Stock Market Volatility and Learning∗

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Abstract

Introducing learning into a standard consumption based asset pricing model with constant discount factor considerably improves its empirical performance. Learning causes momentum and mean reversion of returns and thereby excess volatility, long-horizon return predictability, and low frequency deviations from rational expectations (RE) prices. Learning also generates the possibility of price bubbles and - for overvalued prices - stock market ‘crashes’, i.e., sudden and strong price decreases with prices having a tendency to fall below their RE value. No symmetric stock market increases occur when prices are undervalued. Using first moments of U.S. asset price data (1926:1-1998:4) to estimate the learning model, we find plausible values for the discount factor and the coefficient of relative risk aversion and a surprisingly good match with the second moments of the data. Required deviations from full forecast rationality are small and learning based forecasts often outperform rational forecasts in samples with the same length as our empirical data.

JEL Class. No.: G12

1 Introduction

Stock prices show movements that are difficult to explain within the realm of rational expectation models. The high premium earned for holding stocks, the high volatility of stock prices and the low volatility of dividends, the occurrence of stock market crashes, and the existence of long sustained increases and decreases of stock prices are difficult to explain if stock prices are the discounted value of dividends.

A simple consumption based asset pricing model of learning where today’s stock price is determined by the expectation of tomorrow’s stock price has the

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potential to explain all these observations. The dynamics generated by learning are such that the market will display instability, i.e., a tendency to deviate from rational expectations (RE) prices. Under learning the dynamics of expectations and actual price are such that if the market has been decreasing (increasing), it will have a tendency to decrease (increase) further. Once prices deviate from their RE values, mean reverting forces start to operate and tend to pull prices back to normal levels. Since these reverting forces tend to operate gradually, they generate a slow and protracted return to normal valuation levels over time.

For an increasing market, we find that the upward momentum in returns generated by learning can become stronger than the mean reverting forces and thus give rise to an asset price bubble with unstable price dynamics. Yet, once price embark on a bubble path, small innovations in fundamentals (dividends) can trigger a market ‘crash’, i.e., strong negative returns with prices having a tendency to fall below their RE levels. Since these price decreases can be triggered by small changes in fundamentals and perceptions, we refer to them as stock market ‘crashes’.

As is easy to see, the forces introduced by learning generate excess volatility and long-horizon predictability of returns. These asset pricing features, together with the existence of stock market bubbles and crashes, have been a major puzzle for standard rational expectations models.  

RE models are also not very rich in terms of their interactions between market volatility and various other aspects of the economy such as the conduct of monetary policy, the degree of investors’ risk aversion, or the presence of speculative investors with short investment horizons. We document that under learning low real interest rates, very high and very low degrees of risk aversion, and the presence of speculative investors tend to increase the fluctuations in the market. A model with learning thus suggests a different role for monetary policy and investors’ risk attitude that seems to be consistent with views generally expressed by central bankers, e.g., Papademos (2005).

In addition to studying the qualitative features introduced by learning, we evaluate the ability of the learning model to quantitatively account for the behavior of U.S. stock markets. In particular, we estimate a model with learning using the simulated method of moments and the first moments of quarterly U.S. data (1926:1-1998:4) provided by Campbell (2003). We show that the model quantitatively matches important second moments of the data, such as the volatility and persistence of the price dividend ratio or the evidence on long-term excess return predictability. An estimated standard RE model with time-varying discount factors generated by habit persistence, as used by Abel (1990), performs considerably worse: it grossly fails to capture the second moments.

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1 Santos and Woodford (1997) show that the existence of asset price bubbles rests on rather fragile conditions in infinite horizon RE models.
of the price dividend ratio, does not generate return predictability over long horizons, and does not allow for the possibility of bubbles or crashes.

Models of learning have been used before to explain various asset pricing facts. Timmermann (1993, 1996) and Brennan and Xia (2001), for example, explore the implications of Bayesian learning about the dividend process and show that learning can increase return volatility and may even generate an empirically plausible equity premium. This paper abstracts from learning about the dividend process and considers learning about the stock price process instead. Bullard and Duffy (2001) show that learning dynamics can converge to complicated attractors, whenever the RE equilibrium is unstable under learning dynamics, and thereby give rise to a range of interesting asset pricing phenomena. Brock and Hommes (1998) derive similar results studying a heterogenous agent economy where agents choose between various predictors of future stock price. The present model considers a representative agent economy in which the rational expectations equilibrium is stable under learning dynamics. Recently, Carceles-Poveda and Giannitsarou (2006) find that the presence of learning does not alter significantly the behavior of asset prices in a linearized economy with stationary dividends. The present paper finds strong effects of learning for the corresponding non-linear model with stationary dividend growth.

The paper is organized as follows. Section 2 documents various asset pricing facts from the literature that this paper is concerned with. Section 3 presents a simple learning-based asset pricing model and derives analytical results about the behavior of stock prices under learning. Section 4 extends the setting and discusses various determinants of stock market volatility. In section 5 we estimate the learning model and evaluate its performance. We also show that the learning model generates deviations from RE that would be difficult to detect, even for samples of the length considered in our empirical analysis. Section 6 concludes.

2 Facts and Models

Throughout this paper we are concerned with the asset pricing facts outlined in this section. While these facts are well documented in the literature, we quantify them here using a single data set for the U.S. covering the period 1926:1-1998:4. The empirical evidence provided in this section is used later on to estimate and evaluate the asset pricing model we develop.

2 Stability under learning dynamics is defined in Marcet and Sargent (1989).

3 Cecchetti, Lam, and Mark (2000) determine the misspecification in beliefs about future consumption growth required to match the equity premium and other moments of asset prices.

4 We use lagged dividends to compute the price dividend ratio, causing the effective sample to start in 1927:1. The data is provided by Campbell (2003) and based on NYSE/AMEX value-weighted portfolio returns taken from CRSP stock file indices. It can be downloaded at http://kuznets.fas.harvard.edu/~campbell/data.html.
1. **Equity premium and risk free rate.** Real stock returns - averaged over long time spans - tend to be high, while real returns on short-term bonds tend to be low and very close to zero on average. These asset pricing facts are usually referred to as the equity premium puzzle and the risk free rate puzzle, respectively (Mehra and Prescott (1985), Weil (1989)). The upper panel of table 1 documents these features for the U.S. economy.5

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>Average PD-ratio</td>
<td>105.4</td>
</tr>
<tr>
<td>Average stock return</td>
<td>2.36%</td>
</tr>
<tr>
<td>Average bond return</td>
<td>0.16%</td>
</tr>
<tr>
<td>Second moments</td>
<td></td>
</tr>
<tr>
<td>Std. dev. PD-ratio</td>
<td>35.4</td>
</tr>
<tr>
<td>Auto-correlation PD-ratio</td>
<td>0.95</td>
</tr>
<tr>
<td>Std. dev. stock return</td>
<td>11.5%</td>
</tr>
<tr>
<td>Std. dev. bond return</td>
<td>1.35%</td>
</tr>
</tbody>
</table>

Table 1: Asset pricing facts

2. **Behavior of the price dividend ratio.** The price dividend ratio is high on average, very volatile and displays rather persistent deviations from its mean value. Table 1 shows that the average real dividend yield in the U.S. has been below 1% quarterly and that the mean price dividend ratio is less than 3 times its standard deviation. The autocorrelation of the price dividend ratio implies deviations from the mean to have a half-life between 3 and 4 years. Figure 1 depicts the U.S. price dividend ratio, illustrating that it large low frequency deviations from its sample mean (bold horizontal line in the graph).

3. **Excess volatility.** Numerous studies, starting with Shiller (1981) and LeRoy and Porter (1981), have found that it is difficult to reconcile actual stock price behavior with that implied by rational expectations models with constant discount rates. In the light of such asset pricing models, the price dividend ratio should be constant if dividend growth is white noise, which seems empirically plausible. A look at figure 1 then suggests that stock prices are not close to the present value of dividends. Similarly, asset

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5 Table 1 reports moments for level data. Using data in logs gives rise to a very similar picture. The equity premium, however, then drops to approximately 6.4% p.a. compared to the 8.8% p.a. implied by the table.
pricing models with constant discount rates suggest that stock returns are far too volatile when compared to the volatility of real dividend growth, a fact sometimes referred to as the volatility puzzle. In the U.S. stock returns are about three times as volatile as dividend growth, see tables 1 and 2.\(^6\) A model with constant discount factor would suggest both to have roughly the same volatility.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Average dividend growth</td>
<td>0.346%</td>
</tr>
<tr>
<td>Std. dev. dividend growth</td>
<td>3.63%</td>
</tr>
</tbody>
</table>

Table 2: Dividend behavior

4. Long-horizon predictability. The price dividend ratio has no clear ability to forecast future fundamentals, e.g., future dividends or future earnings, see Campbell (2003). A large body of research suggests, however, that the

\(^6\)Table 2 accounts for seasonalities in dividend payments by averaging actual payments over the last 4 quarters, as in Campbell (2003). At an annual frequency, stock returns are also about three times as volatile as dividend growth.
price dividend ratio is negatively related to future excess stock returns and that there is predictability in excess stock returns over long horizons, e.g., Poterba and Summers (1988), Campbell and Shiller (1988), and Fama and French (1988). Table 3 presents evidence on the predictability of excess returns obtained from regressing cumulated excess returns over a certain horizon on the lagged price dividend ratio.\(^7\) As in other studies, the \(R^2\) increases and the regression coefficients become increasingly negative as the horizon rises.\(^8\) This suggests the presence of low frequency components in excess stock returns, i.e., the presence of long and sustained increases and downturns of stock prices.

<table>
<thead>
<tr>
<th>Years</th>
<th>Coefficient on PD-Ratio</th>
<th>(R^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0017</td>
<td>0.05</td>
</tr>
<tr>
<td>5</td>
<td>-0.0118</td>
<td>0.34</td>
</tr>
<tr>
<td>10</td>
<td>-0.0267</td>
<td>0.46</td>
</tr>
<tr>
<td>15</td>
<td>-0.0580</td>
<td>0.53</td>
</tr>
</tbody>
</table>

Table 3: Excess stock return predictability (1927:1-1998:4)

5. **Stock market crashes.** Stock markets occasionally experience ‘crashes’, i.e., strong and sudden price decreases, which seem to occur after a period of strong asset price increases. Table 4 lists the crashes identified by Mishkin and White (2002) for the S&P 500 over the period 1947:2-1998:4. A stock market crash is defined as a nominal price decrease by more than 20% occurring in a short period of time (generally less than 3 months). There are four episodes with such strong reductions in prices, with the stock market crash in October 1987 probably being the most uncontroversial one. Interestingly, the stock market crashes listed in table 4 are clearly identifiable as decreases of the price dividend ratio in figure 1, which suggests that crashes are not the result of changes in fundamentals (dividends) only.

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\(^7\)The table reports results from OLS estimation of

\[ X_{t,t+s} = c_0 + c_1 \frac{P_t}{D_t} \]

where \(X_{t,t+s}\) is the observed real excess return of stocks over bonds between \(t\) and \(t+s\) and \(\frac{P_t}{D_t}\) the price dividend ratio in \(t\). As in Campbell (2003) the price dividend ratio is the price divided by average dividend payments in the last 4 quarters.

\(^8\)Whether the coefficients are significantly different from zero is a non-trivial question because the price dividend ratio is highly autocorrelated, see the discussion in Campbell and Yogo (2005).
<table>
<thead>
<tr>
<th>Start</th>
<th>End</th>
<th>Total Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dec 1961</td>
<td>June 1962</td>
<td>-22.5%</td>
</tr>
<tr>
<td>Nov 1968</td>
<td>June 1970</td>
<td>-30.9%</td>
</tr>
<tr>
<td>Jan 1973</td>
<td>Dec 1974</td>
<td>-45.7%</td>
</tr>
<tr>
<td>Aug 1987</td>
<td>Dec 1987</td>
<td>-26.8%</td>
</tr>
</tbody>
</table>


It has been noted, e.g., by Fama and French (1988), that asset pricing facts 2, 3, and 4 above can be consistent with RE asset pricing models if one introduces appropriately time-varying stochastic discount factors. Campbell and Cochrane (1999) substantiate this claim presenting a model where the required time-variation is achieved through a sophisticated process for consumption habits. Their habit process, however, introduces a substantial number of ‘free parameters’ into the model, which seems unattractive from a modeling viewpoint. The learning model developed below can match all of the asset pricing facts listed above but introduces on a single ‘free parameter’ into an otherwise standard asset pricing model.

3 A Simple Model of Stock Prices

Using a simple risk-neutral asset pricing model, we now illustrate that introducing learning changes the behavior of stock prices in a way that helps to explain the asset pricing facts discussed in the previous section. The emphasis in this section is on qualitative results that can be obtained from analytical reasoning. Section 4 extends the analysis to the case with risk-averse investors.

Consider the asset pricing equation

\[ P_t = \delta \tilde{E}_t (P_{t+1} + D_{t+1}) \]

where \( P_t \) is stock price, \( \delta \in (0, 1) \) discount factor, and \( D_t \) dividends. \( \tilde{E}_t \) denotes investors’ perceptions about the future as of time \( t \), which we allow to be less than fully rational. Equation (1) can be derived from a Lucas (1978) asset pricing model with linear consumption utility or from simple arbitrage by a risk-neutral investor, if \( \delta \) denotes the inverse of the short-term gross interest rate.

We consider investors that care about predicting future price \( P_{t+1} \) and future dividend \( D_{t+1} \) rather than about predicting the discounted sum of dividends.
\( \sum_j \delta^j D_{t+j} \), which could be obtained by iterating over the right-hand side of equation (1). The previous assumption is plausible if, for example, agents are short-term investors who sell and buy new stocks each period.\(^{10}\)

The process for dividends is

\[
\frac{D_t}{D_{t-1}} = a \varepsilon_t
\]  

(2)

where \( \varepsilon_t \sim \exp N(-\frac{s^2}{2}, s^2) \) is an iid shock to dividend growth with mean zero and \( a \geq 1 \) expected gross dividend growth. As documented in Mankiw, Romer and Shapiro (1985) or Campbell (2003), equation (2) provides a reasonable approximation to the empirical behavior of quarterly dividends in the U.S..

If agents hold rational expectations (RE) about future prices and dividends, i.e., if \( \hat{E}_t [\cdot] = E_t [\cdot] \), equations (1) and (2) imply

\[
P_t = \delta a \frac{D_t}{1 - \delta a}.
\]  

(3)

Equations (2) and (3) together imply that the rational expectations equilibrium misses all five asset pricing facts mentioned in section 2. First, it predicts average stock returns to approximately equal average dividend growth.\(^{11}\) In the data the latter falls far short of the former, illustrating that the model cannot explain the observed equity premium. Second, the model counterfactually predicts a constant price dividend ratio. Third, the model predicts quarterly stock returns to be roughly as volatile as dividend growth, which fails to be the case in the data.\(^{12}\) Forth, the model implies that there is no predictability of excess returns. Lastly, unless there are large changes in fundamentals (dividends), stock prices will not move by much.

3.1 Learning about Stock Returns

We now illustrate the effects of introducing learning into this simple asset pricing model. Since we want to study learning schemes that forecast well within the model, we assume that agents know the process for dividends so that equation (1) can be written as

\[
P_t = \delta \hat{E}_t (P_{t+1}) + \delta E_t (D_{t+1})
\]  

(4)

\(^{10}\)Alternatively, agents may have a longer investment horizon but the ‘law of iterated expectations’ may not hold, which is required for such iteration to be valid. Consistent with the learning model constructed later on, agents may believe to have (slightly) misspecified forecasting models. Iteration of short-term forecasting models to obtain longer term predictions is then known to be sub-optimal, see Bhansali (2002) on this point. Adam (2005) provides experimental evidence on the breakdown of the law of iterated expectations in a situation where agents become aware that they use possibly misspecified forecasting models.

\(^{11}\)This follows from \( \frac{P_{t+2} D_t}{P_{t+1} D_{t+1}} = \frac{P_{t+2}}{P_{t+1}} \frac{D_t}{D_{t+1}} \approx \frac{D_t}{D_{t+1}} \) where \( PD = \frac{\delta a}{1 - \delta a} \) is the quarterly price dividend ratio, which tends to be large.

\(^{12}\)See previous footnote.
where the conditional expectation $E_t(D_{t+1})$ is known to agents.\textsuperscript{13} Agents, however, have to learn about the best way to forecast future prices, so they will form $E_t(P_{t+1})$ using past information to estimate parameters in their forecast function and using some learning scheme.\textsuperscript{14}

We specify the expectations under learning as

$$E_t\left[\frac{P_{t+1}}{P_t}\right] = \beta_t$$

where $\beta_t$ is some estimator of stock price growth based on past observations. Equation (3) shows that for $\beta_t = \alpha$ the learning model generates rational expectations. We suppose that $\beta_t$ is estimated using past observed stock price growth according to

$$\beta_t = (1 - \alpha)\beta_{t-1} + \alpha \frac{P_{t-1}}{P_{t-2}}$$

(5)

where $\alpha \in (0, 1)$ is a ‘gain’ parameter. With this formulation agents’ expectations of stock price growth are a weighted average of past realized growth rates. Observations in the more distant past are thereby discounted relative to more recent observations. Economically, this may reflect agents’ doubts about the stationarity of average stock price growth rates.

Given these perceptions, equation (4) implies that the actual stock price growth under learning is

$$\frac{P_{t+1}}{P_t} = \left(\frac{1 - \delta \beta_t}{1 - \delta \beta_t - \delta \Delta \beta_{t+1}}\right) a_{t+1}$$

(6)

where $\Delta \beta_{t+1} = \beta_{t+1} - \beta_t$.

We can study analytically the behavior of the stock prices under learning by analyzing the so-called $T$-map, i.e., the map from perceived to expected actual stock price growth. From equation (6) follows that the $T$-map is given by

$$T(\beta_t, \Delta \beta_{t+1}) = E_t\left[\frac{P_{t+1}}{P_t}\right] = \frac{1 - \delta \beta_t}{1 - \delta \beta_t - \delta \Delta \beta_{t+1}} a$$

(7)

\textsuperscript{13}Since dividend growth follows an exogenous process, it would be relatively easy for agents to learn the parameters governing it. Implicitly, we assume this learning process to have converged already.

\textsuperscript{14}Timmerman (1993) differs from our analysis in three ways: first, by giving less information to agents assuming that agents do not know the process for dividends and thus not know how to formulate conditional expectations involving future dividends. Second, by assuming Bayesian learning; this is easily accomplished in his setting where agents learn to predict exogenous variables but it would be problematic in our case, where agents predict endogenous variables (future stock price) with endogenous variables (current stock price). Third, by assuming that agents predict discounted dividends and set $P_t = \delta E_t \sum_j \delta^j D_{t+j}$. This is consistent with agents who do not have uncertainties about how stock prices are determined in the future. Timmermann’s analysis can thus be summarized as a case with long-term investors that have to learn about the dividend process.
This map has a number of interesting features. First, there is an asymptote as expected stock price growth increases from below towards $\delta^{-1}$. For higher growth expectations there ceases to be a finite market clearing price. Second, actual stock price growth depends not only on the level of return expectations $\beta_t$ but also on the change in return expectations $\Delta \beta_{t+1}$. We now analyze the implication of these two features for the behavior of stock prices under learning. Thereafter, we relate our findings to the asset pricing facts discussed in section 2.

**Result 1 (excess volatility).** Under learning stock returns are more volatile than dividend growth.

Using $P_{t+1} = \frac{a}{1-\delta \beta_t} D_t$ one finds

$$Var\left( \ln \frac{P_{t+1} + D_{t+1}}{P_t} \right) = Var\left( \ln \frac{1 + \frac{a}{1-\delta \beta_t - \delta \Delta \beta_{t+1}} D_{t+1}}{D_t} \right) \geq Var\left( \ln \frac{D_{t+1}}{D_t} \right)$$

Intuitively, learning generates variance of growth perceptions and this increases the variance of stock returns (in non-trivial ways). Since realized stock price growth feeds back into return perceptions, learning has the potential to generate a quantitatively significant increase in return volatility.

**Result 2 (momentum).** For growth perceptions close to the RE value, stock returns have a tendency to increase (decrease) further once they have started to increase (decrease).

Suppose stock price growth expectations have been increasing ($\Delta \beta_{t+1} > 0$). From equation (7) it follows that in this case the expected actual stock price growth is $T(\beta_t, \Delta \beta_{t+1}) > a$ for all perceptions $\beta_t$. Conversely, if $\Delta \beta_{t+1} < 0$ then $T(\beta_t, \Delta \beta_{t+1}) < a$. Therefore, it is clear that even if agents arrived at the rational expectations value $\beta_t = a$ while their expectations are increasing (decreasing), i.e., they arrived at this value from below (above), the estimated stock price growth has a tendency to keep increasing (decreasing) and to overshoot (undershoot) the rational expectations value. For perceptions close to rational expectations, the learning dynamics thus generate inertia in stock returns.

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15 In standard asymptotic learning analysis with a vanishing gain parameter, one can make the approximation $\Delta \beta_{t+1} \approx 0$. This implies that the effects of changes in perceptions are ignored. These changes will play a crucial role in the present analysis instead.

16 The subsequent derivation exploits the independence of $\beta_{t+1}$ and $D_{t+1}/D_t$, which follows from equation (5).

17 This feedback effect is absent if learning is about exogenous variables as, for example, in Timmermann (1993).
Result 3 (mean reversion). For growth perceptions below the RE value, there is a tendency for perceptions to eventually return to values close to the RE value.

For $\beta_t < a$ and $\Delta \beta_{t+1} > 0$ this follows directly from (7). For $\Delta \beta_{t+1} < 0$ beliefs may continue to decrease for a while, as suggested by Result 2 above, but for all $\beta_t$ sufficiently low $T(\beta_t, \Delta \beta_{t+1}) \geq \beta_t$. Perceptions will thus eventually revert direction and start to increase towards the RE value.

Result 4 (bubbles). For a sufficiently high increase in growth perceptions, there is the possibility of a learning induced asset price bubble with explosive stock price dynamics.

For $\Delta \beta_{t+1} > 0$ sufficiently large, one has $T(\beta_t, \Delta \beta_{t+1}) > \beta_t$ for all perceptions $\beta_t$. Since

$$\lim_{\beta_t \to (\delta^{-1} - \Delta \beta_{t+1})} \Delta \beta_{t+1} = \lim_{\beta_t \to (\delta^{-1} - \Delta \beta_{t+1})} (T(\beta_t, \Delta \beta_{t+1}) - \beta_t) = +\infty$$

the dynamics are unstable for sufficiently high beliefs $\beta_t$. Stock prices will then have the tendency to grow without bound, generating a learning induced asset price ‘bubble’. A learning based asset price bubble also emerged in the hyperinflation study of Marcet and Nicolini (2003), but there - unlike in the current paper - the real value of the asset (money) is decreasing along the bubble path.

To further illustrate the previous finding on asset price bubbles, let us consider a concrete example. Figure 2 draws the $T$-map, i.e., the mapping from agents’ perceptions against the true conditional expectations implied by their perceptions as well as a 45 degree line. The upper panel assumes that $\Delta \beta_{t+1} = .001\%$, e.g. agents went from believing a growth rate of stock prices of 1% to believing a growth rate of stock prices of 1.001%. The figure shows that in this case the range of stable $\beta$ values is still quite large. But if $\Delta \beta_{t+1} = .01\%$, as assumed in the lower panel of the figure, e.g., agents went from believing a growth rate of stock prices of 1% to believing a growth rate of stock prices of 1.01%, actual expected stock price growth is higher than agents’ growth perceptions for any $\beta$ so that there is a tendency for stock prices to grow without bound. There is an asymptote when $\delta(\beta_t + \Delta \beta_{t+1}) = 1$. This shows that even moderate increases in expectations can generate instability in the stock market, with a tendency for stock prices to grow.

Result 5 (crashes). For high growth perceptions, there is the possibility of a ‘market crash’, i.e., a strong and sudden price decrease.

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18 This follows from (7) and the fact that one has $\beta_t + \Delta \beta_{t+1} > 0$ because gross stock price growth drops at most to zero.

19 The figure assumes $\delta = 0.99$ and $a = 1$. 

11
Figure 2: Mapping from perceived to actual expected stock price growth ($T$-map)
When growth perceptions and stock prices are high a small decrease in return expectations, triggered for example by negative shock to dividends, can generate a very strong price decrease. This is illustrated in figure 3, which depicts the $T$-map for $\Delta \beta_{t+1} = +.01\%$ and $\Delta \beta_{t+1} = -.01\%$ together with the 45 degree line. The figure shows that the difference in the $T$-maps is very pronounced when growth perceptions are high, i.e., when prices are overvalued compared to their RE level. High valuation levels, i.e., when $\beta_t + \Delta \beta_{t+1}$ is close to the inverse of the discount factor, therefore imply that stock prices react strongly to even slight reductions in perceived stock price growth. Such changes in perceptions play less of a role when prices are undervalued. The learning model thus implies that small innovations in realized stock price growth can trigger a large fall in price when prices are overvalued, but no symmetric price increase for undervalued prices. Moreover, once prices have started to fall from overvalued levels, Result 1 suggests that they will have a tendency to fall below their RE value. Small changes in fundamentals may thus trigger a ‘stock market crash’.

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20 As before, the figure assumes $\delta = 0.99$ and $\alpha = 1$.
21 In figure 3 this can be seen from the fact that the $T$-map for $\Delta \beta_{t+1} = -.01\%$ intersects with the 45 degree line at a value below one, i.e., below the RE level.
We now informally discuss how the previous results are related to the ability
of the learning model to replicate the asset pricing facts discussed in section
2. A precise quantitative investigation of this relationship will be taken up in
section 5.

Clearly, Results 1 and 5 immediately show that the learning model generates
excess volatility and occasional market crashes. Results 2 and 4 imply that
learning leads to persistence in stock returns that is unrelated to the behavior
of dividends. As will become clear in section 5, this persistence in returns will
have a low frequency component and this increases considerably the likelihood
of the model to generate an equity premium similar to the one observed in the
data. The presence of low frequency components in returns also implies that the
price dividend ratio will be volatile and display persistent deviations from its RE
value. Finally, Result 3 shows that returns display mean reversion, potentially
explaining why the price dividend ratio predicts future excess returns. It thus
appears that the learning model is qualitatively consistent with all the asset
pricing facts documented in section 2.

4 Determinants of Stock Market Volatility

In this section we extend the analysis to the case with risk averse investors
and discuss various factors influencing the stability of the stock market under
learning. Following Lucas (1978) we consider a representative consumer-investor
maximizing

\[
\max_{S_t, C_t} \sum_{t=0}^{\infty} \delta^t \frac{C_t^{1-\sigma} - 1}{1 - \sigma}
\]

s.t.

\[ P_t S_t + C_t = (P_t + D_t) S_{t-1} \]

where \( C_t \) denotes consumption, \( S_t \) the agent’s stock holdings at the end of
period \( t \), and \( \sigma \geq 0 \) the coefficient of relative risk aversion. The investor’s first-
order conditions, the market clearing conditions \( S_t = 1 \) and \( C_t = D_t \), and the
assumption that agents know the conditional expectations of dividends deliver
the asset pricing equation

\[
P_t = \delta \tilde{E}_t \left( \left( \frac{D_t}{D_{t+1}} \right)^\sigma P_{t+1} \right) + \delta E_t \left( \frac{D_t^\sigma}{D_{t+1}^\sigma} \right)
\]

(8)

For the risk-neutral case (\( \sigma = 0 \)) this simplifies to equation (4) studied in the
previous section. To find the REE assume that perceived risk-adjusted stock
price growth is given by \( \tilde{E}_t \left[ (a\varepsilon_{t+1})^{1-\sigma} \frac{D_{t+1}}{D_t} \right] = \beta \). From equation (8) follows

\[
P_t = \frac{\delta E_t \left( (a\varepsilon_{t+1})^{1-\sigma} \right)}{1 - \delta \beta} D_t
\]

(9)
which implies that expected actual risk-adjusted stock price growth is given by 
\( E_t \left( (a\epsilon_{t+1})^{1-\sigma} \right) \). In the REE perceived and actual stock price growth must coincide, i.e.,

\[
\beta^{RE} = E_t \left( (a\epsilon_{t+1})^{1-\sigma} \right) = a^{1-\sigma} e^{-\sigma(1-\sigma)\tilde{\mu}} \quad (10)
\]

Equation (9) then shows that RE stock prices are given by

\[
P_t = \frac{\delta \beta^{RE}}{1 - \delta \beta^{RE}} D_t \quad (11)
\]

which is identical to equation (3) derived in the previous section, except for \( \beta \) now having an interpretation as risk-adjusted stock price growth. Therefore, if agents learn about risk adjusted stock price growth according to

\[
\beta_t = (1 - \alpha)\beta_{t-1} + \alpha \left( \frac{D_{t-1}}{D_{t-2}} \right)^{-\sigma} \frac{P_{t-1}}{P_{t-2}} \quad (12)
\]

the analysis of the previous section extends in a natural way to the case with risk aversion.

We now illustrate that a range of commonly mentioned factors do indeed influence the (in)stability of the stock market under learning. In particular, we assess the effects of long versus short stock holding periods, i.e., the effects of speculative investment behavior, the effects of investors’ risk aversion, and the possible effects of monetary policy. The subsequent analysis we will be deliberately brief. It should be understood as an illustration of the fact that the learning model generates rich interaction effects between these aspects of the environment and stock prices, although these effects are largely absent under rational expectations. We plan to take up these issues in greater detail in future work.

As before, the model has an asymptote as \( \delta \beta \) approaches 1 from below, see equation (11). Since learning generates deviations of perceptions from its rational expectations value \( \beta = \beta^{RE} \), the average price effects of deviations are going to be larger the larger \( \delta \beta^{RE} \), i.e., the ‘closer’ the rational expectations equilibrium is located to the asymptote. Furthermore, since deviations of perceptions from rational expectations are driven by innovations in realized risk adjusted stock price growth, see equation (12), the learning model will generate larger deviations, the larger the variance of realized stock price growth in the rational expectations equilibrium. The latter is given by

\[
VAR \left( \left( \frac{D_{t-1}}{D_{t-2}} \right)^{-\sigma} \frac{P_{t-1}}{P_{t-2}} \right) = a^{2(1-\sigma)} e^{(-\sigma)(1-\sigma)\tilde{\mu}} \left( e^{(1-\sigma)^2 s^2} - 1 \right) \quad (13)
\]

We now proceed and analyze determinants of stock market instability under learning in the light of these observations.
Long versus Short Holding Periods. It is clear that $\delta$ is determined by the holding period of agents, in other words, the number of periods they stay with the stock before they sell it. A holding period of more than one time-period for the stock price would be equivalent with the basic model if $\delta$ would be reinterpreted as the discount factor between the beginning and the end of holding period and if instead of $D_{t+1}$ we put the discounted value of all dividends paid during the holding period. A long holding period is associated with investors, while a short holding period is associated with speculative behavior, interested in buying the stock in order to sell it soon. If agents hold on to the stock for a number of periods $N$, the model corresponds to a lower $\delta$ as $N$ grows. Short holding periods (highly speculative market) thus move the RE equilibrium closer to the asymptote and thereby increase the instability of the market under learning.\textsuperscript{22}

Investors’ Risk Aversion. The coefficient of relative risk aversion $\sigma$ affects both average (risk adjusted) stock price growth $\beta^{RE}$ as well as the variance of realized stock price growth, see equation (13). Figure 4 depicts the mean and variance of stock price growth in the RE equilibrium as a function of $\sigma$, using the dividend growth facts from table 2.\textsuperscript{23} The value of $\beta^{RE}$ reaches its minimum at $\sigma = \frac{1}{3} + \frac{\ln a}{s} \approx 3.13$, while the variance of realized stock price growth reaches the minimum at $\sigma = 1$. The figure thus shows that stock market instability unambiguously increases if risk aversion falls below one. Likewise, instability increases if if risk aversion increases above $\sigma \approx 3.13$. Very low as well as very high degrees of risk aversions thus increase stock market instability under learning.

Monetary Policy. Short-term real interest rates are constant and given by\textsuperscript{24}

$$R_t = \frac{\delta^{-1}}{E_t \left( \left( \frac{D_{t+1}}{D_t} \right)^{-\sigma} \right)} = \delta^{-1} a^\sigma e^{-\sigma(\sigma+1) \frac{\sigma}{2}}$$

(14)

Formally introducing monetary policy into the present model is thus not a straightforward exercise. However, many economic models predict monetary expansions to generate a temporary increase in current output and consumption levels, but - due to long-run neutrality - lower growth rates of output and consumption in subsequent periods. The present model can replicate these effects of monetary expansions by an increase in the current level of dividends coupled with lower dividend growth rates in the future. Equation (14) shows that - consistent with this interpretation of a monetary expansion - short-term real interest rates are predicted to fall. To simplify matters even further, we study a permanent fall in the dividend growth rate. Doing so we most likely

---

\textsuperscript{22}Note that the discount factor does not affect the variance of risk-adjusted stock price growth, see (13).

\textsuperscript{23}The values in the table imply $a = 1.00346$ and $s = 0.03624$.

\textsuperscript{24}This holds for the case with learning agents and agents holding RE.
Figure 4: Mean and variance of risk-adjusted stock price growth in REE
overstate the effects of a monetary expansion, but the qualitative effects should
remain unchanged.

Equations (10) and (13) show that a permanent fall in \( a \) (monetary expan-
sion) increases both the mean and the variance of risk-adjusted dividend growth,
provided \( \sigma > 1 \). When estimating the learning model in section 5, we find \( \sigma > 1 \).
Our preliminary assessment thus suggests that low real interest rates are associ-
ated with increased stock market instability under learning, a finding that seem
to be in line with the conventional wisdom expressed by policymakers.

5 An Estimated Asset Pricing Model with Learning

This section quantitatively evaluates how well our simple asset pricing model
with risk averse and learning investors can match the asset pricing facts docu-
mented in section 2. In particular, we use the simulated method of moments
(SMM) and the first moments of asset prices listed in table 1 to estimate the
parameters \((\delta, \sigma, \alpha)\) characterizing the learning model. We then use the second
moments from table 1 and the results of the return predictability regressions
reported in table 3 to evaluated the ability of the model to fit the data.

5.1 Model Estimation

We choose the parameters \((a, s)\) characterizing the exogenous dividend process
so as to match the values reported in table 2 and then use SMM to estimate
\((\delta, \sigma, \alpha)\). Details of the estimation procedure are reported in appendix A.1.
Table 5 reports the estimation outcome and shows that parameter estimates a
plausible on a priori grounds. In particular, the discount factor is below one
and the degree of relative risk aversion is still in a plausible range.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount factor ((\delta))</td>
<td>0.9767</td>
</tr>
<tr>
<td>Relative risk aversion ((\sigma))</td>
<td>8.30</td>
</tr>
<tr>
<td>Gain parameter ((\alpha))</td>
<td>(1.5 \cdot 10^{-3})</td>
</tr>
</tbody>
</table>

Table 5: Estimates for the learning model

Using the estimated learning model, table 6 reports the first moments of the
price dividend ratio and the stock and bond returns as well as the \([1\%, 99\%]\]
interval for samples of the same length as the underlying U.S. data.\(^{25}\) While
it is not possible to evaluate model performance using table 6, it is still worth

\(^{25}\) The numbers in the current and subsequent tables are based on 10000 simulations of a
length of 288 periods each.
noting that at the learning model is not inconsistent with the ex-post equity premium observed in the data.\textsuperscript{26}

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Average PD-ratio</td>
<td>105.4</td>
<td>96.9 [52.5,177.6]</td>
</tr>
<tr>
<td>Average stock return</td>
<td>2.36%</td>
<td>1.68% [1.16%,2.61%]</td>
</tr>
<tr>
<td>Average bond return</td>
<td>0.16%</td>
<td>0.16% [0.16%,0.16%]</td>
</tr>
</tbody>
</table>

Table 6: First moment of the learning model

5.2 Model Evaluation

Figure 5 depicts typical simulation outcomes for the price dividend ratio using the estimated learning model.\textsuperscript{27} As in the data, see figure 1, the price dividend ratio fluctuates considerably and displays persistent deviations from steady state. The simulations also have occasional spikes, indicating that asset prices can be considerably overvalued when compared to their rational expectations value.

While the results displayed in figure 5 are suggestive, we now proceed with a more formal evaluation of the learning model. Table 7 reports second moments for the stock and bond returns and for the price dividend ratio and confronts them with the moments in the data. The learning model matches the standard deviation and autocorrelation of the price dividend ratio extremely well. With regard to the volatility of stock and bond returns, the evidence is more mixed. While learning increases the volatility of stock returns considerably above that of dividend growth (3.63%), the learning model does not entirely resolve the return volatility puzzle. Regarding the volatility of real bond returns the learning model performs reasonably well, especially if one takes into account that the empirical standard deviation reported in the table is an ex-post measure of the real interest rate based on nominal returns and inflation rates and thus most likely overstates ex-ante variability due to the presence of unpredictable movements in inflation.

\textsuperscript{26} As is well-known, the fact that our model equates investors’ consumption with dividends helps in this respect.

\textsuperscript{27} The model is simulated for 288 periods, i.e., the same number of periods as we have observations in the data.
Figure 5: Simulated behavior of the price dividend ratio in the estimated learning model (different scales)

| Std. dev. PD-ratio | 35.4 | 27.7 | [5.88, 105.17] |
| Auto-correlation PD-ratio | 0.95 | 0.98 | [0.93, 1.00] |
| Std. dev. stock return | 11.5% | 5.70% | [4.07%, 9.94%] |
| Std. dev. bond return | 1.33% | 0.00% | [0.00%, 0.00%] |

Table 7: Data and learning model (second moments)

We next turn to the evidence regarding excess return predictability using the lagged price dividend ratio. Table 8 depicts the regression coefficients and $R^2$ values for the learning model together with the values obtained from U.S. data. Regression coefficients in the learning model are surprisingly consistent with empirical evidence, especially at longer horizons. Moreover, the $R^2$ values are positive and increase with the forecasting horizon.

Overall, we interpret the previous findings as providing considerable support for the empirical performance of our simple learning model.

<table>
<thead>
<tr>
<th>Years</th>
<th>Data</th>
<th>Coefficient on PD-Ratio</th>
<th>$R^2$</th>
<th>Learning Model</th>
<th>Coefficient on PD-Ratio</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0017</td>
<td>0.05</td>
<td>-0.0007</td>
<td>[-0.015, -0.0002]</td>
<td>0.05</td>
<td>[0.00, 0.20]</td>
</tr>
<tr>
<td>5</td>
<td>-0.0118</td>
<td>0.34</td>
<td>-0.0037</td>
<td>[-0.0097, 0.0008]</td>
<td>0.35</td>
<td>[0.03, 0.75]</td>
</tr>
<tr>
<td>10</td>
<td>-0.0267</td>
<td>0.46</td>
<td>-0.0076</td>
<td>[-0.0282, 0.0036]</td>
<td>0.45</td>
<td>[0.02, 0.85]</td>
</tr>
<tr>
<td>15</td>
<td>-0.0580</td>
<td>0.53</td>
<td>-0.0120</td>
<td>[-0.0618, 0.0113]</td>
<td>0.43</td>
<td>[0.00, 0.86]</td>
</tr>
</tbody>
</table>

Table 8: Excess stock return predictability

5.3 Assessing Learning-Based Forecasts

In this section we evaluate the forecasting performance of learning agents. In a first step we consider the statistical performance by comparing the standard error of learning agents’ forecasts to that of rational forecasters. In a second step we compare the risk-adjusted returns generated by learning-based and rational forecasts.

Realized risk-adjusted stock price growth between period $t$ and $t+1$ is given by

$$\left( \frac{D_{t+1}}{D_t} \right)^{-\sigma} \frac{P_{t+1}}{P_t} = 1 - \delta \beta \frac{1}{1 - \delta \beta_{t+1}} (a \epsilon_{t+1})^{(1-\sigma)}$$

A rational agent that knows that other agents are learning, that knows the beliefs of learning agents and how they are updating their beliefs, and that has
Figure 6 compares rational and learning forecasts for the estimated learning model. It depicts the probability distribution for the average standard error of rational forecasts divided by the average standard error of learning-based stock price forecasts. Values above one indicate that learning agents have outperformed rational agents in terms of the average forecast error during a simulation. This happens because of sampling uncertainty and is a not an unlikely outcome: in more than 46% of the simulations learning agents outperform rational agents. Moreover, the distribution is highly positively skewed with rational forecasts occasionally performing extremely bad. The median standard error ratio is 0.99984 and thus very close to one. The mean standard error ratio is 1.0119, which shows that on average the forecast errors of learning agents are lower than those generated by rational forecasters!

Clearly, agents do not directly care about the standard error of forecasts, but instead about the risk-adjusted returns (RAR) associated with an investment strategy that is based on their forecasts. In the learning model the stock price is such that - from the perspective of learning agents - expected risk-adjusted returns of bonds and stocks are equalized. We can thus assume that learning agents are going to be invested in stocks all the time. Let $RAR_L$ denote the (geometric) average of quarterly risk adjusted returns associated with this learning-based investment strategy. Now consider a hypothetical rational agent who can decide whether to invest in stocks or bonds. This agent will invest in stocks if - from her fully rational perspective - risk-adjusted expected returns

\[
E_{t-1} \left( \left( \frac{D_{t+1}}{D_t} \right)^{-\sigma} \frac{P_{t+1}}{P_t} \right) = E_{t-1} \left( \frac{1 - \delta \beta_t}{1 - \delta \beta_{t+1}} (a\epsilon_{t+1})^{(1-\sigma)} \right)
\]

\[
= E_{t-1} \left( \frac{1 - \delta \beta_t}{1 - \delta \left( (1 - \alpha) \beta_t + \alpha \frac{1 - \delta \beta_t}{1 - \delta \beta_{t+1}} a\epsilon_t \right)^{(1-\sigma)}} \right)
\]

\[
= \frac{(1 - \delta \beta_t) a^{(1-\sigma)} \sigma^{(\sigma-1)} \frac{2}{2}}{1 - \delta \left( (1 - \alpha) \beta_t + \alpha \frac{1 - \delta \beta_t}{1 - \delta \beta_{t+1}} a\epsilon_t \right)^{(1-\sigma)} \frac{2}{2}}
\]

(15)

As is the case for learning agents, we assume that rational agents use information up to period $t-1$ when determining expected stock price growth, see equation (5). This implies that $\beta_t$ is part of the $t-1$ information set of rational agents.

This is the case if during a simulation there is a burst of a bubble, i.e., if prices return towards their RE value once they have temporarily embarked on an explosive path.

We assume this agent to have weight zero so that her investment decisions do not affect aggregate price dynamics.
Figure 6: Std. dev. rational forecast errors over std. dev. learning based forecasts errors
Figure 7: Quarterly risk adjusted excess returns from holding learning-based instead of rational expectations

of stocks are higher than those of bonds, i.e., whenever:32

\[ E_{t-1} \left[ \left( \frac{D_{t+1}}{D_t} \right)^{-\sigma} \frac{P_{t+1}}{P_t} \right] + E_t \left[ \left( \frac{D_{t+1}}{D_t} \right)^{-\sigma} \frac{D_{t+1}}{P_t} \right] > \delta^{-1} \]

Otherwise, the investor rationally chooses to invest in bonds. The quarterly risk-adjusted returns of this rational investment strategy are denoted by \( RAR^R \).

Figure 7 depicts the distribution of \( RAR^L - RAR^R \) for the estimated learning model and the same sample length as our empirical sample.33 The realized risk-adjusted returns obtained by learning agents are higher than those of rational agents in more than 1/4 of the simulations. Taking the average across all simulations, the quarterly risk-adjusted returns of rational agents are just 0.06% higher than those of rational agents.

We interpret the previous evidence as showing that based on average forecasting and average risk-adjusted return performance it would be hard to tell empirically whether rational forecasts outperform learning-based forecasts. The performance difference seems small, even though rational agents are endowed with perfect information about the structure of the economy, other agents' learning process, and the beliefs held by learning agents.

32 As before, risk-adjusted stock price growth expectations will be based on information up to \( t-1 \) to avoid giving the rational agent an informational advantage over learning agents.

33 The figure is based on 10000 simulations.
5.4 Comparison to RE Model

In this section we compare the performance of the learning model with that of a RE model with the same number of model parameters as our learning model.

A standard extension of the two parameter constant discount factor model from section 4 in the RE literature is the habit formation model discussed in Abel (1990). The representative consumer maximizes

$$\max_{S_t, C_t} E_0 \sum_{t=0}^{\infty} \delta^t \left( \frac{C_t}{C_{t-1}} \right)^{1-\sigma} - 1$$

s.t.

$$P_t S_t + C_t = (P_t + D_t) S_{t-1}$$

where \( \kappa \geq 0 \) denotes the habit persistence parameter and \( C_{t-1}^p \) the habit stock, which is external to the agent. The investor’s first-order conditions and the market clearing condition \( C_t = D_t \) deliver the asset pricing equation

$$P_t = \delta E_t \left( \frac{D_t^e(\sigma-1)}{D_{t+1}^e(\sigma-1)} \right) \left( \frac{D_t^p}{D_{t-1}^p(\sigma-1)} \right) (P_{t+1} + D_{t+1})$$

Together with the process for dividends (2) this implies that under rational expectations

$$E_t \left( \frac{P_t}{D_t} \right) = \delta E_t \left( \frac{D_t^e(\sigma-1)}{D_{t+1}^e(\sigma-1)} \right) A E_t (a_{t+1}^{(1+\kappa(\sigma-1))} + A^{-1} a^{1-\kappa(\sigma-1)})$$

$$E_t \left( \frac{P_t + D_t}{P_{t-1}} \right) = \delta^{-1} a^{\sigma-\kappa(\sigma-1)} E_t \frac{a^{-1+\kappa(\sigma-1)}}{E_t^{\sigma-\kappa(\sigma-1)}}$$

$$E_t (R_t) = \delta^{-1} a^{\sigma-\kappa(\sigma-1)} E_t \frac{a^{-1+\kappa(\sigma-1)}}{E_t^{\sigma-\kappa(\sigma-1)}}$$

where

$$A = \frac{\delta E_t (a_{t+1}^{(1+\kappa(\sigma-1))} + A^{-1} a^{1-\kappa(\sigma-1)})}{1 - \delta a^{1+\kappa(\sigma-1)} E_t (\epsilon_t^{(1+\kappa(\sigma-1))})}.$$
Parameter | Estimate
---|---
Quarterly discount factor ($\delta$) | 0.9932
Coef. of relative risk aversion ($\sigma$) | 4.95
Habit persistence parameter ($\kappa$) | 0.60

Table 9: Parameter estimates for rational expectations model

The lower panel of table 10 shows that the model strongly fails in accounting for the second moments of asset prices. The price dividend ratio displays insufficient volatility and fails to be autocorrelated. The model does no better than the learning model in matching the volatility of stock returns and grossly overpredicts the volatility of short-term bond returns.

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>First moments</td>
<td></td>
</tr>
<tr>
<td>Average PD-ratio</td>
<td>105.4</td>
</tr>
<tr>
<td>Average stock return</td>
<td>2.36%</td>
</tr>
<tr>
<td>Average bond return</td>
<td>0.16%</td>
</tr>
<tr>
<td>Second moments</td>
<td></td>
</tr>
<tr>
<td>Std. dev. PD-ratio</td>
<td>35.4</td>
</tr>
<tr>
<td>Auto-correlation PD-ratio</td>
<td>0.95</td>
</tr>
<tr>
<td>Std. dev. stock return</td>
<td>11.5%</td>
</tr>
<tr>
<td>Std. dev. bond return</td>
<td>1.33%</td>
</tr>
</tbody>
</table>

Table 10: Data and RE model

Table 11 presents evidence on return predictability for the RE model. While the coefficient on the price dividend ratios are consistent with those found in the data, the regression results for the RE model are entirely spurious. This shows up in the $R^2$ values which, unlike in the data, tend to be very close to zero and fail to increase with the prediction horizon.

Unlike the learning model, the RE model thus fails to account for the second moments of asset prices or the evidence on excess return predictability.
### Table 11: Excess stock return predictability

<table>
<thead>
<tr>
<th>Years</th>
<th>Data Coefficient on PD-Ratio</th>
<th>R²</th>
<th>RE Model Coefficient on PD-Ratio</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0017</td>
<td>0.05</td>
<td>-0.0002 [-0.0037, 0.0041]</td>
<td>0.00 [0.00, 0.02]</td>
</tr>
<tr>
<td>5</td>
<td>-0.0118</td>
<td>0.34</td>
<td>-0.0015 [-0.0115, 0.0156]</td>
<td>0.00 [0.00, 0.03]</td>
</tr>
<tr>
<td>10</td>
<td>-0.0267</td>
<td>0.46</td>
<td>-0.0048 [-0.0273, 0.0368]</td>
<td>0.01 [0.00, 0.03]</td>
</tr>
<tr>
<td>15</td>
<td>-0.0580</td>
<td>0.53</td>
<td>-0.0114 [-0.0656, 0.0622]</td>
<td>0.01 [0.00, 0.04]</td>
</tr>
</tbody>
</table>

### 6 Conclusions

A simple asset pricing model of learning generates rich asset pricing dynamics that are much more in line with the empirical behavior of stock markets than the dynamics generated by equally simple rational expectations models. Since learning-induced deviations form rational expectations are small, the results of this paper show that even slight non-rationalities in expectations can have large implications for the behavior of asset prices. This sensitivity of asset prices to beliefs has been noted before in the RE literature before. Geweke (2001) and Weitzman (2005), for example, show that there fails to be a finite asset price if agents expect dividend growth to have a t-distribution instead of a normal distribution, no matter how close the t-distribution is to normal. In the light of this literature, the contribution of this paper is to show that non-rationalities generated by simple learning mechanisms cause deviations from RE prices of a kind such that simple asset pricing models become consistent with the empirical evidence.

### A Appendix

#### A.1 Details of the Estimation

We use the simulated method of moments to estimate the asset pricing models. Let $\phi$ be a vector of model parameters and $M(\phi)$ denote the model-theoretic moment vector for parameter values $\phi$. Letting $EM$ denote the corresponding vector of empirical moments we estimate

$$
\hat{\phi} = \arg \min_{\phi} (M(\phi) - EM)\Omega(\phi)^{-1}(M(\phi) - EM)
$$

where $\Omega(\phi)$ is the variance covariance matrix of $M(\phi)$.

For the learning model one has $\phi = (\delta, \sigma, \alpha)$. Analytical expressions for the model analogues of the moments in table 3 only exist for real bond returns, see (14), which are constant over time. The latter implies that estimation according to equation (17) requires matching the empirical bond returns exactly. Given
values for $\sigma$ and $\alpha$, we thus choose the discount factor $\delta$ in equation (14) so that this is the case. This leaves us with a system with two parameters $(\sigma, \alpha)$ and two moments. We perform a grid search for $\sigma \in [0, 30]$ and $\alpha \in [0, 3 \cdot 10^{-3}]$ and compute $M(\phi)$ and $\Omega(\phi)$ for each $\phi$ by simulating the model five thousand times for the same number of periods as we have data. If a simulation shows explosive behavior during the sample period we simply repeat it.\(^{35}\) The same applies if the rational expectations of future stock price growth would be infinite during a simulation.\(^{36}\) Simulations are started at the rational expectations solution and 50 model periods are used for initialization.

For the RE model with habit persistence $\phi = (\delta, \sigma, \kappa)$ and we have analytical expression for the theoretical analogues of the moments in table 3, see equations (16). Since there exists a $\phi$ such that $M(\phi) = EM$, analytical expressions for the moment matrix $\Omega$ are not needed. Estimation thus simply consists of numerically finding the zero of a system of nonlinear equations.

References


\(^{35}\) We also restrict consideration to areas of the parameter space where at least 50% of the simulations do not show explosive behavior. For $\alpha > 3 \cdot 10^{-3}$ explosive stock price behavior occurs in more than 50% of the simulations.

\(^{36}\) This is required to have meaningful comparisons of forecast errors between learning and rational forecast in a situation with learning agents.


