Sequential auctions with information about future goods

Robert Zeithammer

October 6, 2008

Abstract: When capacity-constrained bidders have information about a good sold in a future auction, they need to take the information into account in forming today’s bids. The capacity constraint makes even otherwise unrelated objects substitutes, and creates an equilibrium link between future competition and current bidding strategy. This paper proves existence and uniqueness of a symmetric pure-strategy equilibrium under mild conditions on the population distribution of valuations, characterizes general properties of the equilibrium bidding strategy, and provides a simple technique for numerically approximating the bidding strategy for arbitrary valuation distributions. The key property of the equilibrium is that almost all bidders submit positive bids in the first stage, thereby ensuring trade with probability one. Even bidders who strongly prefer the second object submit a positive bid in the first auction, because losing the first auction is informative about the remaining competitors who also lost, and losing with a low bid indicates that these competitors are quite strong. Because of the guaranteed trade, the sequential auction with information about future goods is a very efficient trading mechanism, achieving over 98 percent of the potential gains from trade across a wide variety of settings.

I would like to thank Birger Wernerfelt, Sergei Izmalkov, Sushil Bikhchandani, and seminar participants at UT Austin and CU Boulder for suggestions and help with this paper. Contact info: Robert Zeithammer, UCLA Anderson School of Management, 110 Westwood Plaza, Los Angeles, CA 90095. Email: rzeitham@ucla.edu
Introduction

In many auction markets, similar goods are auctioned off in a pre-announced sequence: Governments and large firms auction off contracts for similar projects on a regular basis, often announcing future projects well in advance to allow bid-preparation. Estate auctions and auctions for property of retiring farmers are usually conducted as a sequence of auctions for the individual objects, with the entire set of auctioned goods available for inspection before the first auction begins. Consumers face sequential auctions with information about future goods on eBay, where upcoming auction-endings are listed several days in advance. In all of the above situations, the individual auctioned goods are not necessarily identical, but they are substitutes to the bidders because of capacity constraints: a consumer buying a digital camera on eBay only has use for one camera, a construction firm can only fulfill a limited number of contracts given its technology, and a farmer may only need one tractor out of the several different tractors auctioned off in the sale of a retiring neighbor’s machinery.

This paper models the sequential auctioning of two objects to unit-capacity bidders. Each object is auctioned by a second-price sealed-bid auction with zero reserve. Each bidder has two non-negative private valuations, one for each object, and the pairs of valuations are drawn independently across bidders from an arbitrary bivariate distribution with a bounded density and full support on a compact rectangle. This distributional assumption is very general because not only can the two valuations be arbitrarily correlated, the distribution need not be symmetric in either support or shape, and it can even be discontinuous. Departing from most previous models of sequential auctions, each bidder knows her private value of the second unit before bidding on the first unit. The main finding of this paper is that there exists a unique symmetric pure-strategy Bayes-Nash equilibrium of the game, and the equilibrium first-stage bidding function is easy to compute numerically thanks to a contraction property of the fixed-point relationship that characterizes the equilibrium. In the equilibrium, no bidders want to abstain from the first auction. This is sharply different from a best-response intuition about a rational bidder facing exogenous competition in the second stage: an exogenous second-stage competition would make a rational bidder abstain from the first auction whenever her option value of losing, i.e. her expected second-stage surplus, exceeded her valuation of the first unit, i.e. her maximum possible first-stage surplus.
There are no first-period abstentions in equilibrium because the option value of the second auction is endogenous to first-period bidding, i.e. directly linked to the first-period bidding strategy through the equilibrium requirements. Let the two stages of the game be “today” and “tomorrow”. The key tradeoff faced by the bidders is between winning today and bidding tomorrow (today’s winner exits the game because of the unit capacity assumption). As in other models of sequential auctions for substitutes, the optimal bidding strategy solves this tradeoff by reducing today’s bid to compensate for the opportunity cost of winning today, which is equal to a loser’s expected surplus from participating in tomorrow’s auction. The difference in the present model is that should I lose today, the competition I will face tomorrow depends on the bid-level at which I lose today: In equilibrium, all bidders reduce their bid today as a function of their values of tomorrow’s object, so losing to a lower bid today makes higher competition tomorrow more likely. At the margin, I therefore need to assess the opportunity cost of winning today not only as a function of my valuation of tomorrow’s object, but also as a function of the bid I submit today. In other words, losing at different bid-levels today is informative about the competition I should expect tomorrow. This paper analyzes this dependence, and shows that it makes any pure abstention strategy unravel.

The received theory of sequential auctions for substitutes focuses either on auctions of several identical units of a good (Milgrom and Weber 2000, Black and de Meza 1992, Katzman 1999), or on auctions of heterogeneous goods without information about future goods (Engelbrecht-Wiggans 1994). When bidders demand more than one identical unit of the same good and have diminishing marginal utility, Black and de Meza (1992) and Katzman (1999) show that even multi-unit-demand bidders will make their first-stage bids contingent only on their valuations of the first unit. In contrast, the bidders in the proposed model always base their first-stage bids on both valuations. Nevertheless, the reasoning based on endogenous option value of the second auction is analogous: In equilibrium, bidders with diminishing marginal utility assume that should they lose the first auction, they will lose to another bidder with the same first-unit valuation, and that competitor is therefore guaranteed to win the second unit.

The proposed model generalizes Milgrom and Weber (2000) by separating the influence of information about future goods from the relationship between the private values of the two objects sold. Unlike in Milgrom and Weber’s model, bidder types are two-dimensional here because first-stage bids are a function of both valuations. Models with multi-dimensional types
are rare in the auction literature because of the technical difficulties they pose. A seminal example is the model of Che and Gale (1998), who study the case of single-object auctions with each bidder facing a privately-known budget constraint in addition to her valuation. The proposed model also generalizes Gale and Hausch (1994), who consider two-dimensional information analogous to the present setting, but solve the model only for two bidders. The extension to more bidders is non-trivial because it guarantees a non-degenerate second-stage competition to the losers of the first auction, and leads to the endogeneity of this competition described above. There is also a qualitative difference in the conclusions – the probability of first-stage trade is less than one with only two bidders. Finally, a version of the proposed model with free disposal (discussed in the Extensions section) is related and complementary to Burguet (2005), who models a sequential ascending right-to-choose auction for two condominiums – one facing East and one facing West. In Burguet’s model, each bidder wants only one condo and not all bidders prefer the same exposure. However, their private information is not truly two-dimensional because sorting the bidders according to their valuations of their preferred exposure automatically sorts them according to their valuations of their less-preferred exposure (the valuation of the less-preferred exposure is a publicly-known monotonic function of the preferred-exposure valuation). This restriction is enough to make a sequential right-to-choose auction efficient – a result that does not necessarily hold in the general setting considered here.

An efficient allocation of the two objects can obviously be achieved by the appropriate Vickrey-Clarke-Groves mechanism (VCG). Budish (2008) uses the computational procedure proposed in this paper to assess the efficiency of the sequential auction with information about future goods. He finds that under several different assumptions about the distribution of bidder valuations, the sequential auction is over 99 percent efficient. Besides applying the computational procedure under a range of distributional assumptions, he also extends the theory by showing that revealing the information about future goods always increases efficiency. In this paper, the computational procedure is used to investigate the effect of within-bidder correlation in valuations on efficiency and distribution. Correlation between the private values of the two objects captures the similarity of the two objects, and hence the degree to which they are substitutes in addition to the substitution forced by the capacity constraint. Consider two fundraiser dinners held simultaneously. When one dinner is for the Republican party and the other for the Democratic party, the personal valuations of the bidders are likely negatively
correlated. In other words, the events are opposites and probably would not be substitutes without the capacity constraints. When one dinner is for the Republican party and the other for a cause unrelated to politics, such as cancer research, then the valuations are uncorrelated – demand for one good is unrelated to demand for the other. Finally, when the two concurrent dinners are both for the Republican party, the valuations are nearly perfectly correlated, and the two events would likely be substitutes even without the capacity constraint. Regardless of the cause supported by each dinner, the two events are substitutes in the model proposed here because of the capacity constraints – each participant can only attend one of them. By simulating arbitrarily correlated goods and computing the bidding functions, I find that the correlation between the valuations decreases both the overall efficiency and the portion of the gains from trade collected by the seller as revenue. Therefore, the auctions for objects with more positively correlated valuations are actually less competitive for the bidders. While the seller revenues are sensitive to the correlation, the overall efficiency varies only slightly as correlation changes and remains over 98 percent of maximum achievable gains from trade. These findings extend to correlated distributions the main conclusion of Budish (2008) that the auction studied here is a very efficient mechanism for selling arbitrary different objects to capacity-constrained bidders.

The paper is organized as follows: Section II presents the model, establishes existence and uniqueness of the equilibrium, and derives properties of the bidding function. Section III then presents the numerical procedure for computing the bidding function, illustrates the model on a special case of the uniform distribution and three bidders, and analyzes the effect of correlated valuations on efficiency and distribution. Section IV discusses both possible extensions and barriers to further generalization. The most important extension is to the case of free disposal – a world in which the winner of the first object can still bid on the second object, and dispose of the less-preferred object should he win both. Interestingly, the free-disposal case is technically very similar to the costly disposal situation of Section II, and the bidding function turns out to be very similar in the uniform example. Section IV also explains why it would be difficult to generalize the proposed model to more than two auctions, and why there would not be a pure-strategy equilibrium should the seller use a reserve inside the support of the first-object valuations. Section V concludes by interpreting the results within a broad framework of choice and sequential search, and by highlighting both the contributions of the present work and potential avenues for future research.
II. Model

Two second-price sealed-bid auctions 1 and 2 are conducted in a sequence, each auction sells
one object with a zero reserve. \( N > 2 \) risk-neutral bidders participate in the sequence of auctions.
Each bidder has a single-object capacity in that she derives no value from a second object.\(^1\) Each
bidder knows her private valuations of the two objects \((v_1,v_2)\) at the start of the game, and the
\((v_1,v_2)\) pairs are drawn independently across bidders from a distribution with a bounded density \(f\)
and full support on a closed and bounded rectangle \([L_1,H_1] \times [L_2,H_2] \subseteq \mathbb{R}^2\) with \( H_i > L_i \geq 0 \).
The cumulative distribution of the valuations is denoted \( F(x,y) = \Pr(v_1 \leq x \& v_2 \leq y) \). It is also
useful to define the cumulative distribution function under a continuous curve \( \psi : [L_2,H_2] \to \mathbb{R} \)
as \( F(\psi,y) \equiv \int_{L_2}^{\psi(y)} \int_{L_1}^{v_2} f(v_1,v_2) dv_1 dv_2 \). By either notation, \( F(H_1,v_2) \) is the marginal cdf of \( v_2 \), and
\( F(v_1,H_2) \) is the marginal cdf of \( v_1 \). Disposal or resale of a purchased object is costly enough
that the winner of the first auction does not bid again.\(^2\) The two auctions are conducted promptly
after each other, so there is no discounting of the second auction’s outcome.

A central result of this paper is that there exists a unique symmetric pure-strategy
equilibrium, in which almost all bidders bid more than \( L_1 \) in the first period. The equilibrium
relies on regularity of the surplus a first-stage loser can expect from the second auction as a
function of his valuation \( v_2 \) and the magnitude of the first-stage winning bid \( c_1 \). Denote this
expected surplus function \( S(v_2,c_1) \), and define regularity as follows:

**Definition:** A function \( S(v_2,c_1) \) on \([L_2,H_2] \times [L_1,H_1]\) is **regular** when it satisfies all of the
following conditions:

A) For every \( v_2 \in [L_2,H_2] \), \( S \) does not decrease in \( c_1 \) faster than unity:

---

\(^1\) For example, the objects are two different cars, and each bidder only has a use for one car. Or the objects are two
different procurement contracts, and each bidder is a firm that can fulfill only a single contract. Or the objects are
tickets to two concurrent events held this evening.

\(^2\) A demand-assumption equivalent to costly disposal would be that the value of the second object contingent on
owning the first object is zero. Thus, the winner of the first car cannot sell or even throw away the first car in order
to free up his garage for the second car. Analogously, the firm awarded the first contract must fulfill it before
bidding again, and sub-contracting is not feasible. Or there simply is no time to turn around and re-sell won tickets
to this evening’s events. The potential impact of free disposal on bidding will be explored in the Extensions section.
∀c < d ∈ (L₁, H₁), S(v₂, d) − S(v₂, c) > (−1)(d − c).

B) S is continuous in c₁ on c₁ ∈ [L₁, H₁].

C) For every v₂ ∈ [L₂, H₂], S(v₂, L₁) = 0.

The symmetric pure-strategy Nash equilibrium equilibrium is constructed in the following steps:

1) The second stage has a dominant strategy to bid valuation v₂ (second-price sealed-bid auction), so the main difficulty is in characterizing first-stage bidding. The first stage has a unique best response function b₁(v₁, v₂) to any regular expected surplus function S(v₂, c₁) arising from second-stage bidding.

2) The full support and boundedness of f imply that for every β ∈ (L₁, H₁), there exists a unique candidate for an isobid curve I(v₂ | β) - a curve in (v₁, v₂) space along which b₁(v₁, v₂) = β. The set of I(v₂ | β) for all β ∈ (L₁, H₁) is the unique set of equilibrium isobids when it implies a regular S(v₂, c₁) on [L₂, H₂] × [L₁, H₁].

3) The full support and boundedness of f together with the properties of the candidate I(v₂ | c₁) curves ensure that the implied S(v₂, c₁) is regular, hence ensuring the existence of a unique symmetric pure-strategy Nash equilibrium function b₁(v₁, v₂).

**First stage best response to a second-stage surplus function**

In the second auction, it is a dominant strategy to bid one’s private value v₂. Consider a single focal bidder, and let c₁ be the highest first-stage competitive bid faced by the focal bidder, i.e. the maximum first bid of N-1 first-stage competitors. Suppose the focal bidder believes c₁ to be distributed according to some distribution G₁ on [0, H₁]. The second stage bidding equilibrium implies that losers of the first auction can expect a non-negative surplus from bidding on the second object. On the margin, this expected surplus is a function of c₁ because the competitors’ v₂S influence first-stage bidding, and so losing the first auction at different bid-levels involves a different systematic selection of the second-stage competitors who also lose the first stage. Let S(v₂, c₁) be the expected second-auction surplus of a first-stage loser with a given v₂ when the
first-stage winner bids \( c_1 \). Given any \( S(v_2, c_1) \), the bidders solve the following problem in the first auction:

\[
b_1(v_1, v_2) = \arg \max_{\beta} \left\{ \int_0^\beta (v_1 - c_1) dG_1(c_1) + \int_\beta^\infty S(v_2, c_1) dG_1(c_1) \right\}
\]

(1)

When \( S \) is regular, the bidder’s surplus-maximization problem is concave at the following first order condition that implicitly characterizes \( b_1(v_1, v_2) \):

\[
FOC : b_1(v_1, v_2) = \beta \quad \text{such that} \quad \beta = v_1 - S(v_2, \beta)
\]

(2)

Regularity of \( S \) is needed to ensure concavity, and it also implies that the implicit function \( b_1(v_1, v_2) \) is well-defined and its image is restricted to \( b_1 \geq L_1 \). All these results are collected in the following proposition (please see the Appendix for detailed proofs of all Propositions):

**Proposition 1 (First-stage best-response characterization):** The best response to any regular surplus function \( S(v_2, c_1) \) is characterized by a bidding function \( b_1(v_1, v_2) \geq L_1 \) defined implicitly as \( b_1(v_1, v_2) = \beta \) such that \( \beta = v_1 - S(v_2, \beta) \).

The form of the bidding function in Proposition 1 provides interesting intuition. Since the first-stage winner does not bid again, the option value \( S \) of the second auction is the opportunity cost of winning the first item. Because of the truth-revealing property of the second-price auction, the bidder thus bids her net value of the first object: \( v_1 - S \), and the optimal bid does not depend on \( G_1 \). When evaluating the option value of the second auction, the bidder assumes that she loses the first stage to a competitive bid that exactly matches her first bid, hence she assumes \( c_1 = b_1 \). This is because \( c_1 = b_1 \) is the only situation in which raising the first bid slightly changes the outcome of the game, and \( S(v_2, b_1) = S(v_2 \mid c_1 = b_1) \) is therefore the opportunity cost relevant at the margin. It remains to be shown that under the present distributional assumptions, there exists a unique equilibrium regular \( S(v_2, c_1) \). The next section presents the key building block of the argument – a reduction of the equilibrium analysis to a single dimension.
Existence and uniqueness of candidates for equilibrium isobids

Any expected surplus function \( S(v_2, c_1) \) is closely linked to the underlying beliefs about second-stage competition. Consider a single bidder who lost the first auction, and let \( c_2 \sim G_2 \) be the highest competitive bid in the second stage, i.e. the maximum second bid of the other first-stage losers. Since the first-stage bids are in general a function of \( v_2 \)'s and \( c_1 \) is an upper bound on others’ first-stage bids, \( G_2 \) needs to be conditioned on \( c_1 \), \( G_2(c_2 | c_1) \). A set of conditional distributions \( G_2(c_2 | c_1) \) for every \( c_1 \) implies the following expected surplus function:

\[
S(v_2, c_1) = \int_{v_2} (v_2 - c_2) dG(c_2 | c_1) = \int_{v_2} G(c_2 | c_1) dc_2
\]

where the second equality follows from integration by parts. This is a familiar result: the expected surplus in a standard auction is the integrated probability of winning.

Given equation (3), analyzing the dependence of the expected surplus on \( c_1 \) can be achieved by analyzing how \( c_1 \) influences \( G_2 \). Consider a single focal bidder in the first auction, suppose all of his \( N-1 \) competitors happen to be bidding according to the same \( \tilde{b}_1(v_1, v_2) \), and suppose further that the competitors’ bids can be captured by “isobids” - lines in \( (v_1, v_2) \) space along which \( \tilde{b}_1(v_1, v_2) \) is constant:

**Definition:** The isobid for bid-level \( \beta \) is a function \( \tilde{b}(v_1, v_2) : [L_2, H_2] \rightarrow R \) s.t. \( \tilde{b}_1(v_1, v_2) = \beta \).

When the isobids are weakly increasing in the bid-level \( \beta \) (as they will be in equilibrium), they allow a succinct characterization of the second-stage surplus \( \tilde{S}(v_2, c_1) \) the focal bidder can expect after losing the first stage to a bid \( c_1 \). The constraint imposed by \( c_1 \) on the valuations of the competing bidders who also lose the first auction is that all their \( (v_1, v_2) \) lie weakly below the \( c_1 \) isobid (the competitor who bids exactly \( c_1 \) wins the first auction and exits the game). Suppose there are \( N-2 \) other first-stage losers because the first-stage winner does not bid again.  

---

3 The next section will show that there is a unique regular \( S \) that satisfies symmetric equilibrium constraints, so Proposition 1 will capture the equilibrium bidding strategy, and hence \( \tilde{b}_1(v_1, v_2) \geq L_i \geq 0 \), so trade is guaranteed.
isobid $\tilde{I}(v_2 \mid c_1)$, the $\tilde{G}_2(c_2 \mid c_1)$ facing the focal bidder is therefore expressed as the distribution of the maximum of $N-2$ independently distributed valuations $v_2$ such that $\tilde{b}_1(v_1, v_2) \leq c_1$. Each of the competitors’ valuations is in turn distributed according to a distribution function defined by the ratio of the $f$ probability mass below $\tilde{I}(v_2 \mid c_1)$ and to the left of $c_2$, and the entire probability mass under the isobid $\tilde{I}(v_2 \mid c_1)$:

$$\tilde{G}_2(c_2 \mid c_1) = \Pr^{N-2}(v_2 \leq c_2 \mid \tilde{b}_1(v_1, v_2) \leq c_1) = \left( \frac{F[\tilde{I}(\cdot \mid c_1), c_2]}{F[\tilde{I}(\cdot \mid c_1), H_2]} \right)^{N-2} = \left( \frac{\int_X f(w) \, dw}{\int_{X+Y} f(w) \, dw} \right)^{N-2}$$  \hspace{1cm} (4)

where $X$ and $Y$ are the pertinent areas under the isobid illustrated in Figure 1.

The equilibrium places a very specific constraint on each isobid: In a symmetric pure-strategy equilibrium, the isobid $I(\cdot \mid c_1)$ implied by the best response of the focal bidder to the $\tilde{G}_2(c_2 \mid c_1)$ in equation (4) must be the same as isobid $\tilde{I}(\cdot \mid c_1)$ that generated the $\tilde{G}_2(c_2 \mid c_1)$. If the $\tilde{G}_2(c_2 \mid c_1)$ in equation (4) integrates to a regular $S(v_2, c_1)$ according to equation (3), then the best response of the focal bidder to $\tilde{I}(\cdot \mid c_1)$ is characterized by Proposition 1, and the implied isobid $I(\cdot \mid c_1)$ is:

![Figure 1: Isobid in (v_1,v_2) space](image-url)
Proposition 1

\[
I(v_2 | c_1) = \{ v_1 : b_1(v_1, v_2) = c_1 \} = c_1 + \bar{S}(v_2, c_1) = c_1 + \int_{L_2} \bar{G}_2(c_2 | c_1) \, dc_2
\]  \hspace{1cm} (5)

Plugging equation (4) for \( \bar{G}_2(c_2 | c_1) \) in equation (5) and requiring that \( \bar{I}(\cdot | c_1) = I(\cdot | c_1) \) finally produces the equilibrium constraint:

\[
I(v_2 | c_1) = T(I | c_1)(v_2) = c_1 + \int_{L_2} \left( \frac{F[I(\cdot | c_1), x]}{F[I(\cdot | c_1), H_2]} \right)^{N-2} \, dx
\]  \hspace{1cm} (6)

Where \( T(J | \beta) \) is a mapping on the space of functions defined by

\[
T(J | \beta)(v_2) = \beta + \int_{0}^{v_2} \left( \frac{F[J, x]}{F[J, H_2]} \right)^{N-2} \, dx ,
\]

and equation (6) restricts any equilibrium \( I(v_2 | c_1) \) to be a fixed point of \( T(\cdot | c_1) \). Specifically, any equilibrium with a regular expected surplus function must satisfy the fixed-point integral equation that links the curvature of \( I \) to probability mass under it. To reveal the relationship between the curvature of \( I \) and the probability mass under it while including the case of \( F[I(\cdot | c_1), H_2] = 0 \), multiply equation (6) by

\[
F[I(\cdot | c_1), H_2],
\]

differentiate the result twice (dropping the conditioning on \( c_1 \) for clarity):

\[
I''(v_2)F^{N-2}(I, H_2) = (N-2)F^{N-3}(I, v_2) \int_{L_2} f(v_1, v_2) \, dv_1 \text{ such that } I(L_2) = c_1, I'(L_2) = 0.
\]  \hspace{1cm} (6')

It remains to be shown that candidates for equilibrium isobids (i.e. curves satisfying equation 6 or 6’) exist and that they indeed give rise to a regular surplus function as needed.

Before showing existence of candidates for equilibrium isobids, I discuss an interesting negative result implied by the equilibrium restriction: candidates for isobids cannot intersect the lower boundary of the support, and so there cannot be a symmetric pure-strategy equilibrium with bidding below \( L_1 \): Consider any bid-level \( \beta < L_1 \). Equation (6) implies that the isobid \( I(v_2 | \beta) \) has to be equal to \( \beta < L_1 \) for \( v_2 = L_2 \). Suppose there is some \( v_2 > L_2 \) such that the above \( I(v_2 | \beta) > L_1 \) (and so the \( (I(v_2 | \beta), v_2) \) bidder bids \( \beta \) in equilibrium). From continuity of isobids, there must be a \( v^* < v_2 \) such that \( I(v^* | \beta) = L_1 \). But from equation (6), the bidder with
$(v_1, v_2) = (L_1, v^*)$ is guaranteed to encounter strictly superior competitors in the second round with probability one (should he lose on the margin at bid-level $\beta$). Therefore, his expected surplus is zero, and his best response is to bid his $v_1 = L_1 > \beta$ in this first round, a contradiction with $(L_1, v^*)$ being on the isobid $I(v_2 | \beta)$. This argument can be summarized in the following result:

**Proposition 2 (No abstentions in equilibrium):** There does not exist any candidate equilibrium isobid $I(v_2 | \beta)$ for $\beta < L_1$ such that $I(v_2 | c_1) > L_1$ for some $v_2 \in [L_2, H_2]$. Therefore, $b_1(v_1, v_2) \geq L_1$ in any symmetric pure-strategy Nash equilibrium.

Equilibrium considerations alone thus restrict our attention to first-round bids at or above the lower bound of the support of $v_1$. This is remarkable because one may expect for some bidders with $v_2 \approx H_2$ and $v_1 \approx L_1$ to bid far below $L_1$, or to even abstain from the first auction to protect their continuation payoff available only should they lose. The reason why such bidders do not bid below $L_1$ is the information contained in losing the first auction on the margin: in a symmetric equilibrium, losing with such a low bid implies that the second-round competition will be very fierce. This result means that no bidders will abstain from the first auction as long as the seller reserve is weakly below the boundary of the valuation support. This is counter-intuitive because a naïve analysis may guess that bidders with $(v_1, v_2) \approx (L_1, H_2)$ would not be willing to bid more than $L_1$ in order to protect their continuation payoff. The result arises because in equilibrium, losing is informative about remaining competitors, especially losing with valuations $(v_1, v_2) \approx (L_1, H_2)$ and a bid near $L_1$: such a loss implies that the marginal competitor, and hence all remaining competitors, have very high second-object valuations, and so the continuation payoff is close to zero for all $v_2$.

Above $L_1$, the situation is surprisingly orderly given the mild distributional assumptions. Unfortunately, the equilibrium condition does not usually have a closed-form solution. However, it is possible to constructively prove that a unique equilibrium $I(v_2 | c_1)$ exists for every $c_1$: 

11
**Proposition 3 (Existence and uniqueness of candidate equilibrium isobids)**: For each \( c_i \in (L_1, H_1) \), there exists a unique nondecreasing 1-Lipschitz function

\[
I(v_2 | c_i) : [L_2, H_2] \rightarrow [c_i, c_i + H_2 - L_2] \quad \text{with} \quad I(L_2 | c_i) = c_i \quad \text{such that} \quad I(L_2 | c_i) = T[I(L_2 | c_i) | c_i].
\]

*Iterations of \( T(\cdot | c_i) \) starting with any nondecreasing 1-Lipschitz function*

\[
I_0 : [L_2, H_2] \rightarrow [c_i, c_i + H_2 - L_2] \quad \text{with} \quad I_0(L_2) = c_i \quad \text{converge to} \quad I(v_2 | c_i) \quad \text{exponentially fast.}
\]

The proof relies on showing that a \( K \)-times repetition of the mapping \( T \) defined in equation (6) is a contraction map on the closed metric space of all nondecreasing 1-Lipschitz functions on the support of \( v_2 \) with \( I(L_2) = c_1 \) (under the supremum metric). Therefore, the Banach Fixed point theorem implies existence and uniqueness of a fixed point of \( T \). Moreover, the contraction property implies an iterative numerical method for computing any candidate isobid, as highlighted in the second part of the Proposition.

Proposition 3 plays an important role in reducing the dimensionality of the problem of finding the two-dimensional equilibrium-function \( b_1(v_1, v_2) \): for a given level of \( b_1 \), the candidate equilibrium isobid \( I(v_2 | b_1) \) is only a one-dimensional function that depends on \( f \) and does not depend on other isobids. To close the argument for existence and uniqueness of a symmetric pure-strategy equilibrium, it remains to be shown that together, the set of equilibrium isobids implies a regular equilibrium expected surplus function.

**Existence and uniqueness of a regular equilibrium expected surplus function**

The full set of candidates for equilibrium isobids \( \{I(v_2 | c_i)\}_{c_i = L_1}^{H_1} \) from Proposition 3 defines a two-dimensional candidate isobid function \( I(v_2, c_i) \) on \([L_2, H_2] \times (L_1, H_1)\), which in turn implies a unique expected surplus function:

\[
S(v_2, c_i) = I(v_2, c_i) - c_i \quad (7)
\]

---

\(^4\) Mere existence follows even easier from the Schauder fixed point theorem because the aforementioned metric space of functions is compact: Arzela-Ascoli Theorem implies that a set of continuous real functions on a compact interval is compact when it is closed, bounded, and equicontinuous. The first two conditions hold by definition, and equicontinuity is implied by the 1-Lipschitz continuity of the image of \( T(I) \) for any 1-Lipschitz \( I \).
Extend the surplus function to the entire closed rectangle by defining it as a limit
\[ S(v_2, L_1) = \lim_{c \to L_1^+} I(v_2, c) - L_1. \]
The properties of candidate isobids translate immediately into the
properties of \( S \) as a function of \( v_2 \). Some of these properties are used extensively in the analysis
to follow, so I state them first:

**Proposition 4**: For every \( c_1 \in (L_1, H_1] \), the equilibrium surplus function \( S(v_2, c_1) \geq 0 \), \( S \) is twice
differentiable in \( v_2 \), 1-Lipschitz and convex in \( v_2 \) on \([L_2, H_2] \). For every \( c_1 \in (L_1, H_1] \),
\[ S(L_2, c_1) = 0 \] and \( \frac{\partial S}{\partial v_2} \bigg|_{v_2 = L_2} = 0 \), and \( S \) increasing in \( v_2 \) for all \( v_2 \in (L_2, H_2] \).

Proposition 4 is immediate from the fact that \( S \) is an integrated cumulative distribution function:
\[
S(v_2, c_1) = \int_0^{v_2} G_2(c_2 | c_1) \, dc_2 \geq 0 \Rightarrow 1 \geq \frac{\partial S}{\partial v_2} = G_2(v_2 | c_1) \geq 0
\]
\[
\Rightarrow \frac{\partial^2 S}{\partial v_2^2} = (N - 2) G_2^{N-3}(v_2 | c_1) \int_{I(v_2,c_1)} f(v_1, v_2) \, dv_1 \geq 0
\] (8)
where all inequalities are strict whenever \( v_2 > L_2 \). The intuition for Proposition 4 is
straightforward: The expected surplus increases in the valuation \( v_2 \) because a higher valuation
makes winning more likely and also increases the actual surplus conditional on winning. Since
these two at-least-linearly-increasing components effectively multiply to produce the expected
surplus, the convexity results. Since increasing \( v_2 \) by a small amount can increase the expected
future surplus at most by that amount (and that only in the case when future prices are
guaranteed to be below \( v_2 \)), the slope of \( S \) in \( v_2 \) is bounded above by unity.

The expected surplus function is thus very well behaved as a function of \( v_2 \). To close the
equilibrium construction argument, it needs to be shown that the expected surplus function is
also well behaved in \( c_1 \), i.e. that it is regular. The three regularity conditions A,B, and C are
demonstrated in turn.

To show that \( S \) does not decrease in \( c_1 \) faster than unity, it is enough to show that two
candidates for equilibrium isobids cannot cross or even touch each other, i.e. that \( I(v_2, c_1) \) is
strictly increasing in both arguments:
**Proposition 5 (Part A of regularity):** For every $c, d \in (L_1, H_1): d > c \Rightarrow I(v|d) > I(v|c)$ for every $v \in [L_2, H_2]$

The result of Proposition 5 is implied by the full support assumption together with the fact that a slope of any candidate for equilibrium isobid $I(v|c)$ at $w$ is $G_2(w|c)$ which is in turn a ratio of probability mass under $I(v|c)$ and left of $w$ and the probability mass under the entire $I(v|c)$.

This equilibrium relation between the candidate isobid and its slope implies that when $d > c$, the isobid $I(v|c)$ that starts out below $I(v|d)$ at $L_2$ cannot have the required higher slope at a potential intersection point with the higher $I(v|d)$. Interestingly, even a tangency is ruled out, so $I(v_2, c_1)$ is strictly increasing in both arguments, and $S(v_2, c_1) = I(v_2, c_1) - c_1$ thus cannot decrease in $c_1$ faster than unity.

The second (B) part of regularity (continuity in $c_1$) follows from uniqueness of candidate isobids and the global bound on their slopes.

**Proposition 6 (Part B of regularity except for continuity at $L_1$):** The equilibrium surplus function $S(v_2, c_1)$ is continuous in $c_1$ at all $c_1 \in (L_1, H_1]$.

The argument proceeds as follows: Together with compact support of the candidate isobids, the global bound on their slopes guarantees that as $\varepsilon$ approaches zero, $I(v_2 | c_1 + \varepsilon)$ converges uniformly to a nondecreasing 1-Lipschitz function $I_+: [L_2, H_2] \rightarrow [c_1, c_1 + H_2 - L_2]$ with $I_+(L_2) = c_1$. Together with the global bound on $f$, uniform convergence of the candidate isobids then implies that when $c_1 \in (L_1, H_1]$, $I_+$ satisfies the equilibrium relation in equation (6), so $I_+$ must be $I(v_2 | c_1)$ because there is a unique function that satisfies that relation. There is an analogous argument for $I(v_2 | c_1 - \varepsilon)$ converging uniformly to the $c_1$ candidate isobid from below, and so continuity results.

The final part of regularity is the most striking: the equilibrium surplus vanishes as the bid-level $c_1$ approaches the lower bound of the support $L_1$, even when the $v_2$ is large relative to
the competition, for example when $v_2 = H_2$. Given equation 7, it is enough to show that candidate isobids $I(v_2 | c_1)$ approach the constant function $I(v_2 | L_1) = L_1$ as $c_1$ approaches $L_1$.

**Proposition 7 (Part C of regularity and continuity at $L_1$):** As $c_1$ approaches $L_1$, $I(v_2 | c_1)$ converge uniformly to $I(v_2 | L_1) = L_1$, and so $S(v_2, c_1)$ converges to $S(v_2, L_1) = 0$.

Note that by defining $I(v_2 | L_1)$ as a limit, Proposition 7 extends the isobid function $I(v_2, c_1)$ and hence the implied surplus function $S(v_2, c_1) = I(v_2, c_1) - c_1$ to be continuous on the entire closed support $[L_2, H_2] \times [L_1, H_1]$. The equilibrium condition (6) is not defined for the function $I(v_2 | L_1) = L_1$ because the mass below it is zero, but the equivalent equilibrium condition (6’) holds, and so $b_1(L_1, v_2) = L_1$ is a candidate for an equilibrium isobid. Proposition 7 essentially shows that it is a unique candidate, and that $I(v_2 | c_1)$ converge to it.

There is a useful metaphor that helps explain the intuition for Proposition 7: Imagine a horizontally-oriented zipper, with the lower side of the zipper firmly fixed to a constant function $[L_2, H_2] \rightarrow L_1$, and the upper side representing the candidate isobid for the $L_1$ level $I(v_2 | L_1) : [L_2, H_2] \rightarrow [L_1, L_1 + H_2 - L_2]$. The zipper is stitched closed at the left end $(L_2, L_1)$ by definition of a candidate isobid. The intuition for isobids of smaller and smaller bid-levels converging to the constant function $L_1$ is that constructing the $I(v_2 | L_1)$ starting at $(L_2, L_1)$ is like a closing the zipper from left to right: The slope of $I(v_2 | L_1)$ at $v_2 = L_2$ is zero (true for all candidate isobids), and so is its curvature (not true for any other candidate isobid) because there is no mass below $L_1$ (see equation 8). Starting with the zipper open, there is mass somewhere under the isobid, but the mass cannot be near $(L_2, L_1)$ because of the zero curvature there.

Technically, a positive mass under $I(v_2 | L_1)$ (open zipper) makes $T(I | L_1)$ well defined, and the proof of Proposition 7 shows that successive applications of $T(\cdot | L_1)$ to any candidate function $I$ (1-Lipschitz $I(v_2 | L_1) : [L_2, H_2] \rightarrow [L_1, L_1 + H_2 - L_2]$ such that $I(L_2 | L_1) = L_1$) contract towards $L_1$ according to a an increasingly more convex function with a smaller and smaller area
underneath: \( T^K (I \mid L_i)(v_2) < L_1 + C^K \frac{v_2^{2K+1}}{(2K+1)!} \) for some positive constant \( C \). Successive applications of \( T(\cdot \mid L_i) \) thus close the zipper from left to right. Therefore, an \( L_1 \) candidate isobid with a positive mass underneath is not possible.

From the perspective of the bidders, an \( L_1 \) isobid \( I(v_2 \mid L_1) \) with a positive mass underneath unravels as follows: Suppose there is a bidder \((v_1, v_2)\) with \( v_1 > L_1 \) who is on the \( I(v_2 \mid L_1) \) isobid line. In expecting a positive expected surplus should she lose, she is relying on other bidders \((\tilde{v}_1, \tilde{v}_2)\) with smaller but still non-minimum valuations \( L_1 < \tilde{v}_i < v_i \) to also bid \( L_1 \) (\( L_i < \tilde{v}_i \) is critical for a positive mass of such competitors). In a symmetric equilibrium, those bidders \((\tilde{v}_1, \tilde{v}_2)\) rely on other bidders with yet smaller but still non-minimum valuations to also bid zero, all the way down to bidders arbitrarily near the point \((L_2, L_1)\). But bidders on \( I(v_2 \mid L_1) \) sufficiently close to \((L_2, L_1)\) realize that they will lose the second auction almost surely because the probability mass under \( I(v_2 \mid L_1) \) and left of a very small \( v_2 \) is zero (since an equilibrium \( I \) must have a slope and curvature of zero at \( L_2 \)). Therefore, the cascade unravels, the \((\tilde{v}_1, \tilde{v}_2) \approx (L_2, L_1)\) competitors bid a small amount strictly greater than \( L_1 \), and the focal bidder \((v_1, v_2)\) thus also bids strictly more than \( L_1 \).

Given Propositions 1-6, it is finally possible to state the main theorem of this paper:

**Theorem 1**: There exists a unique symmetric pure-strategy Nash equilibrium characterized by a continuous bidding function \( b_1(v_1, v_2) \) such that:

\begin{itemize}
  \item \( b_1(L_1, v_2) = L_1 \), \( b_1(v_1, L_2) = v_1 \) and \( b_1(v_1, v_2) > L_1 \) for all \((v_1, v_2) > (L_1, L_2)\)
  \item \( b_1(v_1, v_2) \) is increasing in \( v_1 \) and decreasing in \( v_2 \) for all \((v_1, v_2) > (L_1, L_2)\)
  \item \( b_1(v_1, v_2) \) for converges to \( v_1 \) as the number of bidders approaches infinity
\end{itemize}

The proof of Theorem 1 essentially ties together all the previous results. The properties of a regular \( S \) together with Proposition 4 imply all the properties of \( b_1(v_1, v_2) \), where the comparative statics in valuations follow most easily by implicit differentiation of equation (2).
The intuition for the comparative static in the number of bidders is as follows: The overall number of bidders $N$ controls the extent of the bid-shading in that as the number of bidders increases, it is harder and harder to win the second auction. Meanwhile, the winner’s surplus also shrinks. Therefore, the expected surplus from the second auction shrinks to zero and the two auctions become effectively isolated.

Theorem 1 is notably silent about the comparative static of $S$ in $c_1$. While $S$ is nondecreasing in $c_1$ for the Uniform distribution because $f=1$ and $I’s$ are upward-sloping, this is not necessarily the case in general: It is possible to imagine distributions, for which $S$ locally decreases in $c_1$, albeit never faster then unity. For example, consider an $f$ with most of the mass concentrated near some point $(v^*_1, v^*_2)$ and uniform elsewhere. Then, the isobids will coincide with the uniform isobids for all $c_1$ low-enough that $I(v^*_2 | c_1) \ll v^*_1$ (and $S$ will thus be nondecreasing small $c_1$). However, as $c_1$ increases further, the isobids will become flatter and flatter in the range $[0, v^*_2)$, and have approximately slope 1 in the range $(v^*_2, 1]$. Therefore, $S(v_2, c_1)$ will be decreasing in $c_1$ for $v_2 \in [0, v^*_2)$ for $c_1$ such that $I(v^*_2 | c_1) \approx v^*_1$.

**Example: Three bidders, $f$ Uniform**

In rare situations, like the case of three bidders and $f$ Uniform, the equilibrium condition (6) yields an analytical solution for low-enough $c_1$. With three bidders, $N-2=1$, and with uniform $f(v_1, v_2) = 1$, so the equilibrium condition considerably simplifies. Note that

$$f(v_1, v_2) = 1 \Rightarrow \int_0^{I(v_1, v_2)} f(v_1, v_2) dv_1 = \min[I(v_2 | c_1), 1],$$

suppress the conditioning on $c_1$ for clarity, and consider $c_1$ low enough that $I(v_2 | c_1) < 1$. Then, equation (6') is the following differential-integral equation:

$$I''(v) \int_0^1 I(x) dx = I(v)$$ subject to $I(0) = c_1, I'(0) = 0$. This equation has a unique solution for every $c_1$ which can be characterized in “almost-closed” form as:

$$I(v) = c_1 + \frac{1 + \exp\left[2A(c_1)v\right]}{2\exp\left[A(c_1)v\right]} \text{ where } A(c_1) \text{ satisfies } 1 = c_1A(c_1)\sinh(A(c_1)) \text{. The solution is obtained by first solving } I''(v)\left(\sqrt{A^2}\right) = I(v) \text{ subject to } I(0) = c_1, I'(0) = 0, \text{ for an arbitrary }$$
constant $A$, and then noting that the solution satisfies $\int_0^1 I(v) \, dv = \frac{c_1 \sinh(A)}{A}$, so with $A$ satisfying $1 = c_1 A \sinh(A)$, $I(v)$ satisfies the original differential-integral equation. The implicitly defined function $A(c_1)$ exists, because $A \sinh(A)$ is continuous, monotonically increasing on $[0, \infty)$, zero at $A=0$, and rising without bound as $A$ approaches infinity. This result applies to all $c_1$ in $(0, 0.552)$, above which point $I(1|c_1) > 1$, so the upper bound of $v_1$ takes over, and the equilibrium $G_2$ slowly approaches the uniform cdf.

The differential-equation approach gets quickly complicated as $N > 3$ and other $f$ get considered. It is clear from (6') that a closed-form cdf is a necessary condition for a closed-form solution. Moreover, one can see that for $N > 3$, $(N-3)$ additional differentiations are necessary to create a differential-integral equation with fixed limits of integration, so extending the above “two-step” approach to Uniform and arbitrary $N > 3$ would require solving a differential equation of the $(N - 1)$-st order.

The Uniform example on a unit square brings up the question whether it is possible to solve for $b_i(v_1, v_2)$ on a unit square (after appropriately rescaling the density $f$ to the unit square), and then simply scale up the solution to an arbitrary rectangle $[L_1, H_1] \times [L_2, H_2]$. Unfortunately, this not possible in general, but a scaling procedure is available when both $v_1$ and $v_2$ are subjected to the same affine transform:

**Proposition 8**: Suppose $b_i(v_1, v_2)$ is the equilibrium bidding function for $f(v_1, v_2)$ with support on $[L_1, H_1] \times [L_2, H_2]$. Let $w_i = A + B v_i$ for some $A, B \geq 0$, and let $f_w(w_1, w_2)$ be the distribution of $(w_1, w_2)$. Then, $\tilde{b}(w_1, w_2) = A + B b_i\left(\frac{w_1 - A}{B}, \frac{w_2 - A}{B}\right)$ is the equilibrium bidding function for $f_w(w_1, w_2)$ on $[A + BL_1, A + BH_1] \times [A + BL_2, A + BH_2]$.

The key to preservation of the equilibrium bidding function under an affine transform is that the affine transform scales up both the intercept and the slope of the isobid by the same factor $B$. Thus, the affine transform of the equilibrium isobids in the $(v_1, v_2)$ space produces equilibrium
isobids in the \((w_1, w_2)\) space: 
\[ I_w(w_2 \mid c_1) = A + BL \left( \frac{w_2 - A}{B} \mid \frac{c_1 - A}{B} \right). \]

Note that this would not be the case if one affine transform were used on \(v_1\), and another on \(v_2\). For example, if \(w_1 = A + Bv_1\) and \(w_2 = C + Dv_2\), 
\[ I_w(w_2 \mid c_1) \neq A + BL \left( \frac{w_2 - C}{D} \mid \frac{c_1 - A}{B} \right). \]
Therefore, proving equilibrium existence and uniqueness for a rectangular support is not a trivial extension of proving equilibrium existence for the unit square support.

Besides the comparative statics shown in Theorem 1 and Proposition 8, there is also potential for a comparative static of \(S(v_2, b_1)\) in the correlation between \(v_2\) and \(v_1\), but the effect of the correlation is only discussed qualitatively here. Suppose the distribution \(f\) is symmetric and its support is a square \([L, H]^2\). Consider the “exogenous” competition arising from \(N-2\) bids independently drawn from the marginal population distribution of \(v_i\):
\[ F(v) \equiv F(H, v) = F(v, H). \]
Exogenous competition is the competition anticipated by a forward-looking bidder, who does not appreciate the fact that future competitors are systematically selected in sequential auctions with information about future goods.

When \(v_2\) and \(v_1\) are independent, the equilibrium second-stage competition is stronger than exogenous because it involves the same number of bidders drawn from a distribution that first-order stochastically dominates \(F\). The dominance follows from the fact that isobids are non-decreasing: for every \(c_1\), independence implies that 
\[ F(v) = \frac{F(c_1, v)}{F(c_1, H)} \]
and equation (4) defines the equilibrium distribution as
\[ \frac{F[I(\cdot \mid c_1), v]}{F[I(\cdot \mid c_1), H]} = \frac{F(c_1, v) + A}{F(c_1, H) + A + B} < \frac{F(c_1, v)}{F(c_1, H)} = F(v) \]  
(9)
where \(A\) and \(B\) are the positive probability-masses between \(I(\cdot \mid c_1)\) and \(c_1\) below \(v\) and above \(v\) respectively. In contrast to the independent case, \(\frac{F[I(\cdot \mid c_1), v]}{F[I(\cdot \mid c_1), H]} > F(v)\) when \(v_2\) and \(v_1\) are sufficiently positively correlated, and so the second-stage competition becomes weaker than
exogenous. In the limit as the correlation between $v_2$ and $v_1$ approaches unity, the identical-unit model of Milgrom and Weber (2000) applies, in which the bidder with the highest valuation $v = v_1 = v_2$ exits in the first period, so only the $N-1$ lowest bidders remain – weaker bidders than exogenous. While an analytical result is not available, it is possible to explore the impact of a correlation by numerical approximation of equilibrium bidding. The numerical approximation methodology is discussed next, along with the results.

III. Numerical approximation of $b_1(v_1,v_2)$: Method and example

Given any $f$ for which the equilibrium exists, the equilibrium isobids can be easily numerically approximated, hence allowing quantitative analysis of $b_1(v_1,v_2)$. For a fixed $c_1$, any function $I(v_2)$ can be represented as a vector of function-values on a fine grid of $v_2$’s, and $f$ represented on a fine two-dimensional grid. Proposition 3 then implies that repeated iterations of $T$ starting with any nondecreasing 1-Lipschitz function $I_0 : [L_2, H_2] \rightarrow [c_1, c_1 + H_2 - L_2]$ with $I_0(L_2) = c_1$ converge to the equilibrium $I(v_2 | c_1)$. Performing these iterations separately for every $c_1$ on a fine grid until convergence, one can then construct equilibrium $S(v_2, c_1)$ by simply subtracting $c_1$ from $I(v_2 | c_1)$, and solving for $b_1(v_1,v_2)$ approximately on a grid using equation (2).

In the rest of this section, I restrict attention to a unit-square support, namely $L_i = 0$ and $H_i = 1$. The most efficient direction of proceeding through all possible $c_1$’s makes use of the continuity of $S$ in $c_1$ and the fact that isobids for high $c_1$’s have greater mass of $f$ underneath, and so their $T$ contracts faster. Therefore, it is best to start with $c_1 = 1$ and work down, initializing the iterations of each $T(I | c_1)$ with $I_0(v_2) = c_1 + S(v_2, c_1 + \delta)$, where $\delta$ is the size of a single step of the $c_1$ grid. I find that the iterations of $T$ converge faster than implied by the loose upper bound derived in the proof of Proposition 3, which is fortunate because the $K$ required according to the bound would not be practical for small $c_1$. Nevertheless, dozens of iterations are usually required for low $c_1$. Please see Figure 2 for an illustration of the implied $b_1(v_1,v_2)$ using the example of three bidders and uniform $f$. In this particular example, it is possible to verify the iterative method’s performance because there exists a closed-form solution for a range of small-enough $c_1$. The numerical method indeed converges to the true $I(v_2 | c_1)$. 

20
Figure 2: First-period equilibrium bidding function with $N=3, f=$Uniform$[0,1]$.

Note to Figure 2: Each line represents the first-period equilibrium bid as a function of $v_2$ with $v_1$ fixed. Black solid lines are with costly disposal, magenta dashed lines are with free disposal. The $y$-intercept of each curve is exactly $v_1$ because $b_1(0|v_1) = v_1$.

Building on the theoretical and computational foundation of this paper, Budish (2008) uses the above numerical procedure to assess the allocative efficiency of the sequential auction with information about future goods. He finds that the sequential auction is surprisingly efficient – realizing over 98 percent of the potential gains from trade for a wide range of distributions with $v_1$ and $v_2$ uncorrelated. The efficiency distribution does have a thin but long left tail: in the 3-bidder Uniform example, Budish finds the most inefficient draw of the valuations to be $(1,0.99), (0.58,0),(0,1)$: instead of receiving the first object, the first bidder just barely loses...
the first auction to the second bidder, resulting in only 79 percent efficiency and 0.42 social surplus left on the table. The rest of this section explores the efficiency properties of the auction in correlated settings, and illustrates the comparative static of equilibrium option values demonstrated in the end of the previous section.

To obtain an intuitive family of correlated distributions on a unit square that are similar in other respects but differ in the correlation, I used the restriction to the unit square of a bivariate Normal density with mean \((0.5, 0.5)\) and standard deviations equal to 0.5. To ensure that the densities had not only theoretically but also practically full support, I added a small constant to the resulting density. Because of this correction and also because of the restricted support, the magnitude of the resulting correlations is smaller than the correlation of the generating Normal. Nevertheless, a wide range of correlations arises. Budish (2008) argues and shows that the efficiency of the sequential auction increases in the number of bidders. It is therefore sufficient to compute the efficiency and revenue of the auction for \(N=3\) bidders. To compute the efficiency and revenue properties as a function of the correlation, I first used the computational procedure to obtain the bidding function, and then simulated 1 million groups of three bidders drawn iid from the appropriately correlated distribution (random draws achieved by inverse-cdf method on the grid). For every simulated auction, I computed the bids and allocations via the sequential auction, as well as the efficient allocation and supporting payments in the appropriate Vickrey-Clarke-Groves mechanism.

The computations of the equilibrium bidding function were difficult for high positive correlations. The reason is that isobids bunch up very close together just after crossing the \(v_1 = v_2\) diagonal, and so the numerical integration on the grid gets too imprecise. Please see the Appendix for a discussion of this “bunching up” in the limiting case of perfect correlation, when the model coincides with the model of Milgrom and Weber (2000). Finer and finer grids are thus required for convergence of the of \(T\) mapping as correlation approaches unity, demanding more and more memory from the computer. Therefore, I was only able to obtain the results for correlations in the \([-1, 0.83]\) range. This is an important limitation of the grid method, and itself an interesting future challenge for a computational game-theorist. While very high positive correlations cannot be evaluated using the simple grid approach, the limiting case of perfect correlation is easy to compute by simulating the Milgrom and Weber (2000) model.
The results are summarized in Table 1, I discuss the efficiency findings first: The general message of Budish (2008) seems robust to correlation: for all correlations examined, the sequential auction is almost perfectly efficient. However, it is also evident that efficiency decreases as correlation increases. This relationship is one of the new findings of this paper, and the rest of this paragraph provides the intuition for it: When the correlation is -1, i.e. \( v_1 = 1 - v_2 \), the auction is perfectly efficient because there is no equilibrium surplus to be had: the equilibrium strategy of every bidder is to bid \( v_i \) in the first auction, because should he lose in a tie (which is the marginal situation), he will meet an equal match in the second auction with probability one. As correlation increases but remains negative, this intuition holds approximately, resulting in only very slight inefficiencies, and very little shading of the first-auction bids below \( v_i \): for example, the modal bidder with \((v_1, v_2) = (0.5, 0.5)\) expects to meet such strong competitors in the second auction that his equilibrium second-stage surplus \( S(v_2, b_1(v_1, v_2)) \) is only 39 percent of the “exogenous surplus”, i.e. the surplus he could expect if he replaced the surviving competitor with another one drawn randomly from the same distribution. When the correlation is zero, Table 1 illustrates the point of Equation (9): the expected equilibrium second-stage surplus is only 75 percent of the exogenous level. On the efficiency front, the 99.3 percent figure for zero correlation is exactly in line with Budish’s findings for uncorrelated Beta distributions similar in shape to the distribution studied here. As correlation becomes positive and increases towards unity, efficiency declines. This is because low-level isobids become steeper and steeper, reaching high \( v_1 \) values for \( v_2 = 1 \). For example, with correlation of only 0.8, the 0.01 isobid reaches the point \((v_2 = 1, v_1 = 0.59)\). This means that among the bidders \( \{(0.01 + \epsilon, 0), (0.59, 1), (0.59, 1)\} \), the first bidder wins the first object, while it would clearly be socially best for him to win nothing. The amount of social surplus left on the table is 0.58 – more than the maximum amount of about 0.52 in the uncorrelated case. In the limit as correlation approaches 1, the 0.01 isobid approaches the diagonal from below, so the amount of social surplus left on the table approaches 1. There is a clear discontinuity at 1, where the auction again becomes perfectly efficient as the model becomes equivalent to Milgrom & Weber (2000).
Table 1: Efficiency and revenue performance of the auction with correlated values

<table>
<thead>
<tr>
<th>Corr</th>
<th>Avg</th>
<th>Sequential auction with information about future goods</th>
<th>Sequential auction with information about future goods</th>
<th>Bidder option value relative to exo. 2nd stage</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Social surplus</td>
<td>Seller revenue</td>
<td>Equil. second-stage surplus of modal bidder</td>
</tr>
<tr>
<td></td>
<td></td>
<td>VCG</td>
<td>VCG</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>1.23</td>
<td>100.0%</td>
<td>100.0%</td>
<td>0% 195%</td>
</tr>
<tr>
<td>0.84</td>
<td>1.28</td>
<td>98.8%</td>
<td>82% 18%</td>
<td>0.53 111.6% 37% 128%</td>
</tr>
<tr>
<td>0.69</td>
<td>1.32</td>
<td>98.9%</td>
<td>82% 16%</td>
<td>0.65 95.8% 111.6% 37% 128%</td>
</tr>
<tr>
<td>0.47</td>
<td>1.35</td>
<td>99.0%</td>
<td>82% 15%</td>
<td>0.68 95.2% 108.1% 35% 115%</td>
</tr>
<tr>
<td>0.00</td>
<td>1.42</td>
<td>99.3%</td>
<td>84% 11%</td>
<td>0.73 94.9% 105.3% 32% 101%</td>
</tr>
<tr>
<td>-0.47</td>
<td>1.42</td>
<td>99.7%</td>
<td>90% 7%</td>
<td>0.80 95.0% 101.5% 28% 75%</td>
</tr>
<tr>
<td>-0.68</td>
<td>1.43</td>
<td>99.8%</td>
<td>94% 5%</td>
<td>0.90 95.4% 98.3% 23% 39%</td>
</tr>
<tr>
<td>-0.83</td>
<td>1.43</td>
<td>99.9%</td>
<td>97% 3%</td>
<td>0.94 96.5% 98.2% 21% 23%</td>
</tr>
<tr>
<td>-1.00</td>
<td>1.46</td>
<td>100.0%</td>
<td>100% 1%</td>
<td>0.96 98.3% 98.7% 21% 12%</td>
</tr>
</tbody>
</table>

The division of social surplus between the seller and the bidders also depends on the correlation: as discussed in the previous paragraph, the perfect negative correlation effectively isolates the two auctions, forcing the bidders to bid all the way to their valuations in both rounds. Not surprisingly, this intense competition leaves the seller with 68 percent of the gains from trade. At the opposite end of the correlation distribution, the highest-value bidder wins the first auction, so the remaining bidders expect weak competition, which in turn leads everyone to shade their first-auction bids below their valuations. This hurts the seller, leaving him with only 43 percent of the social surplus. At the opposite end of the correlation distribution, the highest-value bidder wins the first auction, so the remaining bidders expect weak competition, which in turn leads everyone to shade their first-auction bids below their valuations. This hurts the seller, leaving him with only 43 percent of the social surplus.

Table 1 also illustrates that the distribution of seller revenue from the auction relative to the VCG payments has a long and thin right tail: while the average VCG payment generally
exceeds the average auction revenue, there are situations for which VCG generates much smaller
revenue in both absolute and relative terms. Moreover, it is clear that these situations are more
likely with higher positive correlations because they correspond to inefficient allocations. With a
correlation of 0.8, a large absolute difference occurs with \( \{(0.26,0.15),(0.8,0.96),(0.8,0.96)\} \): the auction is quite inefficient because the first bidder wins the first auction, but it generates 0.52
more seller revenue than VCG’s 0.69. In relative terms, the difference in revenue becomes large
when bidders are \( \{(x,x),(y,0),(0,0)\} \) because the VCG generates no revenue while the auction
can generate revenue \( \min[y,b_1(x,x)] > 0 \). When \( x=1 \), the \( y \) maximizing the difference is 0.46.
This last result is not surprising – it is well known that VCG revenues can be zero or very small
when the bidder competition is weak. It is interesting to note that while generating less revenue
on average, the sequential auction has a much lower variance of revenue and still generates
revenue when VCG does not.

IV. Extensions of the modeling assumptions

Free disposal

The model of section II assumes that the first-stage winner cannot dispose of the first unit if she
wins the second unit. This is realistic for situations where the goods are binding contracts for
specialized procurement of products or services that are difficult to subcontract: the capacity of
the first-contract winner is then tied up by fulfilling the first contract. How would the results
change if the goods were easily disposable, for example two different consumer digital cameras?
Let all assumptions remain exactly as in Section II, but assume that disposal of any won object is
free, and the utility of owning both objects is \( \max(v_1,v_2) \). Then, the winner of the first unit will
sometimes want to bid in the second auction, because the second unit may provide additional
value. In particular, the first-stage winner will bid \( b_{2,w}(v_1,v_2) = \max(v_1,v_2) - v_1 \) in the second
stage, which is positive whenever \( v_2 > v_1 \). The first-stage losers, of course, will continue bidding
\( b_{2,l}(v_1,v_2) = v_2 \) as in the model with costly disposal. In this subsection, I show that free disposal
does not automatically guarantee trade in the first period. Instead, symmetric equilibrium
arguments analogous to Proposition 2 are still necessary to show that trade occurs almost surely.
The concepts of a regular surplus function and an isobid extend quite readily to the free-disposal setting, and make the analysis tractable.

Given the chance to throw away the first object and bid again, it seems costless to simply bid a tiny $\varepsilon > 0$ in the first period even when $H_2 \approx v_2 > v_1 \approx L_1$: winning the first auction merely reduces the bidder’s bid in the second auction rather than preventing bidding altogether. Instead of foregoing the entire expected surplus $S(v_2)$ by winning the first object, the winner with free disposal only foregoes the opportunity cost of $S(v_2) - S(v_2 - v_1)$, which should be less than $v_1 - \varepsilon$ the benefit of winning the first auction with a tiny $\varepsilon > 0$. Therefore, it seems that free disposal guarantees trade almost surely in any symmetric pure-strategy Nash equilibrium. Unfortunately, this intuition is incomplete because winning vs. losing also impacts the second-period competition: the winner faces stiffer competition than the losers. Contingent expected-surplus functions are necessary to characterize the equilibrium, as discussed next.

Use equation 3 to define expected second-period surpluses as integrals of the different winning probabilities in the cases of losing and winning the first auction: First, let $S_L(v_2, c_1) = \int_0^{v_2} G_L(c_2 | c_1) dc_2$ be the expected second-period surplus of the first-stage loser, where $G_L$ is the cdf of $c_2$ in the case of losing. Second, let $S_W(v_2 - v_1, c_1) = \int_0^{v_2 - v_1} G_W(c_2 | c_1) dc_2$ be the expected surplus of the first-stage winner with $G_W$ defined analogously to $G_L$. Since the winner faces stiffer second-stage competition than a loser (both face $N-2$ losers and one additional competitor, where the additional competitor is also a loser in $G_W$, as opposed to a winner in $G_L$, and winners bid less than losers), it is immediate that $G_W$ stochastically dominates $G_L: G_L \geq G_W$. Given any $S_L$ and $S_W$, the bidders solve the following problem in the first stage:

$$b_1(v_1, v_2) = \arg \max_\beta \left\{ \int_0^{v_2} \left[ v_1 - c_1 + (v_2 > v_1) S_W(v_2 - v_1, c_1) \right] dG_L(c_1) + \int_0^\infty S_L(v_2, c_1) dG_W(c_1) \right\}$$

(10)

Compared to equation 3, winning the first auction involves the additional surplus $S_W(v_2 - v_1, c_1)$. The first-order conditions are:

$$\beta = \left[ v_1 - \int_{\max(0, v_2 - v_1)}^{v_2} G_L(c_2 | \beta) dc_2 \right] - \int_0^{\max(0, v_2 - v_1)} \left[ G_L(c_2 | \beta) - G_W(c_2 | \beta) \right] dc_2$$

(11)
The incomplete intuition of the previous paragraph noted that since $G_L \leq 1$, the bracketed term in equation (11) must be positive. However, $G_L \geq G_W$ makes the second term negative, and it is not clear from equation (11) whether a solution with $\beta > 0$ is guaranteed for almost every $(v_1, v_2)$. An argument analogous to the one for Propositions 2 and 3 is needed to guarantee first-period trade almost surely, along with the appropriate regularity conditions on $S_L$ and $S_W$. I do not develop the existence and uniqueness proof in detail, it is analogous to that in Section II.

The isobids of the free-disposal model are weakly bounded below by the isobids of the costly disposal model of Section II. It is clear from equation (11) that for all $c_1$ such that $I(v_2 | c_1) \geq v_2$ for all $v_2$, the isobids of the two cases coincide exactly: the second term in the equation vanishes, and $S_L(v_2, c_1) = S(v_2, c_1)$ of Section II because the winner does not bid again when his $v_i \geq v_2$. When $c_1$ is low-enough that $I(v_2 | c_1) < v_2$ with costly disposal, $S_L(v_2, c_1) < S(v_2, c_1)$ because of the additional competition from the winner. The equilibrium condition becomes

$$\tilde{I}'(v_2 | c_1) = G_L(v_2 | c_1) - I(v_2 | v_2)G_W \left[ v_2 - \tilde{I}(v_2 | c_1) \right]$$

and it is clear that an $I(v_2 | c_1)$ of Section II does not coincide with $\tilde{I}(v_2 | c_1)$, and $\tilde{I}(v_2 | c_1) < I(v_2 | c_1)$ because the isobids shade the same intercept and $\tilde{I}$ has lower slope everywhere. Note that because $v_1$ appears in the limits of integration in (11), the equilibrium condition (12) is less elegant than that in (6). The reason is that the constraint imposed by $c_1$ on the first-auction-winner’s valuations is that her $(v_1, v_2)$ lie on the $c_1$ isobid contour (as opposed to weakly underneath it as in the case of other losers). Even without solving the model further, it is obvious from $\tilde{I}(v_2 | c_1) \leq I(v_2 | c_1)$ that when disposal is free, the equilibrium first-stage bid $b_i(v_1, v_2)$ is at least as high as the corresponding first-stage bid given the same $(v_1, v_2)$ in the model with costly disposal.

At least in the [Uniform,N=3] example used throughout this paper, the first-stage bidding strategies are surprisingly similar with and without free disposal, with the biggest absolute differences predictably occurring in the $(v_1 < v_2)$ region of the parameter space. Please see Figure 2 for an illustration.
**Reserve prices above \( L_1 \)**

Let all assumptions remain exactly as in Section II, but assume that the seller uses a reserve-price \( R > L_1 \) in the first period. Trade is obviously no longer guaranteed. More surprisingly, the reserve not only makes some bidders abstain from the first auction, it also rules out a pure-strategy equilibrium and changes the bidding strategy of those bidders who participate. This observation is interesting because the truth-revealing nature of a standard single-shot 2PSB suggests that a reserve price should not affect bidding.

There is a positive measure of bidders who abstain: the probability of abstention is at least \( F(R, H_2) \) because bidders with \( v_1 < R \) cannot derive a positive surplus from the first auction. Suppose that there is a symmetric pure-strategy equilibrium, and let the upper bound of abstaining in \((v_1, v_2)\) space be a curve \( I^A(v_2) \). \( I^A(v_2) \) must satisfy:

\[
I^A(v_2) = R + S^A(v_2) = R + \frac{v_2}{L_1} \left( \frac{F(I^A, x)}{F(I^A, H_2)} \right)^{N-1} dx + (1 - \alpha) \int_{L_2} F^{N-2}(H_1, x) dx \tag{13}
\]

where \( \alpha \) is the probability that all bidders abstain, \( S^A(v_2) \) is the expected future surplus given abstaining, and \( F(H_1, v_2) \) is the marginal probability distribution of \( v_2 \) in the population.

Equation (13) follows from the two possibilities for future competition that can occur after an abstention: First, if someone bids \( R \) or more in the first period and one winner exits the game, there will be \( N-2 \) competitors drawn from \( F(H_1, v_2) \). Second, if everyone else also abstains, there will be \( N-1 \) competitors, all drawn from below \( I^A(v_2) \). Unfortunately for pure strategies, a bidder on \( I^A(v_2) \) has an incentive to deviate:

**Proposition 9:** When the seller uses a reserve price above \( L_1 \) in the first period, there is no symmetric pure-strategy equilibrium.

Proposition 9 follows immediately from equation (13): Despite the fact that \( v_1 - R = S^A(v_2) \) defines the curve, a bidder with valuations on \( I^A(v_2) \) is not actually indifferent between bidding \( R \) and abstaining. When

\[
\int_{L_1}^{v_2} \left( \frac{F(I^A, x)}{F(I^A, H_2)} \right)^{N-1} dx < \int_{L_2}^{v_2} F^{N-2}(H_1, x) dx,
\]

the bidder actually prefers to...
bid \( R \) because it either results in exactly the same payoff as abstaining (if there is at least one competing bid above \( R \)), or avoids the stiff future competition in the all-abstain case by winning the first object for \( R \) and receiving \( v_i - R > S^d(v_i) \). Therefore, a positive measure of bidders below \( I^d(v_i) \) actually prefer to bid \( R \). Analogously, if \( \int_{L_1}^{v_i} \left( \frac{F(I^d(x), x)}{F(I^d, H_2)} \right)^{N-1} dx > \int_{L_1}^{v_i} F^{N-2}(H_1, x) dx \), bidders slightly above \( I^d(v_i) \) will want to abstain rather than bid \( R \). Therefore, \( I^d(v_i) \) cannot be the indifference curve between bidding and abstaining unless the two surpluses are identical.

It is easy to see on a Uniform example that they cannot be identical – the all-abstain future competition involves more bidders drawn from a stochastically dominant distribution.

In summary a strategic reserve in the first period results in mixed entry strategies for bidders who are approximately indifferent between bidding the reserve and abstaining: the best response to everyone abstaining is to bid reserve, and the best response to fewer people abstaining is to abstain more often. Further analysis of reserve pricing is beyond the scope of this paper.

**More than two auctions**

The analysis of more than two auctions in a sequence is complicated by the asymmetry and learning issues first raised by Milgom & Weber (2000). Suppose there are three auctions instead of only two, with each bidder having private single-item valuations \( v_1, v_2, v_3 \), and realistically suppose further that only the winner and the price \( p_1 \) of the first unit are revealed before the second stage. Then, second-stage bidders have asymmetric beliefs about the \( v_3 \) of the remaining competitors, because one of the remaining bidders bid exactly \( p_1 \) while the other bidders bid strictly below \( p_1 \). Even if this asymmetry is resolved by revealing all first-stage bids as suggested by Milgom & Weber, the information about future goods would make it necessary to explicitly model second-stage beliefs about \( v_3 \) since first-stage bids are a function of \( v_3 \), and bidders thus may have an incentive to mislead their competitors into thinking their \( v_3 \) is very high by bidding very low in the first stage. Even the three-auction series would thus be a very strategically-rich environment, the price \( p_1 \) would enter second-stage bids, and it is not clear whether symmetric pure-strategy equilibria would exist in the first stage.
V. Discussion

This paper is motivated by the information structure of both the eBay marketplace, and the existing industrial-procurement marketplaces. In particular, the proposed models investigate the impact of public information about characteristics of future goods on equilibrium bidding. As these market institutions become economically more significant, it is important to learn about the properties of their demand side implied by forward-looking bidding strategies. The present results provide a first step towards understanding the buyer behavior in such information-rich multi-object marketplaces.

Sequential-auction models share much of their mathematics and intuition with the optimal-stopping models of sequential search. The unit-demand bidder must decide how much effort to put into purchasing the current unit, i.e. which probability of winning to select. There is an auction for a known object today (a chance to stop the “search”), and losing that auction (continuing the “search”) allows the bidder to advance to tomorrow’s auction and get some expected surplus. Since winning today (stopping the “search”) makes tomorrow’s surplus either unavailable (costly disposal) or lower (free disposal), a rational bidder does not try to win today as aggressively as she would in the absence of tomorrow’s auction. This paper analyzes what happens to this sequential-search intuition when the specific object sold in tomorrow’s auction is already known today. Under these rich informational conditions, the sequential-search metaphor is blended with a metaphor of choice that is inherently simultaneous: There are now two auctions, and the bidder already knows which of the auctions would give her a higher chance of surplus in isolation, and so it would be trivial to choose which of the auctions to participate in. However, the problem the bidder is facing is not a simple choice, but the first-auction bidding problem. The optimal bidding strategy thus needs to not only compensate the bidder for the opportunity cost of “stopping the search”, but also steer her towards the auction that is more lucrative for her. Naturally, the bidding strategy in the first auction is therefore a function of both valuations, increasing in the valuation of today’s object and decreasing in the valuation of tomorrow’s object. The bidder is thus trying to make the “choice” at least probabilistically within the sequential framework.

In a symmetric equilibrium, the bidder also needs to consider the fact that all other bidders are doing the same thing. When all bidders decrease their bids today as a function of their valuations of tomorrow’s good, there arises a non-trivial relationship between today’s
bidding strategy and tomorrow’s competition: tomorrow’s competition depends on the future valuations of the bidders who are more likely to lose today’s auction, which is endogenously determined by today’s bidding strategy that in turn depends on tomorrow’s competition. Therefore, the option value of tomorrow’s auction is intertwined with the bidding strategy in today’s auction. In equilibrium, the strategy is thus determined by an expected-surplus function that takes itself into account when estimating the expected level of competition tomorrow.

The main finding of this paper is that under mild assumptions about the distribution of valuations in the bidder population, there exists a unique symmetric pure-strategy Bayesian-Nash bidding equilibrium. A key property of this equilibrium is that there will almost always be trade in the first stage. Besides arguing for the existence of a well-behaved symmetric pure-strategy equilibrium, this paper also documents some general qualitative properties of equilibrium bidding. Equilibrium bids increase in the value of the first object and decrease in the value of the second object in the following way: bidders bid their private values of today’s object minus the expected surplus of tomorrow’s auction. The bidding strategy is thus fully characterized by the expected surplus function, which is shown to always increase convexly in the valuation of tomorrow’s unit and, less intuitively, to also depend on the valuation of today’s unit. The expected surplus function depends on the valuation of today’s unit because it is optimal for each bidder to assume that she is pivotal to the outcome of the game, and hence should she lose today’s auction, she would lose in a tie with today’s maximum competitive bid. This seeming complication, i.e. the fact that equilibrium expected future surplus depends on the valuation of today’s unit, is actually crucial in enabling a pure-strategy equilibrium to exist in the first place. If, for example, tomorrow’s competition were exogenously drawn from the probability distribution of valuations, there would be a chance of no trade today, and the purity of equilibrium strategies would likely break down because \( b_1 = 0 \) bidders would deviate up to avoid high future competition associated with everyone abstaining from the current auction.

In addition to this characterization of equilibrium bidding, this paper also demonstrates a number of general properties of the bidding function, including stability under affine transformation of the valuations, and a comparative statics in the number of bidders and the correlation of valuations. The equilibrium surplus function not only regular, but also differentiable despite the possibly discontinuous density of the valuation distribution because it involves a double integral of the distribution. As the number of bidders increases, the surplus
The existence and uniqueness of the above equilibrium follows from a contraction property of the equilibrium restriction on isobids. This contraction property guarantees convergence of a numerical-approximation technique based on iterating the equilibrium restriction. Using this numerical technique, quantitative statements can be made about the efficiency and distributional outcomes for a particular distribution of valuations. In the case of three bidders and the uniform distribution of the valuations, the new informational assumption has a noticeable impact on the overall economic outcomes: since “future-informed” bidders essentially align their bids with their relative preferences, the resulting allocation mechanism is more efficient than the alternative mechanisms with less information about the upcoming good. In fact, even with only three bidders, the mechanism is within one percent of realizing all potential gains from trade, suggesting that sequential auctions with information about future goods are a nearly efficient way of selling substitutes. This observation was also made by Budish (2008), and this paper extends the near-perfect efficiency finding to correlated distributions. A more thorough inquiry into the impact of information about future goods on the economic outcomes of auction-driven trading should be a fruitful area of future work.

Generalizing the model to more than two stages under the current assumptions would be cumbersome. When there are more than two auctions in a sequence, the bidders will still underbid in expectation of a positive future surplus. However, the optimal bidding strategies are no longer symmetric, and no longer depend only on each bidder’s private information, because there is useful information in prices and other experiences encountered along the way. Milgrom and Weber (2000) discuss the resulting problems extensively, and it is possible that the bidders might have incentives to deceive other bidders through artificially low early bids. However, at least a limited extension of the proposed models to many stages would be desirable. It may be possible to characterize steady-state equilibrium bidding in an infinite-horizon model with one-period look-ahead, or a finite-horizon model with a restrictive assumption that prices are secret until all objects are sold. Similarly, a generalization to allow strategic reserve prices would be difficult because existence of a pure-strategy equilibrium is no longer guaranteed.
Appendix: Proofs of propositions

Proof of Proposition 1: The objective function of a bidder with valuations \((v_1, v_2)\) and an expected surplus function \(S(v_2,c_1)\) is: 

\[
\Pi(\beta | v_1, v_2, S) = \int_0^\beta (v_1 - c_1) dG_i(c_1) + \int_{\beta}^{\infty} S(v_2, c_1) dG_i(c_1).
\]

The first-order conditions are \(dFOC = v_i - \beta - S(v_2, \beta)\). The function \(\Pi(\beta | v_1, v_2, S)\) is globally concave when does not decrease in \(c_1\) faster than unity. This is most easily demonstrated by assuming that \(S\) is partially differentiable: 

\[
\frac{d^2 \Pi}{d \beta^2} = -1 - \frac{\partial S}{\partial \beta}(v_2, \beta) < 0 \iff \frac{\partial S}{\partial \beta}(v_2, \beta) > -1.
\]

When \(S\) is not partially differentiable, it is clear that \(S(v_2,d) - S(v_2, c) > (1)(d - c)\) is sufficient for concavity of \(\Pi(\beta | v_1, v_2, S)\).

Figure A1: Bidding function \(b_1(v_1, v_2)\) is well defined by equation 2

Note to Figure: The solid diagonal 45 degree line is the LHS of equation (2) characterizing the best response to \(S\). The two dashed lines illustrate two possible RHS of equation (2): the lower line is an \(S\) that increases in bid-level, the upper line is an \(S\) that decreases but at a speed less than unity.

To see that the \(b_1(v_1, v_2)\) is well defined by the FOC, consider the following implications of regularity (and see Figure A1 for an illustration): Part A) implies that the RHS of equation (2) does not increase in \(\beta\) faster than unity. Part C) fixes the intercept of the RHS at \(v_1 - S(v_2, L_1) = v_1 \geq L_1\). Since the LHS of equation (2) is an identity function, continuity of \(S\) in \(c_1\) (part B of regularity) implies (via the Intermediate Value Theorem) that the RHS must eventually
intersect the LHS at some $\beta \geq L_i$ that solves equation (2). Such a solution is unique because the slope of $v_1 - S(v_2, \beta)$ in $\beta$ is less than unity everywhere, and so a second intersection is impossible. QED Proposition 1.

**Proof of Proposition 3:**

Fix any $c \in (L_i, H_i)$ and denote the global bound on $f$ by $\bar{f}$. Let $B(c_i)$ be a set of non-decreasing 1-Lipschitz functions $I : [L_2, H_2] \to [c_i, c_i + H_2 - L_2]$ such that $I(L_2) = c_i : B(c_i) = \{ I : [L_2, H_2] \to [c_i, c_i + H_2 - L_2], I(L_2) = c_i, \forall x > y, 0 \leq I(x) - I(y) < x - y, \}$

$B(c_i)$ is a closed subset of the complete metric space of all bounded continuous functions from $[L_2, H_2]$ to $[0, H_1 + H_2 - L_2]$ with the supremum metric: $d(I, J) = \max_{v \in [L_2, H_2]} \| I(v) - J(v) \|$, so it is a complete metric space with the same metric. It is immediate from equation 6 that $T$ projects $B(c_i)$ into itself: $T : B(c_i) \to B(c_i)$. The remainder of the proof shows that for every $c \in (L_i, H_i)$, there exists a $K \geq 1$ such that $T^K(I)$ is a contraction map, i.e. there exists a $q<1$ such that $d(T^K(I), T^K(J)) < q d(I, J)$ for all $I, J \in B(c_i)$. By the Banach Fixed Point Theorem, this is enough to show that $T$ has a unique fixed point in $B(c_i)$ and iterations of $T$ starting at any point in $B(c_i)$ converge to the unique fixed point exponentially fast. The unique fixed point is the unique isobid $I(v_2 | c_i)$.

It is enough to consider $N=3$, $d(T(I), T(J))$ obviously decreases in $N$. Pick any distance $\delta > 0$ and any $I_i \in B(c_i)$. Let $I_2(v) = \arg\max_{f \in B(c_i)} d(I^K(I_i), T^K(J)) = I_1(v) + \min[\delta, v]$, and note that $d(T^K(I_i), T^K(I_2)) < d(T^K(I_i), T^K(I_1 + \delta))$. Therefore, it is sufficient to show $d(T^K(I), T^K(I_1 + \delta)) < q \delta$ for some $q<1$. The following Lemma is critical for the result:

**Lemma 1:** $\forall \alpha \in \mathbb{R}_+ \& \text{integer } M \geq 0: |T(I(x) + \frac{\alpha x^M}{M!})(w) - T(I(x))(w)| < \frac{\alpha \bar{f}^M(w)}{(M + 2)!F(c_1, 1)}$

**Proof:**

$$
|T(I(x) + \frac{\alpha x^M}{M!})(w) - T(I(x))(w)| = \left| \int_{L_2}^{w} f(v_1, v_2) dv_1 dv_2 - \int_{L_2}^{w} f(v_1, v_2) dv_1 dv_2 \right| 
< \frac{\int_{L_2}^{w} \int_{L_2}^{w} f(v_1, v_2) dv_1 dv_2}{F(c_1, 1)} \frac{\alpha \bar{f}^M(w)}{(M + 2)!F(c_1, 1)} 
< \frac{\alpha \bar{f}^M(w)}{(M + 2)!F(c_1, 1)} 
$$
Lemma 1 implies that $T$ may not be a contraction for small-enough $c_1$; setting $\alpha = \delta$ and $M = 0$ in Lemma 1 implies $d(T(I), T(I + \delta)) = \frac{\delta^2H_2^2}{2F(c_1, 1)}$, and may not be less than unity. However, note that $T$ bounds the difference between the images of $I$ and $I + \delta$ quadratically as a function of $v$: $|T(I)(v) - T(I + \delta)(v)| < \frac{\delta^2v_2^2}{2F(c_1, 1)}$. The quadratic bound strengthens to quartic when one applies $T$ twice, i.e. when $T$ is applied to $T(I + \delta)$ and $T(I)$:

$$\left|T^2(I + \delta)(w) - T^2(I)(w)\right| < \left|T\left(J(v) + \frac{\delta^2v_2^2}{2F(c_1, 1)}\right) - T(J(v))(w)\right| < \frac{\delta^4v_2^4}{4!} \left(\frac{\delta^2H_2^2}{F(c_1, 1)}\right)^2$$

where $J = T(I)$. By induction, $d(T^K(I), T^K(I + \delta)) < \frac{\delta}{(2K)!} \left(\frac{\delta^2H_2^2}{F(c_1, 1)}\right)^K$. Since $\lim_{K \to \infty} \frac{C^K}{(2K)!} = 0$ for every positive constant $C$, this proves that there exists a $K$ high enough that $T^K$ is a contraction.

**QED Proposition 3.**

**Proof of Proposition 5:** Suppose there is a pair $d > c$ and a $\tilde{v} \in [L_2, H_2]$ such that $I(v|d) \leq I(v|c)$, and denote $I_c(v) = I(v|c)$ for brevity. Continuity of isobids implies that the two isobids must intersect at or below $\tilde{v}$. Let the smallest intersection point be $v^*$, i.e. $I_d(v) > I_c(v)$ for all $v < v^*$ and $I_d(v^*) = I_c(v^*)$. Therefore, $I_c(v)$ intersects $I_d(v)$ from below at $v^*$, and so the slope of $I_c(v)$ at $v^*$ must be weakly higher than the slope of $I_d(v)$ at $v^*$: $I'_c(v^*) \geq I'_d(v^*)$. Equilibrium together with full support of $f$ rule this ordering of slopes at $v^*$ out, there are three distinct cases.

**Case 1:** $I_d(v) \leq I_c(v)$ on $(v^*, H_2]$ (see dotted line in Figure A2 accompanying this proof). From equation 6, $I'_d(v^*) = G_2(v^*|b) = \left(\frac{F(b, v^*)}{F(b, H_2)}\right)^{N-2}$. Let $A = F(\min[I_c, I_d], H_2)$, $A1 = F(I_c, v^*)$, and $X = F(I_d, v^*) - F(I_c, v^*)$. From full support of $f$, $X > 0$. Since $I_d(v) \leq I_c(v)$ on $(v^*, H_2]$, $I'_d(v^*) = \left(\frac{A1 + X}{A + X}\right)^{N-2} > \left(\frac{A1}{A}\right)^{N-2} \geq \left(\frac{A1}{A + B + Y}\right)^{N-2} = I'_c(v^*)$, where $B + Y \geq 0$ is the probability mass between the two isobids on $(v^*, H_2]$, and it is non-negative because of full support and $I_d(v) \leq I_c(v)$ on $(v^*, H_2]$: $B + Y = \left[ F(I_c, H_2) - F(I_c, v^*) \right] - \left[ F(I_d, H_2) - F(I_d, v^*) \right]$. This is a contradiction with the ordering of slopes necessary for an intersection at $v^*$. Intuitively, a single intersection at $v^*$ forces the conditional probability of a $v_2 < v^*$ greater for the higher and flatter isobid, and this conditional probability happens to be the slope of the isobid at $v^*$.
Figure A2: Cases 1 and 2 of the $I(v_2|d)$ isobid in Proposition 5

Note to Figure: The solid convex function is the $I(v_2|c)$ isobid. The dotted line is the Case 1 possibility for the $I(v_2|d)$ isobid, and the remaining solid line is the Case 2 possibility for the $I(v_2|d)$ isobid. All marked areas denote probability mass between curves.

**Case 2:** Suppose the ordering of the isobids on $(v^*, H_2)$ is ambiguous. Then, there must be at least one more intersection point $w^* > v^*$, and it must be true that $I'_c(w^*) \leq I'_d(w^*)$. For equilibrium isobids, this ordering of slopes at $w^*$ is ruled out by the ordering of slopes at $v^*$ and the full support. Let $A2 + B1 = F(I_d, w^*) - F(I_d, v^*)$, $Y = [F(I_c, w^*) - F(I_c, v^*)] - [F(I_d, w^*) - F(I_d, v^*)]$, and $Q = [F(I_d, H_2) - F(I_d, w^*)] - [F(I_c, H_2) - F(I_c, w^*)]$ (see Figure A2). From full support of $f$, $A2 + B1 > 0$ and $Y > 0$, but the sign of $Q$ is ambiguous. Nevertheless, $I'_c(v^*) \geq I'_d(v^*)$ can be expressed as: $I'_c(v^*) = \left( \frac{A1}{A + B + Y} \right)^{N-2} \cdot \left( \frac{A1 + X}{A + B + X + Q} \right)^{N-2} = I'_d(v^*)$. To obtain the implied slopes at $w^*$, add $A2 + B1 + Y$ to the RHS numerator and only $A2 + B1$ to the RHS numerator. These additions clearly preserve the inequality: $I'_c(w^*) = \left( \frac{A1 + (A2 + B1 + Y)}{A + B + Y} \right)^{N-2} \cdot \left( \frac{A1 + X + (A2 + B1)}{A + B + X + Q} \right)^{N-2} = I'_d(w^*)$. This is a contradiction with the necessary ordering of slopes for a second intersection at $w^*$: $I'_c(w^*) < I'_d(w^*)$.  

36
Case 3: The only remaining possibility that \( v^* \) is in fact a point of tangency, i.e. \( I'_c(v^*) = I'_d(v^*) \) and \( I_d(v) > I_c(v) \) everywhere other than at \( v^* \). Since any intersections are ruled out by Cases 1 and 2, tangency at \( v^* \) means that the tangency holds for all \( z \in [c,d] \), and the curvatures of all the \( I_z \) at \( v^* \) must be non-decreasing in \( z \): \[ \frac{dI''_z(v^*)}{dz} \geq 0. \] Once again, equilibrium implies that this ordering of curvatures cannot happen: Let the slope at \( v^* \) be equal to some \( \lambda \). Then, for every \( z \in [c,d] \), \[ F^{N-2}(I_z, z^*) = G_2(v^* | z) = \lambda, \] and so \( I_z(v) = z + \lambda \int_0^v F^{N-2}(I_z, x) \, dx \). Differentiate twice to obtain \[ I''_z(v) = \frac{\lambda(N-2)}{F^{N-2}(I_z, v^*)} I_z(v) \Rightarrow I''_z(v^*) = \frac{\lambda(N-2)}{F(I_z, v^*)} I_z(v^*) \] Since \( I_z(v^*) \) is a constant for \( z \in [c,d] \) and \( F(I_z, v^*) \) strictly increases in \( z \) because of full support, this means that \( I''_z(v^*) \) strictly decreases in \( z \), a contradiction with \( \frac{dI''_z(v^*)}{dz} \geq 0 \). QED

Proposition 5.

Proof of Proposition 6: It is enough to show that \( I(v_2, c_1) \) is upper semi-continuous at \( c_1 \), the proof of lower semi-continuity is analogous. Fix \( c_1 \) and consider any monotonically decreasing sequence of \( \delta_n > 0 \) such that \( \lim_{n \to \infty} \delta_n = 0 \). The corresponding sequence of isobids \( \{I(v_2 | c_1 + \delta_n)\}_{n=1}^{\infty} \) is uniformly bounded because it projects into \( [c_1, H_1 + H_2 - L_2] \), and it is equicontinuous because all isobids are non-decreasing and have slopes less than unity (1-Lipschitz). Therefore, the Arzela-Ascoli Theorem implies that \( \{I(v_2 | c_1 + \delta_n)\}_{n=1}^{\infty} \) has a uniformly convergent subsequence, and its limit is some 1-Lipschitz function \( I_+(v_2 | c_1) \). Monotonicity of the original sequence (Proposition 5) implies that it also converges uniformly to \( I_+(v_2 | c_1) \), because when \( I(v_2 | c_1 + \delta_n) \) is an element of the convergent subsequence, all \( I(v_2 | c_1 + \delta_{n+k}) \) are between \( I(v_2 | c_1 + \delta_n) \) and \( I_+(v_2 | c_1) \) in the supremum metric \( d(I, J) = \max_{v \in [L_2, H_2]} |I(v) - J(v)| \):

For every \( \varepsilon > 0 \), there is therefore a \( \delta > 0 \) such that for every \( \delta < \delta \),
\[ d\left( I_+ (\cdot | c_1), I_+ (\cdot | c_1 + \delta) \right) < \varepsilon. \]

Monotonicity further implies that for every \( \delta, I_+(v_2 | c_1) \leq I_+(v_2 | c_1 + \delta) \). It remains to be shown that \( I_+(v_2 | c_1) = I(v_2 | c_1) \). This follows from \( f \) having a bounded density \( f < \tilde{f} \) and from the uniqueness of isobids. The bounded density together with the uniform convergence of \( \{I(v_2 | c_1 + \delta_n)\}_{n=1}^{\infty} \) implies that the implied sequence of distributions \( \{G_2(c_2 | c_1 + \delta_n)\}_{n=1}^{\infty} \) also approaches the implied \( G_2(c_2 | I_+(c_1)) \) uniformly:
Pick any $\varepsilon > 0$. Uniform convergence of isobids $\Rightarrow \exists \delta > 0$ such that $\forall \delta < \delta$, 

$$
\left| I(v_2 | c_1 + \delta) - I_+ (v_2 | c_1) \right| \leq \frac{F(c_1, H_2) \varepsilon}{f(H_2 - L_2)}, \ \forall \delta < \delta, \ \text{this implies } \left| G_2 (x | c_1 + \delta) - G_2 (x | I_+) \right| =
$$

$$
= \left( \frac{F \left[ I \left( \cdot | c_1 + \delta \right), x \right]}{F \left[ I \left( \cdot | c_1 + \delta \right), H_2 \right]} \right)^{N-2} \left( \frac{F \left[ I_+ (\cdot | c_1), x \right]}{F \left[ I_+ (\cdot | c_1), H_2 \right]} \right)^{N-2} \left| \frac{F \left[ I \left( \cdot | c_1 + \delta \right), x \right]}{F \left[ I \left( \cdot | c_1 + \delta \right), H_2 \right]} - \frac{F \left[ I_+ (\cdot | c_1), x \right]}{F \left[ I_+ (\cdot | c_1), H_2 \right]} \right| <
$$

$$
\int_{L_2}^{H_2} \int_{L_2}^{H_2} \frac{f(v_1, v_2)}{F(c_1, H_2)} dv_1 dv_2 < \frac{\int \int \left[ I(v_2 | c_1 + \delta) - I_+ (v_2 | c_1) \right] dv_2}{F(c_1, H_2)} < \varepsilon
$$

The uniform convergence of $\{ G_2 \left( c_2 | c_1 + \delta_n \right) \}_{n=1}^{\infty}$ in turn implies that the equilibrium relation from equation (6) $I(v_2 | c_1 + \delta) = T \left[ I \left( v_2 | c_1 + \delta \right) | c_1 + \delta \right]$, which holds for every member of the sequence, is preserved in the limit.

$$
I_+ (v_2 | c_1) = \lim_{\delta \to 0} I_+ (L_2 | c_1 + \delta) + \lim_{\delta \to 0} \int_0^{v_1} G_2^{N-2} (x | c_1 + \delta) dx =
$$

$$
= c_1 + \int_0^{v_1} \lim_{\delta \to 0} G_2^{N-2} (x | c_1 + \delta) dx = T \left[ I_+ (v_2 | c_1) | c_1 \right]
$$

Since $I_+ (v_2 | c_1) = T \left[ I_+ (v_2 | c_1) | c_1 \right]$, it must be true that $I_+ (v_2 | c_1) = I (v_2 | c_1)$ because each isobid is unique (Proposition 3). QED Proposition 6.

**Proof of Proposition 7**: By the same arguments used in the proof of Proposition 6, any monotonic sequence $\{ I \left( v_2 | L_1 + \delta_n \right) \}_{n=1}^{\infty}$ converges uniformly to some 1-Lipschitz function $J (v_2)$ such that $L_1 \leq J (v_2) < I \left( v_2 | L_1 + \delta \right)$ and $J \left( L_2 \right) = L_1$. This proof shows that $J (v_2)$ must be the constant function $J (v_2) = L_1$. Suppose $J (v_2) > L_1$ for some $v_2 > L_2$, and so there is a positive mass under $J : F (J, H_2) > 0$ (from continuity of $J$ together with full support of $f$).

$F (J, H_2) > 0$ in turn implies (see proof of Proposition 6) that the RHS equation (6) is well-defined, and so $J = T (J | L_1)$, i.e. that $J$ is a valid isobid for the bid-level $L_1$. The rest of this proof shows that there cannot be a $J : [L_2, H_2] \to [L_1, L_1 + H_2 - L_2]$ such that $F (J, H_2) > 0$, and $J = T (J | L_1)$. The following Lemma 2 is crucial for proving the impossibility:
Lemma 2: Suppose \( F(J, H_2) > A > 0 \) and \( 0 < f < \bar{f} \). Then, \( \forall \alpha \in R_+ \) and integer \( M \geq 0 \):

\[
J(v_2) \leq L_1 + \alpha \frac{v_2^M}{M!} \Rightarrow T(J|L_1)(v_2) < L_1 + \left( \frac{\alpha \bar{f}}{A} \right) \frac{v_2^{M+2}}{(M+2)!}
\]

Proof: \( T(J|L_1)(v_2) = L_1 + \frac{\int \int \int f(v_1, w) dv_1 dw dz}{F(J, H_2)} < L_1 + \frac{\int \int \int f(v_1, w) dv_1 dw dz + \alpha \int \int \int \bar{f} v_2^M}{A} \frac{v_2^{M+2}}{(M+2)!} \frac{v_2^{M+2}}{(M+2)!}
\]

Lemma 2 implies that successive applications of \( T(\cdot|L_1) \) must eventually contract \( J \) arbitrarily close to \( L_1 \) – a contradiction with a fixed point of \( T(\cdot|L_1) \) satisfying \( F(J, H_2) > A > 0 \): From \( J(v_2) \) being a 1-Lipschitz function, \( J(v_2) \leq L_1 + v_2 \). Therefore,

\[
J(v_2) \leq L_1 + v_2 \Rightarrow T(J|L_1)(v_2) < L_1 + \left( \frac{\bar{f}}{A} \right) \frac{v_2^3}{3!} \Rightarrow T^2(J|L_1)(v_2) < L_1 + \left( \frac{\bar{f}}{A} \right)^2 \frac{v_2^5}{5!} ...
\]

\( (\text{induction}) \)

\[
T^K(J|L_1)(v_2) < L_1 + \left( \frac{\bar{f}}{A} \right)^K \frac{v_2^{2K+1}}{(2K+1)!} \Rightarrow \forall v_2 \in [L_2, H_2]: \lim_{K \to \infty} T^K(J|L_1)(v_2) = 0
\]

It is important to note that there is a crucial difference between Lemmas 1 and 2: Lemma 1 shows that for \( c_i > L_1 \), enough repetitions of \( T(\cdot|c_i) \) eventually contract the images of two candidate functions \( T^K(I_1|c_i) \) and \( T^K(I_2|c_i) \) to be closer to each other (in the supremum metric) than \( I_1 \) and \( I_2 \). In contrast, Lemma 2 shows that for \( c_i = L_1 \), enough repetitions of \( T(\cdot|c_i) \) eventually contract the image of any single candidate function \( I_1 \) to be arbitrarily close to \( L_1 \). However, \( L_1 \) is not a fixed point of \( T(\cdot|L_1) \) because \( T(L_1|L_1) \) is not defined: \( T(\cdot|L_1) \) is not a continuous function on the space of 1-Lipschitz functions \( J:[L_2, H_2] \to [L_1, L_1 + H_2 - L_2] \).

QED Proposition 7

Proof of Theorem 1: Proposition 3 shows that the (symmetric pure-strategy Nash) equilibrium conditions imply the existence of a unique isobid function for each bid-level \( c_i \in (L_i, H_i) \). These equilibrium isobids in turn imply a unique expected surplus function \( S(v_2, c_i) = I(v_2|c_i) - c_i \) defined on \([L_2, H_2] \times (L_1, H_1)\). The rest of the Propositions (4-6) then show that the equilibrium \( S(v_2, c_i) \) is regular on the closure of \([L_2, H_2] \times (L_1, H_1)\). Specifically, Proposition 5 orders the different isobids monotonically in their bid-level, establishing part A of regularity. Proposition 6 then shows continuity of \( S \) on \( c_i \in (L_i, H_i) \), and Proposition 7 finds that \( S(v_2, L_1) \) defined as a limit at \( L_1 \) (and hence continuous there, completing part B) must be equal to zero, which proves part C of regularity. Since the candidates for equilibrium isobids on \( c_i \in (L_i, H_i) \) thus imply a
unique regular surplus function $S(v_2,c_1)$ on $[L_2,H_2] \times [L_1,H_1]$, Proposition 1 implies that there is also a unique symmetric pure-strategy bidding function $b_1(v_1,v_2) = v_1 - S(v_2,b_1(v_1,v_2))$ that is a best response to itself.

Properties of $b_1$: First, $b_1(L_1,v_2) = L_1$ because part C implies $L_1 = L_1 - S(v_2,L_1)$. Second, Proposition 4 implies $b_1(v_1,L_2) = v_1$ and $v_1 > b_1(v_1,v_2) > L_1$ away from $(L_1,L_2)$. Third, regularity and Proposition 4 imply the comparative statics in valuations: suppose for simplicity that $S$ is partially differentiable in $c_1$, and implicitly differentiate of the first-order condition (2):

$$\frac{\partial b_1}{\partial v_2} = \frac{\partial S}{\partial v_2} < 0$$

away from $(L_1,L_2)$ because the numerator is negative by Proposition 4 (and the denominator is positive by part A). Similarly, $\frac{\partial b_1}{\partial v_1} = \left(1 + \frac{\partial S}{\partial c_1|_{c_1 = h}}\right)^{-1} > 0$, so $b_1(v_1,v_2)$ is increasing in $v_1$, but not necessarily at a constant speed equal to unity. The curvature of $b_1(v_1,v_2)$ is ambiguous.

Finally, the comparative static in the number of bidders arises because $S(v_2,c) < E(v_2)$, where $E(v_2)$ is the expected surplus from facing (in a second-price sealed-bid auction) $N-2$ bidders drawn randomly from the $N-1$ lowest bidders of the original $N$ bidders. Let the valuations of the $N-1$ lowest bidders of the original $N$ bidders have a distribution function $G$. Then,

$$E(v_2) = \int_{L_2}^{v_2} G^{N-2}(x) dx$$

which follows from integration by parts as in equation (3).

The integrand converges to zero uniformly as $N$ approaches infinity as long as $G(v_2) < 1$. From full support of $f$, this holds for all $v_2 < H_2$. Since $G^{N-2}(x)$ approaches zero on $[L_2,H_2]$ pointwise, its integral $E(v_2)$ must approach zero even $v_2 = H_2$. QED Theorem 1.

**Proof of Proposition 8:** The equilibrium isobids can be simply scaled up by while preserving the equilibrium relation in equation (6). Specifically, when $I(v_2 \mid c_1)$ is the equilibrium isobid on $[L_1,H_1] \times [L_2,H_2]$, then $I_v(w_2 \mid A + Bc_1) = A + BI\left(\frac{w_2 - A}{B} \mid c_1\right)$ is the equilibrium isobid of the valuations transformed to $[A + BL_1, A + BH_1] \times [A + BL_2, A + BH_2]$ support. The key to the result is that the conditional probability under the transformed $I$ remains unchanged:
\[
F_w \left[ A + BI \left( \frac{w_2 - A}{B} \mid c_1 \right), A + Bx \right] = \int_{A + Bz}^{A + Bz} \frac{1}{B^2} f \left( \frac{w_1 - A}{B}, \frac{w_2 - A}{B} \right) \text{d}w_1 \text{d}w_2 = \frac{\left( \frac{w - A}{B} \right)}{v(v_1, v_2)} f(v_1, v_2) \text{d}v_1 \text{d}v_2 = F \left[ I(\cdot \mid c_1), x \right] \quad \text{where} \quad f_w(w_1, w_2) = \frac{1}{B^2} f \left( \frac{w_1 - A}{B}, \frac{w_2 - A}{B} \right)
\]

Therefore, \( I_w \) is the fixed point of the appropriately defined \( T \) from equation (6):

\[
T_w \left[ I_w \mid A + Bc_1 \right](w_2) = A + Bc_1 + \int_{A + Bz_2}^{A + Bz_1} \frac{F_w \left[ A + BI \left( \frac{w_2 - A}{B} \mid c_1 \right), z \right]}{F_w \left[ A + BI \left( \frac{w_2 - A}{B} \mid c_1 \right), A + BH_2 \right]} \text{d}z = A + BI \left( \frac{w_2 - A}{B} \mid c_1 \right) = I_w \left( w_2 \mid A + Bc_1 \right)
\]

The equilibrium surplus function is therefore \( S_w(w_2, A + Bc_1) = BS \left( \frac{w_2 - A}{B}, c_1 \right) \), and when \( \beta \) is a solution to \( \beta = v_1 - S \left( \frac{w_2 - A}{B}, \beta \right) \), then multiplying both sides of the equation by \( B \) and adding \( A \) leads to \( A + B\beta = A + Bv_1 - BS \left( \frac{w_2 - A}{B}, \beta \right) = w_1 = BS_w(w_2, A + B\beta), so A + B\beta \) is a solution to \( \gamma = w_1 - BS_w(w_2, \gamma) \). QED Proposition 8

Appendix: Relationship with Milgrom & Weber (2000)

The proposed model nests the identical-good model of Milgrom and Weber (2000) (hereafter MW). MW assume that \( v_1 = v_2 \), so the relevant \( f \) is a diagonal ridge in the unit square:

\[
f(v_1, v_2) = \begin{cases} f(v) & \text{when } v_1 = v_2 \\ 0 & \text{when } v_1 \neq v_2 \end{cases}
\]

This \( f \) violates the full-support assumption of this paper, but an adjustment in the measure of integration accommodates it as follows: The development in equations (1) - (3) is still valid because it does not directly rely on the properties of \( f \). The isobids can still be considered to occur in the entire unit square, and it is clear that for every \( c_1 \), \( I(v_2 \mid c_1) \) intersects the diagonal line at some point \( v^*(c_1) \) such that bidders who bid \( c_1 \) have value \( v_1 = v_2 = v^*(c_1) \), and \( I(v_2 \mid c_1) \) coincides with the diagonal line beyond that point (in other words, \( v^*(c_1) \) is the smallest valuation for which \( G(v \mid c_1) = 1 \)). The “probability mass under the isobid” idea of equation (4) is still valid, but with the integration only over the support of \( f \), i.e. the diagonal line \( v_1 = v_2 \): adjust equations (4) and (6) by redefining...
\[
F\left[ I (\cdot | c_1) , v \right] = \int_{L_2}^{v} 1 (I(v | c_1) > v) f(v) dv = \int_{L_2}^{v} 1 (v < v^*(c_1)) f(v) dv
\]
Therefore, for every \(v_2 < v^*(c_1)\), equation (6) can be replaced with
\[
I(v_2 | c_1) = c_1 + \int_{0}^{v_2} \frac{\int_{0}^{v} f(v) dv}{\int_{0}^{v^*(c_1)} f(v) dv} \ dx = c_1 + \int_{0}^{v_2} \Pr (v < x | v < v^*(c_1)) v^{N-2} \ dx .
\]

Following MW in defining \(Y_k^{(m)}\) to be the \(k\)-th highest of \(m\) values drawn \(iid\) from \(F\), and specifically \(Y_1 = Y_1^{(N-1)}\) and \(Y_2 = Y_2^{(N-1)}\) to be the top two order-statistics of competitive values, the isobid can be written as
\[
I(v_2 | c_1) = c_1 + \int_{0}^{v_2} \Pr (Y_2 < x | Y_1 = v^*(c_1)) dx ,
\]
and the first-stage bid satisfies \(b(v) = v - \int_{0}^{v} \Pr (Y_2 < x | Y_1 = v) dx\) because \(v^*(b(v)) = v\). Integration by parts then shows
\[
\int_{0}^{v} \Pr (Y_2 < x | Y_1 = v) dx = \int_{0}^{v} (v - x) d \Pr (Y_2 < x | Y_1 = v) = v - E (Y_2 | Y_1 = v) ,
\]
satisfying \(b(v) = E (Y_2 | Y_1 = v)\), which is the more familiar form of the result.

In the example of \(f\) Uniform, the above process can be illustrated in closed form. Fix a \(c_1\) and suppress it from notation. Since \(f(v, v) = 1\),
\[
I(v_2) = c_1 + \int_{0}^{v_2} \frac{X_{v^*}}{v^*} v^{N-2} \ dx = c_1 + \frac{v^{N-1}}{(N-1)v^*} ,
\]
so the point of intersection can be solved for as:
\[
c_1 + \frac{v^{N-1}}{(N-1)v^*} = v^* \Rightarrow v^* = \frac{N-1}{N-2} c_1 ,
\]
implying the diagonal equilibrium expected-surplus function
\[
S (v, c_1) = S (v_2, c_1 | v = v_2 = v_1 = v^*) = \frac{c_1}{N-2} ,
\]
and so solving (2) along \(v=v_1=v_2\) involves
\[
b_1 = v - \frac{b_1}{N-2} \Rightarrow b_1 (v) = \frac{N-2}{N-1} v = E (Y_2 | Y_1 = v, f \text{ Uniform}) .
\]
References


