)\right) m^e$$

$$N = 1 - C\phi^e m^e$$

$$U'(Y)\left(V_H(\phi^e) - V_L(\phi^e)\right) = C$$

$$\phi^e \geq \bar{\phi},$$

where $\tilde{V}_i(\phi^e)$ is defined as the value function imposing $r = 0$ and respecting the evolution of reputation and exiting in equilibrium. Why is Y constructed with these redefined value functions the right measure of steady-state output? This is because, when the interest rate is zero, then the expected discounted value of profits (output since marginal cost is zero) for all trees at entry is equal to the integral across the cross section of profits in the steady state. What about endogenous exit? The computation of the value functions takes the impact of endogenous exit on the cross section of output into account.

For notation purposes, define $\tilde{y}(\phi^e)$ by

$$\tilde{y}(\phi^e) = \tilde{V}_H(\phi^e)\phi^e + \tilde{V}_L(\phi^e)(1-\phi^e).$$

Hence, $Y = \tilde{y}(\phi^e)m^e$ and

$$\tilde{y}'(\phi^e) = \tilde{V}_H(\phi^e) - \tilde{V}_L(\phi^e) + \tilde{V}_H'(\phi^e)\phi^e + \tilde{V}_L'(\phi^e)(1-\phi^e).$$

Also define the implicit function $m^e(\phi^e)$ as the solution to the constraint

$$U'(\tilde{y}(\phi^e)m^e(\phi^e)))\left(V_H(\phi^e) - V_L(\phi^e)\right) = C.$$

Note that the derivatives of the original objective with respect to the two arguments ϕ^e and m^e are, with respect to ϕ^e,

$$U'(Y)\tilde{y}'(\phi^e)m^e - Cm^e,$$

and with respect to m^e,

$$U'(Y)\tilde{y}(\phi^e) - C\phi^e,$$

and that the derivatives of the constraint with respect to the two arguments are, with respect to ϕ^e,

$$U''(Y)\tilde{y}'(\phi^e)m^e\left(V_H(\phi^e) - V_L(\phi^e)\right) + U'(Y)\left(V_H'(\phi^e) - V_L'(\phi^e)\right),$$

and with respect to m^e,

$$U''(Y)\tilde{y}(\phi^e)\left(V_H(\phi^e) - V_L(\phi^e)\right).$$

These two derivatives of the constraints imply that

$$m^{e\prime}(\phi^e) = -\frac{U''(Y)\tilde{y}'(\phi^e)m^e\left(V_H(\phi^e) - V_L(\phi^e)\right) + U'(Y)\left(V_H'(\phi^e) - V_L'(\phi^e)\right)}{U''(Y)\tilde{y}(\phi^e)\left(V_H(\phi^e) - V_L(\phi^e)\right)}.$$

Note that in the Brownian motion and good news cases, $V_H'(\bar{\phi}) = V_L'(\bar{\phi}) = 0$ due to smooth pasting, so at $\phi^e = \bar{\phi}$

$$\frac{m^{e\prime}(\bar{\phi})}{m^e(\bar{\phi})} = -\frac{\tilde{y}'(\bar{\phi})}{\tilde{y}(\bar{\phi})}.$$

Note that this statement is simply that Y stays constant due to smooth pasting.

Now consider the change in the objective at $\phi^e = \bar{\phi}$. Using the derivatives above, this is given by

$$\left(U'(Y)\tilde{y}'(\bar{\phi})m^e(\bar{\phi}) - Cm^e(\bar{\phi})\right)d\phi^e + \left(U'(Y)\tilde{y}((\bar{\phi})) - C\bar{\phi}\right)m^{e\prime}(\bar{\phi})d\phi^e.$$

Using the constraint itself and dividing by m^e, we have that this derivative has the same sign as the following expression:

$$\tilde{y}'(\bar{\phi}) - \left(V_H(\bar{\phi}) - V_L(\bar{\phi})\right) + \left(\tilde{y}((\bar{\phi})) - \bar{\phi}\left(V_H(\bar{\phi}) - V_L(\bar{\phi})\right)\right)\frac{m^{e\prime}(\bar{\phi})}{m^e(\bar{\phi})}.$$

Using the expression for the derivative of m^e, we then have that this derivative has the same sign as

$$\tilde{y}'(\bar{\phi}) - \frac{\tilde{y}(\bar{\phi})}{\bar{\phi}} = \tilde{V}_H'(\bar{\phi})\bar{\phi} + \tilde{V}_L'(\bar{\phi})(1-\bar{\phi}) - \frac{\tilde{V}_L(\bar{\phi})}{\bar{\phi}}.$$

In order to use the value functions derived above with the transformed variable $\bar{l} = \frac{1-\bar{\phi}}{\bar{\phi}}$, recall $\tilde{V}_i'(\bar{\phi}) = \tilde{V}_i'(\bar{l})\frac{\partial \bar{l}}{\partial \bar{\phi}} = -\frac{\tilde{V}_i'(\bar{l})}{\bar{\phi}^2}$. Then, multiplying the whole expression by ϕ^e, we can reexpress the change in welfare evaluated at $\phi^e = \bar{\phi}$ as

$$-[\tilde{V}_H'(\bar{l}) + \bar{l}\tilde{V}_L'(\bar{l}) + \tilde{V}_L(\bar{l})].$$

Brownian Motion

Value functions $\tilde{V}_L(\phi)$ and $\tilde{V}_H(\phi)$ and their derivatives have the same structure than equations (63) and (65), but since $r = 0$, they are computed with $1 \leq \bar{\gamma} < \gamma$ and $\bar{K} > K$ and the same χ and ψ that were computed under $r > 0$.

Since $\chi = \bar{l}$, $\bar{A}_L(\bar{l}) = A_L(\bar{l}) = 0$. Similarly,

$$\bar{B}_L(\bar{l}) = \int_0^1 \theta^{\bar{\gamma}-1} q(\theta\bar{l}) d\theta > B_L(\bar{l}) = 0$$

since $\frac{\partial \theta^{\gamma-1}}{\partial \gamma} = \theta^{\gamma-1}\ln(\theta) < 0$ for all θ. This implies that

$$\tilde{V}_L(\bar{l}) > V_L(\bar{l}) = 0$$

and

$$\bar{l}\tilde{V}'_L(\bar{l}) < \bar{l}V'_L(\bar{l}) = 0$$

Combining the last two terms

$$\bar{l}\tilde{V}'_L(\bar{l}) + \tilde{V}_L(\bar{l}) = \bar{K}[(1-\bar{\gamma})\bar{B}_L - \bar{\gamma}\bar{A}_L] < 0.$$

Similarly,

$$\bar{l}\tilde{V}'_H(\bar{l}) < \bar{l}V'_H(\bar{l}) = 0,$$

which implies that the change in the welfare function at $\bar{\phi}$ is positive.

Bad News

From Lemmas 3 and 4, and by definition of $\bar{\phi}$, the change in welfare evaluated at $\bar{\phi}$ is strictly positive for all $\hat{r} \geq 0$. In particular, for $\hat{r} = 0$,

$$\tilde{V}_L(\bar{l}) = \frac{1}{\lambda}\left[a_1 \frac{\log(1+\bar{l})}{\bar{l}} - a_0\right] > 0$$

$$\tilde{V}'_L(\bar{l}) = \frac{a_1}{\lambda}\left[\frac{1}{\bar{l}(1+\bar{l})} - \frac{\log(1+\bar{l})}{\bar{l}^2}\right] < 0$$

and

$$\tilde{V}'_H(\bar{l}) = -\frac{a_1}{a_0(1+\bar{l})} < 0,$$

Then

$$\tilde{V}'_H(\bar{l}) + \bar{l}\tilde{V}'_L(\bar{l}) + \tilde{V}_L(\bar{l}) = -\frac{a_0}{\lambda} + \left(\frac{a_1}{\lambda} - \frac{a_1}{a_0}\right)\bar{\phi} < 0$$

since $a_1\bar{\phi} - a_0 < 0$.

Good News

From equation (41), and by definition of $\bar{\phi}$, $\tilde{V}'_H(\bar{\phi}) = \tilde{V}'_L(\bar{\phi}) = \tilde{V}_L(\bar{\phi}) = 0$, then the first-order change of the objective function at $\bar{\phi}$ is zero. Here we show the second-order change of the objective function at $\bar{\phi}$ is always positive, implying that $\bar{\phi}$ is a constrained local minimum, and welfare always increases initially, and then attaining an entry reputation ϕ^e above $\bar{\phi}$, using $F > 0$, is always beneficial.

The second derivative of the objective function with respect to ϕ^e is given by

$$\begin{aligned} & U''(Y)\left(\tilde{y}'(\phi^e)m^e(\phi^e) + \tilde{y}(\phi^e)m^{e\prime}(\phi^e)\right)^2 \\ + \quad & U'(Y)\left(\tilde{y}''(\phi^e)m^e(\phi^e) + 2\tilde{y}'(\phi^e)m^{e\prime}(\phi^e) + \tilde{y}(\phi^e)m^{e\prime\prime}(\phi^e)\right) \\ - \quad & C\left(2m^{e\prime}(\phi^e) + \phi^e m^{e\prime\prime}(\phi^e)\right). \end{aligned}$$

In the good news case, because of smooth pasting, at $\phi^e = \bar{\phi}$, $dY = 0$, or equivalently

$$\tilde{y}'(\phi^e)m^e(\phi^e) + \tilde{y}(\phi^e)m^{e\prime}(\phi^e) = 0.$$

Hence, the first term in this second derivative is zero at $\phi^e = \bar{\phi}$. Dividing the remaining terms by $U'(Y)$ (which is strictly positive) and substituting in the entry constraint, we have that the second derivative of social welfare at $\phi^e = \bar{\phi}$ has the same sign as the expression

$$\begin{aligned} & \tilde{y}''(\bar{\phi})m^e(\bar{\phi}) + 2\tilde{y}'(\bar{\phi})m^{e\prime}(\bar{\phi}) + \tilde{y}(\bar{\phi})m^{e\prime\prime}(\bar{\phi}) \\ - \quad & \left(V_H(\bar{\phi}) - V_L(\bar{\phi})\right)\left(2m^{e\prime}(\bar{\phi}) + \bar{\phi}m^{e\prime\prime}(\bar{\phi})\right). \end{aligned} \tag{71}$$

To sign this term, recall

$$\tilde{y}(\phi^e) = \phi^e\tilde{V}_H(\phi^e) + (1-\phi^e)\tilde{V}_L(\phi^e)$$

and

$$\tilde{y}'(\phi^e) = \tilde{V}_H(\phi^e) - \tilde{V}_L(\phi^e) + \phi^e\tilde{V}_H'(\phi^e) + (1-\phi^e)\tilde{V}_L'(\phi^e)$$

and second derivatives are

$$\tilde{y}''(\phi^e) = 2\left(\tilde{V}_H'(\phi^e) - \tilde{V}_L'(\phi^e)\right) + \phi^e\tilde{V}_H''(\phi^e) + (1-\phi^e)\tilde{V}_L''(\phi^e).$$

Evaluating these expression at $\bar{\phi}$

$$\tilde{y}(\bar{\phi}) = \bar{\phi}\tilde{V}_H(\bar{\phi}) > 0 \tag{72}$$

$$\tilde{y}'(\bar{\phi}) = \tilde{V}_H(\bar{\phi}) > 0 \tag{73}$$

and

$$\tilde{y}''(\bar{\phi}) = \bar{\phi}\tilde{V}_H''(\bar{\phi}) + (1-\bar{\phi})\tilde{V}_L''(\bar{\phi}) > 0, \tag{74}$$

where this last inequality follows from equation (42).

Now consider the derivatives of $m^e(\phi^e)$. This function is defined implicitly by the constraint

$$U'(\tilde{y}(\phi^e)m^e(\phi^e))\left(V_H(\phi^e) - V_L(\phi^e)\right) = C.$$

Thus, the first derivative $m^{e\prime}(\phi^e)$ is solved from

$$U''(Y)\left(\tilde{y}'(\phi^e)m^e(\phi^e)+\tilde{y}(\phi^e)m^{e\prime}(\phi^e)\right)\left(V_H(\phi^e)-V_L(\phi^e)\right)+U'(Y)\left(V_H'(\phi^e)-V_L'(\phi^e)\right)=0$$

and the second derivative $m^{e\prime\prime}(\phi^e)$ is solved from

$$\begin{aligned} & U'''(Y)\left(\tilde{y}'(\phi^e)m^e(\phi^e)+\tilde{y}(\phi^e)m^{e\prime}(\phi^e)\right)^2\left(V_H(\phi^e)-V_L(\phi^e)\right) \\ + \; & U''(Y)\left(\tilde{y}''(\phi^e)m^e(\phi^e)+2\tilde{y}'(\phi^e)m^{e\prime}(\phi^e)+\tilde{y}(\phi^e)m^{e\prime\prime}(\phi^e)\right)\left(V_H(\phi^e)-V_L(\phi^e)\right) \\ + \; & 2U''(Y)\left(\tilde{y}'(\phi^e)m^e(\phi^e)+\tilde{y}(\phi^e)m^{e\prime}(\phi^e)\right)\left(V_H'(\phi^e)-V_L'(\phi^e)\right) \\ + \; & U'(Y)\left(V_H''(\phi^e)-V_L''(\phi^e)\right)=0. \end{aligned}$$

Again, from equation (41), at $\phi^e=\bar{\phi}$, we have $V_L(\bar{\phi})=0$, $V_H'(\bar{\phi})=V_L'(\bar{\phi})=0$, and $dY=0$ so we have that $m^{e\prime}(\bar{\phi})$ is the solution to

$$\tilde{y}'(\bar{\phi})m^e(\bar{\phi})+\tilde{y}(\bar{\phi})m^{e\prime}(\bar{\phi})=0 \tag{75}$$

and $m^{e\prime\prime}(\bar{\phi})$ is the solution to

$$U''(Y)\left(\tilde{y}''(\bar{\phi})m^e(\bar{\phi})+2\tilde{y}'(\bar{\phi})m^{e\prime}(\bar{\phi})+\tilde{y}(\bar{\phi})m^{e\prime\prime}(\bar{\phi})\right)\left(V_H(\bar{\phi})-V_L(\bar{\phi})\right)+$$
$$U'(Y)\left(V_H''(\bar{\phi})-V_L''(\bar{\phi})\right)=0,$$

which implies that

$$\tilde{y}''(\bar{\phi})m^e(\bar{\phi})+2\tilde{y}'(\bar{\phi})m^{e\prime}(\bar{\phi})+\tilde{y}(\bar{\phi})m^{e\prime\prime}(\bar{\phi})=\frac{U'(Y)}{U''(Y)}\frac{V_H''(\bar{\phi})-V_L''(\bar{\phi})}{V_H(\bar{\phi})-V_L(\bar{\phi})}.$$

Since $V_H''(\bar{\phi})=V_L''(\bar{\phi})$, from equation (42),

$$\tilde{y}''(\bar{\phi})m^e(\bar{\phi})+2\tilde{y}'(\bar{\phi})m^{e\prime}(\bar{\phi})+\tilde{y}(\bar{\phi})m^{e\prime\prime}(\bar{\phi})=0. \tag{76}$$

With this last expression, we can rewrite (71) as

$$-\left(V_H(\bar{\phi})-V_L(\bar{\phi})\right)\left(2m^{e\prime}(\bar{\phi})+\bar{\phi}m^{e\prime\prime}(\bar{\phi})\right).$$

Hence, we have that the second derivative of welfare at $\phi^e=\bar{\phi}$ has the same sign as

$$-2m^{e\prime}(\bar{\phi})-\bar{\phi}m^{e\prime\prime}(\bar{\phi}).$$

Using (75) we have that this expression equals

$$2\frac{\tilde{y}'(\bar{\phi})m^e(\bar{\phi})}{\tilde{y}(\bar{\phi})}-\bar{\phi}m^{e\prime\prime}(\bar{\phi}).$$

Using (76) we have that

$$m^{e\prime\prime}(\bar{\phi}) = -\frac{\tilde{y}''(\bar{\phi})m^e(\bar{\phi})}{\tilde{y}(\bar{\phi})} - 2\frac{\tilde{y}'(\bar{\phi})m^{e\prime}(\bar{\phi})}{\tilde{y}(\bar{\phi})}.$$

Using (75) again, we have that

$$m^{e\prime\prime}(\bar{\phi}) = -\frac{\tilde{y}''(\bar{\phi})m^e(\bar{\phi})}{\tilde{y}(\bar{\phi})} + 2\left(\frac{\tilde{y}'(\bar{\phi})}{\tilde{y}(\bar{\phi})}\right)^2 m^e(\bar{\phi}).$$

Hence, we have that the second derivative of welfare at $\phi^e = \bar{\phi}$ has the same sign as

$$2\frac{\tilde{y}'(\bar{\phi})}{\tilde{y}(\bar{\phi})} + \bar{\phi}\frac{\tilde{y}''(\bar{\phi})}{\tilde{y}(\bar{\phi})} - 2\bar{\phi}\left(\frac{\tilde{y}'(\bar{\phi})}{\tilde{y}(\bar{\phi})}\right)^2.$$

Using (72) and (73), we can write this expression as

$$\frac{2}{\bar{\phi}} + \bar{\phi}\frac{\tilde{y}''(\bar{\phi})}{\tilde{y}(\bar{\phi})} - \frac{2}{\bar{\phi}} > 0.$$

Then $\bar{\phi}$ is a constrained local minimum, and welfare always increases initially.