Evolution of Division Rules

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Abstract

Proportional division is the most prominent of the several adhoc rules that appear in the literature for dividing a bankrupt estate. The adhoc rules differ from each other because of the axioms that are imposed in addition to efficiency and claims boundedness. Efficiency requires that the estate be completely divided between the claimants, and claims boundedness requires that no claimant be awarded more than her initial contribution. This paper tries to show that an adhoc rule can be rationalized as the unique self-enforcing long run outcome of Young’s (1993) evolutionary bargaining model by using certain intuitive rules for the Nash demand game. If the agents bargain in the framework of the usual demand game, the long run stochastically stable equilibrium turns out to be equal division of the estate. If, in addition to the usual rules, demanding more than one’s initial claim leads to a zero payoff (claims boundedness) then the long run equilibrium corresponds to the constrained equal awards rule. If the rules of the demand game capture both efficiency and claims boundedness, then the long run equilibrium corresponds to the division proposed by the truncated claims proportional rule. The main insight from the analysis is that the proportional rule emerges as the long run equilibrium only when the left over estate is greater than the highest contribution.

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1 Introduction and literature survey

"...the unjust is what violates the proportion; for the proportional is intermediate, and the just is proportional."

_Nicomachean Ethics, Aristotle_

The literature on non-cooperative bargaining has primarily focused on the question of how agents would divide a given amount of surplus. The simplest game theoretic representation of the problem is the Nash demand game involving two agents and certain rules to map demands of agents into payoffs. The multiplicity of Nash equilibria in the demand game led to the development of the axiomatic approach (Nash 1950, 1953) and extensive form non-cooperative models (Rubinstein 1982) to select one out of the several Nash equilibria.

The surplus over which bargaining takes place is assumed to be exogenous in most of the studies. However, even in the most common examples alluded to in the bargaining literature (landlords and tenants, workers and management), the surplus is _created_, and the involved parties have an idea of their claims over the surplus. This paper deals with situations that could be termed as bargaining under the shadow of claims. It only deals with those bargaining problems where the initial claims of both parties are unambiguous and common knowledge, and the final surplus is not large enough to honor all the initial claims completely. The simplest example is bargaining among creditors to divide up a bankrupt estate.

The model considered in this paper has two populations (low and high claimants) of equal size, each characterized by an exogenous level of claim \((c_l, c_h)\). Every period \(N\) pairs are formed with each pair comprising of one low and one high claimant. During each period, agents in every pair bargain over the same amount \(e(\leq [c_l + c_h] = 1)\) within the framework of a _modified_ Nash demand game. The agents are assumed to be myopic best responders, who sometimes exhibit inertia, and sometimes experiment with non-best response strategies. The paper tries to analyze the long run outcome of this dynamic process. Specifically, it aims to come up with a possible explanation for
the adhoc division rules (in particular, proportionality) from non-cooperative bargaining in this evolutionary setting (a framework similar to Young (1993)).

Young (1993) embeds the demand game in an evolutionary framework and utilizes the idea of stochastic stability to select the unique long run equilibrium. He specifies the evolutionary dynamic in a manner that makes the Nash equilibria of the demand game non-absorbing thereby allowing for transitions among the various equilibria; and then identifies that Nash equilibrium which, in some sense, is easy to get to but difficult to escape. Binmore, Samuelson and Young (BSY 2003) clarify the role of the various adaptive dynamics that have been employed in the literature to model the behavior of agents in evolutionary models of bargaining. They also provide an alternative way to identify the long run stochastically stable equilibria. However, Young (1993) and BSY (2003) assume the surplus to be exogenous and do not deal with the claims of the involved parties.

Moulin (2003) provides an excellent survey of the adhoc division rules that have been proposed in the literature to divide an amount of surplus that is no more than the total amount that went into creating it. The constrained equal awards rule divides the remaining estate equally subject to the constraint that no claimant gets more than her initial contribution. The most common way of dividing a surplus that is insufficient to completely satisfy all the (well defined) original contributions is to divide it proportionally to the initial contributions. The use of proportionality is widespread in both formal and informal (Ellickson 1991) environments. The bankruptcy laws of most countries dictate compensating creditors within a particular priority class proportionally to their initial contributions. The York-Antwerp Rules guiding maritime commerce have proportionality as the motivation behind the general average rule that is used to divide the cargo losses suffered during travel (Knight 1992). The truncated claims proportional rule is derived from the proportional rule through a simple modification. It first redefines the claim of an agent to be the minimum of her initial contribution and the size of the estate; and then divides the estate proportionally to the redefined (truncated) claims.
The existing literature (Aumann and Maschler 1985, Thomson 2004) has looked extensively into the connection between the adhoc division rules and the axiomatic/cooperative bargaining solutions. There exist a few studies that try to come up with non-cooperative games that will have the division suggested by a particular adhoc division rule as the equilibrium. For example, Dagan, Serrano, and Volij (1997) assume that there exists a socially accepted rule to solve claims problems involving two agents. Under this assumption, they devise a non-cooperative game involving any finite number of agents which has the n-person generalization of this bilateral rule as its subgame perfect equilibrium. It is important to note that they do not answer how the society comes up with the particular bilateral rule.

Ellingsen and Robles (2002) and Troeger (2002) develop an evolutionary model in which two agents bargain (in the non-cooperative framework of the Nash demand game) over a surplus that is created by one agent’s investment. However, they aim to establish that evolution eliminates the hold up problem. To understand what is unique about a division rule from a non-cooperative perspective, should we focus on the ex-ante incentives for investors that a particular rule for dividing the surplus would create? Or, should we focus on the bargaining after the surplus is realized; or, both? Ellingsen and Robles (2002) show that efficient investment can be sustained in the long run even if the ultimatum game (with the investor being the responder) is used to model the bargaining interaction. Thus, it seems that several rules of the demand game can lead to efficient investment in the long run. At the same time, each set of rules for the demand game leads to a different division of the surplus. Hence, our analysis focuses on the bargaining interaction and assumes investments to be exogenously given. (It must be emphasized that both Ellingsen and Robles (2004) and Troeger (2002) allow only one agent to invest and the return function is riskless).

Gächter and Riedl (2004) report that experimental subjects acting as a third party allocate the remaining estate (which is less than the sum of initial claims) in proportion to the initial claims. But, when two subjects having different initial claims engage in unstructured anonymous bargaining over the remaining estate, the results are statistically different from the division that is
proportional to initial claims. This result motivates us to look deeper into the psychological differences that arise when a person is asked to act as a third party versus when he happens to be a bargainer himself.

Having considered the results of Troeger (2002) and Gachter and Riedl (2004), this paper takes off by first asking: what must be the considerations of a third party that suggests proportional division of a bankrupt estate; then tries to come up with the rules for the Nash demand game that reflect these considerations; and finally establishes the long run prediction of the evolutionary process using these rules for the demand game. It is shown that several of the adhoc division rules can be obtained as the unique long run prediction of the evolutionary model by suitably changing the rules of the demand game. The reason behind the inability to obtain some of the adhoc rules (that include the proportional rule) as the long run outcome will be discussed.

The structure of the paper is as follows. Section 2 describes the model in detail. A modified set of rules for the demand give is also given. Section 3 of the paper illustrates in detail the steps involved in finding out the long run stochastically stable equilibrium. Section 4 shows the importance of rules of the demand game in determining the long run outcome. If the agents bargain in the framework of the usual demand game, the long run stochastically stable equilibrium turns out to be equal division of the estate. If, in addition to the usual rules, demanding more than one's initial claim leads to a zero payoff (claims boundedness) then the long run equilibrium corresponds to the constrained equal awards rule. If the rules further specify that agents will obtain positive payoffs only if the sum of their demands equals the estate size (efficiency), then the long run divisions are those prescribed by the truncated claims proportional rule.

To motivate the analysis in Sections 4.2 and 4.3, suppose we ask a person to act as a third party and divide an amount \( e \leq 1 \) between two agents who had initially contributed \( c_l = 0.4 \), and \( c_h = 0.6 \). Two things will most likely be observed. First, the division proposed by the third party will never give any agent more than her initial contribution for any \( e \leq 1 \). For example, if \( e = 0.9 \), no person acting as a neutral third party will violate claims boundedness (i.e., suggest
giving the low (high) claimant more than 0.4(0.6). Second, the proposed divisions will be efficient (for example, the third party is unlikely to suggest the division [0.4,0.4] for e = 0.9, and let the remaining amount go waste). Instead of asking why third parties behave in this manner, the paper takes these two presumptions as helpful cues to come up with the rules that should be used to structure the bargaining between the two agents.

The rules that capture both efficiency and claims boundedness lead to the emergence of truncated-proportional division as the unique long run outcome. Consider the case with \((c_l,c_h,e) = (0.4,0.6,0.5)\). The maximum feasible payoff to the high claimant is \(e = 0.5\), and she will end up losing at least \([c_h - \min(c_h,e)] = 0.1\). This is referred to as the sunk claim as it is beyond recovery. The equilibrium division turns out to be as if the claims of agents have been truncated from \(c_j\) to \(\min(c_j,e)\), and then the estate is being divided proportionally to these truncated claims.

The emergence of truncated-proportionality clarifies the reason behind the failure to obtain exact-proportionality as the long run outcome within this framework. This framework disregards claims that become sunk. On the other hand, proportionality requires dividing the leftover estate in proportion to the original contributions, even when a part of the contribution is beyond recovery for one or both the agents. It is only when the remaining estate is large enough to feasibly compensate even the high claimant, that exact-proportionality is the stochastically stable long run equilibrium.

2 The Model

The model considers a family of economies with each economy indexed by the tuple \((c_l,c_h,e)\), where \(e \leq (c_l + c_h) = 1\). Each economy in this family is assumed to consist of two distinct populations (low claimants and high claimants). The size of each population is \(N\). Each bargaining pair consists of a \(L\)-claimant and a \(H\)-claimant. The decision of agents regarding whether to contribute, and if so how much, is not modeled explicitly. Only one level of investment for each population is considered. In every pair the low (high) claimant is assumed to have contributed
$c_l$ ($c_h$). The agents come together in the form of a bargaining pair only after the realization of bankruptcy to decide upon the division of the remaining estate. It is assumed that for every pair during each time period in a particular economy, the size of the pie that remains after bankruptcy equals $e \in (0,1]$. Thus, in a particular economy (belonging to the family) $c_l$, $c_h$, and $e$ take the same numerical values across all pairs for all time periods. For example, there will be an economy in the family with $(c_l, c_h, e) = (0.4, 0.6, 0.5)$. So, in this economy during each time period $N$ pairs are formed with the low (high) claimant in each pair having contributed 0.4 (0.6). Every such pair during each time period has to decide how to divide 0.5. The interest lies in figuring out the unique long run division that will emerge in each economy in the family, and then comparing them to the adhoc division rules for claims problems.

A seemingly serious drawback of the set up is the assumption of fixed values of $c_l$, $c_h$, and $e$ for a particular economy, irrespective of pairings and time. The first move towards greater realism might be to let $e$ vary. However, if we allow $e$ to take different values, then there will be no simple way of specifying the best response dynamic. For example, let $(c_l, c_h) = (0.4, 0.6)$ for all pairs at all times in an economy; but let $e$ take two values 0.5 or 0.9. Should the two agents in a pair that are bargaining over $e = 0.5$ during the current period be allowed to draw inferences from the past play in cases with $e = 0.9$? If not, and players respond only according to the plays in the previous period that had $e = 0.5$, then allowing for two values of $e$ does not help in any way. Troeger (2002) takes this route when he assumes that agents in a pair state their current optimal demands by consulting the distribution of past demands in only those cases that had the same amount of surplus to be divided as this pair has in the current period. (This question does not even arise in Young (1993) and BSY (2003) as the bargaining always takes place over one unit of exogenously given surplus). Unfortunately, it is very difficult to model the other case where we wish to allow agents to best respond to all the observed cases. Thus, the assumption of fixed values of $c_l$, $c_h$, and $e$ for a particular economy is partly for innocuous convenience and partly because of the lack of a simple theory to model learning across similar (but not the same) situations.
The bargaining interaction of a pair is modeled as a modified Nash demand game—the blame game. The demand game involves the two agents in a pair stating their demands simultaneously. The rules of the game specify what happens in case the sum of demands does or does not exceed the pie to be divided. The usual specification involves the agents being awarded their demands in case the sum of their demands does not exceed the pie, and the agents obtaining nothing if the sum of their demands exceeds the pie. Let \( d_l \) (\( d_h \)) be the demand of the low (high) claimant, and \( x_l \) (\( x_h \)) be her payoff. Demands greater than \( e \) are ruled out. The rules of the blame game try to capture the fact that the bargainers have well defined initial contributions. The rules are first described in words and then defined formally.

i) The agents are awarded their demands in case the sum of their demands does not exceed \( e \).

ii) If sum of the demands exceeds \( e \), then two cases have to be considered.

a) If both agents demand more or both demand less than their original contributions, then both get zero. This reflects the thought that we can not pin down the responsibility for the sum of demands exceeding the estate size on either agent.

b) If one demands more than her claim and the other less than her claim, then the former receives zero and the latter receives her demand.

\[
(x_l, x_h) = \begin{cases} 
(d_l, d_h) & \text{if } d_l + d_h \leq e \\
(0, 0) & \text{if } d_l + d_h > e, d_l > c_l, d_h > c_h \\
(0, 0) & \text{if } d_l + d_h > e, d_l \leq c_l, d_h \leq c_h \\
(d_l, 0) & \text{if } d_l + d_h > e, d_l \leq c_l, d_h > c_h \\
(0, d_h) & \text{if } d_l + d_h > e, d_l > c_l, d_h \leq c_h 
\end{cases}
\]  

(1)

The best response functions of the two agents are shown in Figure 1 for the cases that emerge
depending upon the value of \( e \).\(^1\) Pairs of demands of the form \( (d^*_l, d^*_h) = (d^*_l, e - d^*_l) \) that completely exhaust the pie constitute the Nash equilibria. Let \( D^*_l \) denote the range of equilibrium payoffs to the low-claimant. Formally,

\[
d^*_l \in D^*_l = \begin{cases} 
[0, e] & \text{if } e \leq c_l \\
[0, c_l] & \text{if } c_l < e \leq c_h \\
[e - c_h, c_l] & \text{if } c_h < e 
\end{cases}
\] (2)

The minimum equilibrium payoff to the lower claimant is the remainder that will be left if the higher claimant is fully compensated, subject to the constraint that it should be positive. Similarly, the maximum payoff to the lower claimant equals her original contribution, subject to the constraint that \( e > c_l \). This reveals that the equilibria of the modified demand game incorporate

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\(^1\)The best responses are actually correspondences. Some of the Nash equilibria that are irrelevant for the calculation of the stochastically stable equilibrium are not shown in the figure.
some elements of forward induction type of reasoning. The general expression is

\[
d_t^* \in D_t^* = [(D_t^*)_{\text{min}}, (D_t^*)_{\text{max}}] = [\max(0, e - c_h), \min(e, c_l)]
\]

and

\[
d_h^* = (e - d_t^*)
\]

In a one shot interaction the modified Nash demand game has multiple pure strategy Nash equilibria \(^2\) with the equilibrium payoff to the low claimant varying from \((D_t^*)_{\text{min}} = \max(0, e - c_h)\) to \((D_t^*)_{\text{max}} = \min(e, c_l)\). The technique developed by Foster and Young (1990) is utilized to find the stochastically stable equilibria. In order to do so several assumptions have to be made which are described next.

2.1 The unperturbed dynamic

The agents make their demands from the discrete and finite set \([\delta, 2\delta, \ldots, e - \delta]\) having cardinality \(K\) (assuming \(e = (K + 1)\delta\)), where \(\delta\) can be thought of as the least count of the monetary scale used in the economy. With a discrete strategy space, any efficient division of \(e\) will be of the form \(((K+1-k)\delta, k\delta)\), where \(k \in [1, K]\). An efficient division \(((K+1-k)\delta, k\delta)\) that happens to be a Nash equilibrium of the blame game will be referred to as the \(k\)-equilibrium. The low (high) claimants will be relatively better off in an equilibrium with low (high) value of \(k\). Evolution of the process occurs in discrete time. Assuming both populations are of equal size \(N\), in each time period \(N\) random bargaining pairs consisting of a high and a low claimant will be formed. Each pair has complete knowledge of the original contributions, and faces the same problem of dividing \(e\) within the framework of the modified demand game. The state at the end of period \(t\) is \(s_t = (n^l, n^h)_{1t}\), where \(n^j\) is a \(K\) dimensional vector representing the number of agents in population \(j \in (L, H)\) playing the pure strategy \(k \in [1, K]\). The unperturbed adjustment dynamic is assumed to be

\(^2\)It is only when \(e = 1\) that there exists a unique Nash equilibrium for the one shot game with each agent demanding her original contribution, a result that motivated the choice of rules for the demand game.
the best response dynamic such that the current period demand of every agent maximizes her expected payoff given the previous period distribution of demands in the opponents’ population. This dynamic specification can be concisely represented as a Markov chain \( M_{(0,0)} \) on the finite state space \( S \) consisting of all pairs \( s = (n^l, n^h) \in R^K \times R^K \), with \( \sum n^l_k = \sum n^h_k = N \). The transition matrix for the process is denoted by \( T_{(0,0)} = [p_{ij}] \), where \( p_{ij} \) is the probability that the process lands in state \( i \) at time \( (t + 1) \) given it was in state \( j \) at time \( t \). The process is time homogeneous as the transition probabilities do not depend on time.

Restricting the demands of the agents to the finite set \( [\delta, 2\delta, \ldots, e - \delta] \) allows us to proceed with the calculation of the stochastically stable equilibria by making the underlying Markov chain, \( M_{(0,0)} \), finite. It also ensures that the best response of agents in the blame game will be a function and not a correspondence. Best response functions are more likely to lead to singleton absorbing sets for \( M_{(0,0)} \) thereby resulting in the blame game being weakly acyclic.

The best response behavior of agents has at least two unpleasant implications regarding the evolution of the process, given the motivation of this study. First, suppose the process starts with all low claimants playing the same pure strategy \( k \), and all high claimants playing the same pure strategy \( k' \), with \( k \neq (K + 1 - k') \). Given this initial state, under the best response dynamic, the process will keep cycling and the two populations will end up mis-coordinating for ever. Second, the pure strategy strict Nash equilibria will be recurrent states as the probability that the process returns to this state at some time in future, given that it is (or, was) in this state, is unity. Our interest lies in the long run behavior summarized by the stationary probability distribution over states. A probability distribution over the states is stationary if, once realized at some time, the probability distribution over states at all subsequent times remains the same. Since the recurrent class is not unique, the stationary distribution depends on the initial state. Let \( v_{(0,0)}(s|s_0) \) be the relative frequency of the occurrence of state \( s \) till time \( t \), given the initial state is \( s_0 \). Then,

\[
\lim_{t \to \infty} v_{(0,0)}(s|s_0) = \mu_{(0,0)}(s|s_0)
\]
In other words, the process is non-ergodic and can converge to any of the several pure strategy Nash equilibria depending on the initial state, and once the process reaches any such state it gets stuck there. As a result, issues pertaining to equilibrium selection in the long run can not be addressed. Both of these problems (perpetual mis-coordination and initial state dependence) can be eliminated if deviations from best response behavior on part of the agents are introduced.

2.2 The perturbed dynamic

Following BSY (2003), a perturbed best response dynamic is defined that first incorporates inertia to eliminate continual mis-coordination; and then allows the agents to play experimental non-best response strategies to overcome initial state dependence of the long run outcome. Let the probability that an agent states the same demand as in the previous period be $\lambda \in (0, 1)$, and the probability with which she best responds be $(1 - \lambda)$. We can now define the time homogeneous transition matrix $T(\lambda, 0)$. It can be proved that there always exists a $\lambda_{\min} \in (0, 1)$ that will get rid of the perpetual mis-coordination. When inertia is added to the best response dynamic the blame game becomes weakly acyclic. In other words, only the pure strategy Nash equilibria (Young 1998) will be the absorbing sets of $M(\lambda, 0)$. Intuitively, even a small amount of inertia breaks the cycle of mis-coordination by moving the state from the corners to the interior of the state space. The process will be aperiodic but not irreducible for all $\lambda \in (0, 1)$. As every Nash equilibrium is still an absorbing state, the stationary distribution still depends on the initial state, and no meaningful discussion of equilibrium selection is as yet possible. Formally,

$$\lim_{t \to \infty} v(\lambda, 0)(s|s_0) = \mu(\lambda, 0)(s|s_0)$$  \hspace{1cm} (6)$$

To make the Nash equilibria non-absorbing it is further assumed that when an agent goes on to state a demand that is different from the one in previous period, then she experiments with

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3If there exist absorbing sets that are not singletons, then it is also possible that the process reaches an absorbing set without actually converging to a pure strategy Nash equilibrium.

4Since the absorbing sets are singletons, we can now refer to them as the absorbing states.
probability $\epsilon > 0$. While experimenting, the agent is equally likely to state any demand from the feasible set. Thus, in each period an agent responds inertially with probability $\lambda$, plays a best response with probability $(1 - \lambda)(1 - \epsilon)$, and engages in random experimentation with probability $(1 - \lambda)\epsilon$. The time homogeneous transition matrix for the resulting process is denoted by $T_{(\lambda,\epsilon)}$.

The strategies an agent is allowed to play while experimenting might alter the equilibrium that will be selected as the (ultra) long run outcome. If the experimental strategies are chosen at random from the set of feasible strategies then every state is accessible from every other state in a finite number of periods. The process becomes irreducible, with the unique recurrent class being the whole state space. Also, the process is aperiodic because there does not exist any state to which the process will continually return with a fixed time period (greater than one). This helps us in two ways. Irreducibility implies that the process can potentially escape even a Nash equilibrium because in presence of experimentation Nash equilibria cease to be the absorbing states. Irreducibility, together with aperiodicity, implies that the stationary probability distribution over states will be unique and independent of the initial state.

It might be reasonable to assume that agents engage in state dependent experimentation. If agents experiment rationally then they will never play a strategy which (if established as the equilibrium) would give them a lower payoff than what they obtain in the currently established equilibrium. The time homogeneous transition matrix for this specification will be denoted by $T_{(\lambda,\epsilon,R)}$. This process is aperiodic but we do need to argue that it is irreducible. Recall that the low claimants receive their maximum payoff in the 1-equilibrium, and the high claimants receive their maximum payoff in the $K$-equilibrium. Suppose, the process is in the $k$-equilibrium at time $t$, with $1 < k < K$. There is a positive probability that all low (high) claimants rationally experiment in period $(t + 1)$ by playing the strategy 1 ($K$) which (if established as the equilibrium) will provide them their maximum possible payoff. Now, in period $(t + 2)$ there will be a positive probability that all agents in both populations best respond. This will lead each low claimant to play strategy $K$, and each high claimant to play strategy 1. At this point agents in both populations are playing
their minimum payoff strategies. With rational experimentation, every state now becomes accessible. Thus, even with rational experimentation the process is irreducible for every \( \epsilon > 0 \) and will have a unique stationary distribution that is independent of the initial state. However, the two stationary distributions can be different. Formally,

\[
\lim_{t \to \infty} v(\lambda, \epsilon)(s|s_0) = \mu(\lambda, \epsilon)(s) \\
\lim_{t \to \infty} v(\lambda, \epsilon_R)(s|s_0) = \mu(\lambda, \epsilon_R)(s)
\] (7) (8)

2.3 Stochastic Stability

Stochastic stability relates to the limit of the stationary distribution as the probability of experimentation goes to zero. The state \( s \) is stochastically stable if

\[
\lim_{\epsilon \to 0} \mu(\lambda, \epsilon)(s) > 0
\] (9)

The stochastically stable states are those that are most likely to be observed in the long run as experimental play by agents becomes exceedingly rare. The states that can be reached via experimental play by few agents, but escaped only if a large number of agents experiment, are the prime candidates for being the stochastically stable states. The predictive power of the analysis is greater the fewer is the number of states that receive positive probability weight under the above limit.

The algorithm for identifying the stochastically stable states first requires calculating the stationary distribution of the process for an arbitrary \( \epsilon > 0 \), and then finding the states that receive positive probability weight as \( \epsilon \) approaches zero. The limiting operation is easy but the usual technique for calculating the stationary distribution is very cumbersome if the state space is large as it involves solving a huge system of equations.

The next section begins with some useful definitions from graph theory and describes how to identify the stochastically stable state(s) in a relatively straightforward manner using directed
graphs (Friedlin and Wentzell (1984), Young (1993 and 1998)).

2.4  The Minimal Tree

A graph comprises of two types of elements: nodes and edges. An edge connects a pair of nodes. A graph in which the edges have a sense of direction are called directed graphs. A path is a collection of alternate nodes and edges such that each node in this collection is incident to a minimum of zero and a maximum of two edges in this collection. The (in) out-degree of a node is the number of edges directed (in) out-ward at the node. A path in which all the interior nodes have in and out degree of one is called a directed path. A node is reachable from some other node in the graph if there is a directed path that starts at the latter and ends at the former. A cycle in a graph is a collection of alternate nodes and edges such that each node in this collection has in and out degree of one. A graph is acyclic if it contains no cycles, and unicyclic if it contains exactly one cycle. The graph is connected if it is possible to establish a path from any node to any other node in the graph. A tree is a connected acyclic graph. Rooted trees are directed acyclic graphs with $(|S| - 1)$ edges such that each edge is directed towards the root node, and from every node there is one and only one directed path to the root node. However, there can be several trees rooted at the same node. A weighted graph associates a real number with every edge in the graph. The weight of a path in a weighted graph is the sum of the weights of the edges in the path.

Let $G_0$ be the complete directed graph constructed by using each of the $K$ absorbing states (i.e., each of the pure strategy Nash equilibria) of the time homogenous Markov chain $M_{(0,0)}$ as a node. The weight on the directed edge $(k \rightarrow k')$ is taken to be the minimum number of experimenting agents required to move the process from the $k$-equilibrium to the $k'$-equilibrium, often referred to as the resistance of this transition. Consider any one of the trees rooted at node $k \in [1, K]$. It will feature a directed path having $K - 1$ directed edges. The resistance of this rooted tree is defined as the sum of the resistances of the edges along its path. The resistance of each tree, rooted at each of the $K$ nodes, can be calculated in a similar manner. Let $\Gamma_0$ represent the collection of all the
trees in $G_0$. The stochastically stable state(s) is the one that serves as the root of the tree having minimal resistance (the minimal tree) among all the rooted trees in $\Gamma_0$ (Friedlin and Wentzell, 1984). However, if $K$ is large then figuring out the minimal tree by explicitly calculating the resistance of each rooted tree becomes very tedious. The following section relies heavily on Young (1998) and BSY (2003) in establishing that the stochastically stable division of a bankrupt estate corresponds to the divisions proposed by the constrained equal awards rule, if the underlying interaction is assumed to be the blame game described in Section 2.

3 The long run equilibrium

It has been argued in the previous section that the absorbing sets of $M_{(\lambda,0)}$ are singletons. This implies that $M_{(\lambda,0)}$ satisfies the no cycling condition (BSY 2003). Since every finite time-homogenous process reaches an absorbing set, $M_{(\lambda,0)}$ will eventually reach a Nash equilibrium, as only the Nash equilibria are the (singleton) absorbing sets of $M_{(\lambda,0)}$. Since we are only interested in identifying the stochastically stable equilibria we need to focus solely on $M_{(\lambda,\epsilon)}$ as $\epsilon$ tends to zero. This in turn implies that the minimal tree will be rooted at a Nash equilibrium. Hence, all we need to do is to find the tree rooted at a Nash equilibrium that has minimum total resistance. Those Nash equilibria that serve as the root of the trees having minimum total resistance will be the stochastically stable states of $M_{(\lambda,\epsilon)}$.

The equilibria of the blame game have been illustrated at the beginning of Section 2 for continuous strategy space. When the strategy space is discrete and demands are restricted to lie between $\delta$ and $(e - \delta)$, the Nash equilibria are

$$d^*_l \in D_l^* = \begin{cases} [\delta, e - \delta] & \text{if } e \leq c_l \\ [\delta, c_l] & \text{if } c_l < e \leq c_h \\ [e - c_h, c_l] & \text{if } c_h < e \end{cases} \quad (10)$$

16
Recall that \( d_l^* \) denotes an equilibrium payoff to the low claimant, and \( D_l^* \) denotes the set of equilibrium payoffs to the low claimant in the blame game.

In the following discussion only those strategies that are integral multiples of \( \delta \) will be considered. Suppose, the process is currently in the equilibrium \((x, e-x)\). Let

\[
X^+ = [d_l^* : d_l^* > x] \quad \text{and} \quad X^- = [d_l^* : d_l^* < x]
\]

Thus, \( x^+ \in X^+ \) represents an equilibrium payoff to the lower claimant higher than \( x \). The interest lies in figuring out the equilibrium that is most easily accessible from the current equilibrium at \( x \). This most likely transition can either be on the left or on the right of \( x \). We separately figure out the most easily accessible equilibrium to the right of \( x \), and the most easily accessible equilibrium to the left of \( x \). The easier of these two will in turn be termed as the most easily accessible equilibrium from the equilibrium at \( x \). The relevant 2 \( \times \) 2 games that need to be considered are shown in Figure 2. The game labeled \( x^+ > x \) has two pure strategy Nash equilibria: \((x, e-x)\) and \((x^+, e-x^+)\). Suppose the economy is currently in the equilibrium \((x, e-x)\). The equilibrium \((x^+, e-x^+)\) can emerge if a sufficient number of L-claimants experiment with the higher demand of \( x^+ \). The \( 2 \times 2 \) game helps us calculate the fraction of agents in the L-population that should randomly experiment with the higher demand of \( x^+ \) such that the best response for agents in the H-population is to demand \( e-x^+ \).

Let \( f_l[x \rightarrow x^+] \) denote the minimum fraction of L-agents that need to experiment and demand \( x^+ > x \) such that the best response for H-agents is to demand \( e-x^+ \). Formally,

\[
f_l(e-x^+) + (1-f_l)(e-x^+) = f_l.0 + (1-f_l).e-x
\]

\[
\Rightarrow \quad f_l[x \rightarrow x^+] = \frac{x^+ - x}{e-x}
\]

Similarly, let \( f_h[(e-x) \rightarrow (e-x^+)] \) denote the minimum fraction of H-agents that will have to experiment with \((e-x^+)\) such that the best response for L-agents is to demand \( x^+ \). We have
\[ f_h.x^- + (1 - f_h).x^- = f_h.0 + (1 - f_h).x \]

\[ \Rightarrow f_h[(e - x) \to (e - x^+)] = \frac{x}{x^+} \]

### 3.1 The transition \((x \to x^+)\)

The main proposition is arrived at through a sequence of simple results. Let \((x_i^+, e - x_i^+)\) be the equilibrium towards the right of \(x\) that is most easily accessible as a result of experiments initiated by agents in the \(L\)-population. Then

**Result 1(a)** *The least costly transition initiated by \(L\)-claimants towards \(x^+ > x\) is the local transition.*

This is because

\[
x_i^+ = \arg\min_{x^+} f_i[x \to x^+] = \arg\min_{x^+} \left( \frac{x^+ - x}{e - x} \right) = (x^+)_\min = (x + \delta)
\]

**Result 1(b)** *The resistance to the most likely transition towards \(x^+ > x\) initiated by experiments on*
part of L-claimants is

\[ f_l[x \rightarrow x^+_l] = \frac{\delta}{e - x} \]  

(12)

**Result 2(a)** The least costly transition initiated by H-claimants towards \( x^+ > x \) is the extreme transition. This is because

\[ x_h^+ = \arg\min_{x^+} f_h[(e - x) \rightarrow (e - x^+)] = \arg\min_{x^+} (\frac{x}{x^+}) = (x^+)_{\max} = \begin{cases} e - \delta & \text{if } e \leq c_l \\ c_l & \text{if } e > c_l \end{cases} \]

(13)

**Result 2(b)** The resistance to the most likely transition towards \( x^+ > x \) initiated by experiments of H-claimants is

\[ f_h[(e - x) \rightarrow (e - x_h^+)] = \begin{cases} \frac{e}{x - \delta} & \text{if } e \leq c_l \\ \frac{e}{c_l} & \text{if } e > c_l \end{cases} \]

(14)

**Result 3(a)** The most likely transition from the equilibrium at \( x \) towards the right will be to the equilibrium at \( x^{++} \), where

\[ x^{++} = \begin{cases} x + \delta & \text{if } f_l[x \rightarrow x^+_l] \leq f_h[(e - x) \rightarrow (e - x_h^+)] \\ e - \delta & \text{if } f_l[x \rightarrow x^+_l] > f_h[(e - x) \rightarrow (e - x_h^+)] & e \leq c_l \\ c_l & \text{if } f_l[x \rightarrow x^+_l] > f_h[(e - x) \rightarrow (e - x_h^+)] & e > c_l \end{cases} \]

(15)

---

5Let \((x_h^+, e - x_h^+)\) be the equilibrium towards the right of \( x \) that is most easily accessible as a result of experiments initiated by agents in the \( H \)-population.
(i) \( e \leq c_i \)

(ii) \( c_i < e \leq c_h \) and \( e \leq 2c_i \)

(iii) \( c_i < e \leq c_h \) and \( e > 2c_i \)

Figure 3: Nash Products
(iv) $e > c_h$ and $e \leq 2c_i$

(v) $e > c_h$ and $e > 2c_i$

Figure 3: Nash Products
**Result 3(b)** The resistance for the least costly transition for the equilibrium at $x$ will be

$$r^+(x) = \begin{cases} \frac{\delta}{e-x} & \text{if } f_l[x \to x^+_l] \leq f_h[(e-x) \to (e-x^+_h)] \\ \frac{x}{e-\delta} & \text{if } f_l[x \to x^+_l] > f_h[(e-x) \to (e-x^+_h)] \land e \leq c_l \\ \frac{x}{c_l} & \text{if } f_l[x \to x^+_l] > f_h[(e-x) \to (e-x^+_h)] \land e > c_l \end{cases} \quad (16)$$

**Result 4** The least costly transition towards the right will be the local transition initiated by experiments of L-agents.

For this result to be true it has to be proved that the most likely transition towards the right of any established equilibrium which is initiated by experiments of L-claimants requires less number of experimenting agents than the most likely transition initiated by H-claimants. Formally, we require

$$f_l[x \to x^+_l] \leq f_h[(e-x) \to (e-x^+_h)] \Rightarrow \begin{cases} x(e-x) \geq \delta(e-\delta) & \text{if } e \leq c_l \\ x(e-x) \geq \delta c_l & \text{if } e > c_l \end{cases} \quad (17)$$

It can be easily verified that both these inequalities hold true. The term $[x(e-x)]$ is the Nash product at the current equilibrium $(x, e-x)$. If $e \leq c_l$, the minimum value of Nash product $[\delta(e-\delta)]$ occurs when $d^*_l$ is $\delta$, or $e-\delta$. Hence, $x(e-x)$ will not be less than $\delta(e-\delta)$ if $e \leq c_l$. If $e > c_l$, then $x(e-x) \geq \delta(e-\delta) \geq \delta c_l$. The second part of this inequality holds because $e-\delta \geq c_l$.

The first part holds because $\delta(e-\delta)$ is the minimum value of the Nash product. This completes the proof of Result 4. Refer to Figure 3 for a graphical description of the inequalities.

### 3.2 The transition $(x^- \leftarrow x)$

The procedure for calculating the equilibrium towards the left of the current equilibrium $x$ that requires fewest experiments is the same. Calculations show that
\[
f_l(x^- \leftarrow x) = \left(\frac{e-x}{e-x^-}\right)
\]  
(18)
\[
f_h((e-x^-) \leftarrow (e-x)) = \left(\frac{x-x^-}{x}\right)
\]  
(19)

**Result 5** The least costly transition initiated by L-claimants towards \(x^- < x\) is the jump to the left most extreme. This is because

\[
x_l^- = \text{argmin}_{x^-} f_l[x^- \leftarrow x] = \text{argmin}_{x^-} \left(\frac{e-x}{e-x^-}\right) = (x^-)_{\text{min}} = \begin{cases} 
\delta & \text{if } e \leq c_h \\
 e - c_h & \text{if } e > c_h
\end{cases}
\]  
(20)

**Result 6** The least costly transition initiated by H-claimants towards \(x^- < x\) is the local transition to \((x-\delta)\). This holds because

\[
x_h^- = \text{argmin}_{x^-} f_h[(e-x^-) \leftarrow (e-x)] = \text{argmin}_{x^-} \left(\frac{x-x^-}{x}\right) = (x^-)_{\text{max}} = (x-\delta)
\]  
(21)

**Result 7** The least costly transition from any equilibrium \(x\) towards the left will be the local transition initiated by the experiments of H-claimants.

To establish this result it has to be proved that \(f_h[(e-x^-) \leftarrow (e-x)] \leq f_l[x_l^- \leftarrow x]\). Note that

\[
f_h((e-x^-) \leftarrow (e-x)) \leq f_l[x_l^- \leftarrow x] \Rightarrow \begin{cases} 
x(e-x) \geq \delta(e-\delta) & \text{if } e \leq c_h \\
x(e-x) \geq \delta c_h & \text{if } e > c_h
\end{cases}
\]  
(22)

The two inequalities indeed hold true as can be verified from Figure 3. The most likely transition towards left is again the local transition. However, it is initiated by the experiments of H-claimants. The resistance for moving from any equilibrium towards the most easily accessible
Figure 3: Resistances

e \leq 2c_l

e \leq 2c_l

e > 2c_l

e > 2c_l

Resistances

\[ r^{-}(x) = \frac{\delta}{x} \]

Result 8  It is easily verified that \( r^{+}(x) \) is monotonically increasing in \( x \), and \( r^{-}(x) \) is monotonically decreasing in \( x \). \( r^{+}(x) < r^{-}(x) \) for all \( x \) that support a Nash equilibrium if \( e > 2c_l \). \( r^{+} \) intersects \( r^{-}(x) \) at \( \frac{1}{2}e \) if \( e \leq 2c_l \). From an existing equilibrium \( x \) the least costly transition is to the equilibrium at \((x + (-)\delta)\) if \( r^{+}(x) < (> \) \( r^{-}(x) \). Figure 4 summarizes this result.

Proposition 1  The stochastically stable equilibrium corresponds to the constrained equal awards rule if bargaining among the claimants takes place under the rules of the blame game.

The minimal tree is given by the lower envelope of \([r^{+}(x), r^{-}(x)]\) when all least cost transitions are local (Young 1998). This minimal tree is rooted at \( \frac{1}{2}e \) or \( c_l \) depending upon whether \( e \) is smaller or greater than \( 2c_l \). The constrained equal awards rule divides the estate equally subject to the constraint that no claimant gets more than her original contribution. Both agents get half of the estate if the estate is less than \( 2c_l \); the low claimant gets \( c_l \) if the estate is more than \( 2c_l \). Hence, the stochastically stable equilibrium exactly corresponds to the constrained equal awards rule.
If we only allow for rational experimentation by agents then all transitions that involve a jump to an extreme will apriori be ruled out. However, the stochastically stable equilibrium remains unchanged as the least costly transitions for the demand game considered in this section always happen to be the local transitions initiated by rational experiments.

It is worth noting that Proposition 1 can also be obtained by a different choice of rules for the demand game. Consider the usual Nash demand game. Suppose, we add to it the rule that an agent demanding more than her initial contribution gets nothing, irrespective of the demand of her opponent. This can be interpreted as imposing claims boundedness on the final payoffs. The reader can verify that the Nash equilibria of the one shot demand game that imposes claims boundedness to the usual demand game are the same as those of the blame game. This gives us the following corollary to Proposition 1.

**Corollary 1** The stochastically stable equilibrium corresponds to the constrained equal awards rule if the requirement of claims boundedness is added to the rules of the usual Nash demand game.

### 4 Importance of rules of the demand game

The rules of the demand game determine the payoffs resulting from the demands of agents and consequentially affect the analysis in two important ways. First, the rules determine the set of Nash equilibria of the one-shot demand game. Second, they bear upon the criterion that determines the most likely transition from an established equilibrium. This section analyzes the long run behavior of the process under some reasonable rules of the demand game. First the usual demand game is considered, and then the rules are modified to incorporate the idea of efficiency and claims boundedness as discussed in Section 1.
4.1 The usual demand game

Let us consider the same basic setup as in the previous section with the only change being that the rules of the underlying game are those used in the usual Nash demand game.

i) Agents are awarded their demands if the sum of their demands does not exceed $e$.

ii) They obtain nothing if the sum of their demands exceeds $e$.

For any $e \in (\delta, 1]$, the Nash equilibrium strategy vector will be of the form $(x, e-x) = (d^*_l, d^*_h) = (d^*_l, e - d^*_l)$, where $d^*_l \in [\delta, e-\delta]$. The search for the minimal tree will involve the consideration of exactly the same two games shown in Figure 2. The analysis is much easier because the equilibrium payoffs to the agents no longer vary with the value of $e$ in relation to $c_l$ and $c_h$. Note that the minimum value of the Nash product will be $[\delta(e-\delta)]$, at $x = \delta$, and $(e - \delta)$. Thus, the Nash product at any $x \in [2\delta, e - 2\delta]$ will be greater than the Nash product at the extremes. This leads to the following simple result.

**Result**  Local transitions initiated by rational experiments are least costly. Formally,

\[
x(e-x) \geq \delta(e-\delta) \Rightarrow f_l[x \rightarrow x^+_l] \leq f_h[(e-x) \rightarrow (e-x^+_h)] \Rightarrow x^{++} = (x+\delta) \Rightarrow r^+(x) = \frac{\delta}{e-x}
\]

\[
x(e-x) \geq \delta(e-\delta) \Rightarrow f_h[(e-x) \leftarrow (e-x^+_h)] \leq f_l[x \leftarrow x^+_l] \Rightarrow x^{--} = (x-\delta) \Rightarrow r^-(x) = \frac{\delta}{x}
\]

**Proposition 2**  Equal division is the stochastically stable equilibrium if we employ the rules of the usual demand game.

The lower envelope of $[r^+(x), r^-(x)]$ gives the minimal tree, and the intersection of the two curves (if it exists) serves as root of the minimal tree since all least cost transitions are local. The minimal
tree is rooted at $\frac{1}{2}e$, and thus equal division is the stochastically stable equilibrium. This is because

$$r^+ (x) = r^- (x) \Rightarrow \frac{\delta}{e-x} = \frac{\delta}{x} \Rightarrow x = \frac{1}{2}e$$

4.2 Efficiency

Within the same dynamic framework, now consider the following rules for the demand game.

i) Agents are awarded their demands if the sum of demands equals $e$.

ii) They obtain nothing if the sum of demands exceeds or falls short of $e$.

It is straightforward to see that the Nash equilibria of this demand game in a one-shot interaction will be of the form $(d^*_l, e - d^*_l)$. Since all the off-diagonal payoffs are zero, we will have to redo the analysis for identifying the stochastically stable equilibrium by considering the two games shown in Figure 5. All the notations carry the same meaning as in Section 3. We will have

$$f_l[x \rightarrow x^+] = \frac{(e-x)}{(e-x) + (e-x^+)} \Rightarrow x^+_l = (x^+)_\text{min} = (x + \delta)$$

$$f_h[x \rightarrow x^+] = \frac{x}{x + x^+} \Rightarrow x^+_h = (x^+)_\text{max} = (e - \delta)$$

The least costly transition from any $x$ towards the right will be the transition to the extreme right initiated by the H-claimants experimenting with their lowest payoff strategy. This is because $f_l[x \rightarrow x^+_l] \geq f_h[x \rightarrow x^+_h]$, for all $x \in [\delta, e - \delta]$.

Similar calculations for the transitions towards left of $x$ give

$$f_l[x^- \leftarrow x] = \frac{(e-x)}{(e-x) + (e-x^-)} \Rightarrow x^-_l = (x^-)_\text{min} = \delta$$

$$f_h[x^- \leftarrow x] = \frac{x}{x + x^-} \Rightarrow x^-_h = (x^-)_\text{max} = (x - \delta)$$
The least costly transition from any \( x \) towards the left will be the transition to the extreme left initiated by the L-claimants experimenting with their lowest payoff strategy. This is because 
\[ f_l[x \rightarrow x_l] < f_h[x \rightarrow x_h], \text{ for all } x \in [\delta, e - \delta]. \]
Thus, the resistance functions are
\[
    r^+(x) = \frac{x}{x + (x^+)_{\text{max}}} = \frac{x}{x + (e - \delta)} \\
    r^-(x) = \frac{(e - x)}{(e - x) + (e - (x^-)_{\text{min}})} = \frac{(e - x)}{(e - x) + (e - \delta)}
\]
Given an existing equilibrium at \( x \), the least costly transition will be to the equilibrium on extreme right if
\[
    r^+(x) \leq r^-(x) \Rightarrow \frac{x}{(x^+)_{\text{max}}} \leq \frac{(e - x)}{(e - (x^-)_{\text{min}})} = \frac{(e - x)}{(e - x)_{\text{max}}}
\]
The kalai-Smorodinsky solution for this bargaining problem would be the payoff vector \((x_{ks}, e - x_{ks})\), such that
\[
    \frac{x_{ks}}{x_{\text{max}}} = \frac{(e - x_{ks})}{(e - x)_{\text{max}}}
\]
Thus, if the existing equilibrium provides the low claimant a payoff lower (higher) than what she would get in the Kalai-Smorodinsky solution, then the least costly transition will be to the equilibrium on extreme right (left). It is clear from the above calculations that $r^+(x)$ and $r^-(x)$ intersect at $x_{ks}$. We might be tempted to conclude that the lower envelope of $[r^+(x), r^-(x)]$ gives the minimal tree which in turn is rooted at $x_{ks}$, and thus the stochastically stable equilibrium should be $(x_{ks}, e - x_{ks}) = (\frac{1}{2}e, \frac{1}{2}e)$. It was mentioned earlier that this reasoning is applicable only when the least costly transitions are local. However, Proposition 10 of BSY (2003) tells us that the stochastically stable equilibrium is indeed $(x_{ks}, e - x_{ks}) = (\frac{1}{2}e, \frac{1}{2}e)$. The result is summarized in the following proposition.

**Proposition 3**  Equal division of the bankrupt estate is the stochastically stable equilibrium if the rules of the demand game ask for efficiency.

4.3 Efficiency and Claims Boundedness

Efficiency alone is not enough for proportionality to emerge. Consider the following set of rules that are designed to capture both claims boundedness and efficiency.

\[
(x_l, x_h) = \begin{cases} 
(d_l, d_h) & \text{if } d_l + d_h = e \\
(d_l, 0) & \text{if } d_l + d_h = e, d_l \leq c_l, d_h > c_h \\
(0, d_h) & \text{if } d_l + d_h = e, d_l > c_l, d_h \leq c_h \\
(0, 0) & \text{if } d_l + d_h \neq e
\end{cases}
\]  \hfill (24)

The set of Nash equilibria will consist of some allocations in which one agent gets zero payoff. Since such equilibria will be trivially easy to escape, stochastic stability calculations will be unaffected if these equilibria are ignored. The subset of Nash equilibria with strictly positive payoffs
to the low claimants are given by

\[
d^*_l \in \begin{cases} 
[\delta, e - \delta] & \text{if } e \leq c_l \\
[\delta, c_l] & \text{if } c_l < e \leq c_h \\
[e - c_h, c_l] & \text{if } c_h < e 
\end{cases} 
\] (25)

The relevant games that need to be considered are again those illustrated in Figure 4. Following the same procedure and using the same notations we have

\[
f_l[x \rightarrow x^+] = \frac{(e - x)}{(e - x) + (e - x^+)} \Rightarrow x^+_l = (x^+)^{\text{min}} = (x + \delta)
\]

\[
f_h[x \rightarrow x^+] = \frac{x}{x + x^+} \Rightarrow x^+_h = (x^+)^{\text{max}} = \begin{cases} 
\delta & \text{if } e \leq c_h \\
c_l & \text{if } c_l < e 
\end{cases} 
\] (26)

Similarly,

\[
f_l[x^- \leftarrow x] = \frac{(e - x)}{(e - x) + (e - x^-)} \Rightarrow x^-_l = (x^-)^{\text{min}} = \begin{cases} 
\delta & \text{if } e \leq c_h \\
e - c_h & \text{if } c_h < e 
\end{cases} 
\]

\[
f_h[x^- \leftarrow x] = \frac{x}{x + x^-} \Rightarrow x^-_h = (x^-)^{\text{max}} = (x - \delta)
\]

It turns out that from any \( x \) the most likely transitions in either direction are the extreme transitions for all possible cases. Formally, \( f_l[x \rightarrow x^+] \geq f_h[x \rightarrow x^+_h] \) and \( f_h[x^-_h \leftarrow x] \geq f_l[x^-_l \leftarrow x] \)

This further implies that the resistances take the same form as in the previous case. It was established in the previous case that from any established equilibrium at \( x \) the most likely transition will be to the extreme right (left) if the low claimants are receiving a lower (higher) payoff than
suggested by the kalai-Smorodinsky solution. The Kalai-Smorodinsky division, which will also be the stochastically stable equilibrium, is
\[
(x_{s,l}^{ss}, x_{s,h}^{ss}) = \begin{cases} 
\left( \frac{1}{2} e, \frac{1}{2} e \right) & \text{if } e \leq c_l \\
(\left\lfloor \frac{c_l}{c_l+e} \right\rfloor e, \left\lfloor \frac{c_l}{c_l+e} \right\rfloor e) & \text{if } c_l < e \leq c_l \\
(c_l e, c_h e) & \text{if } c_h < e
\end{cases}
\] (28)

The stochastically stable equilibria exactly correspond to the truncated claims proportional rule. This rule first truncates the claim of each agent from \(c_j\) to \(\min(c_j, e)\); and then divides the estate proportionally to the truncated claims. This result in summarized in the following proposition.

**Proposition 4** The division suggested by the truncated claims proportional rule is the stochastically stable equilibrium if the rules of the demand game require both efficiency and claims boundedness.

4.4 Why not proportional?

The results obtained in the previous section help us understand why the proportional division does not emerge as the stochastically stable equilibrium. For ease of exposition let us consider an example with \(c_l = 0.4, c_h = 0.6, \) and \(e = 0.5\). Since \(c_h > e\), the high claimant will definitely end up losing the amount of \((c_h - e)\), henceforth referred to as the sunk claim. (It is easy to see that one, both, or none of the agents might have some sunk claims depending on the particular values of \(c_l, c_h, \) and \(e\)). If we consider this particular set of values then the proportional division would be \((2, 3)\). The usual demand game (Section 3.3), the modified demand game (Section 3.1-2), and the demand game requiring absolute efficiency (Section 4.1) will predict \((2.5, 2.5)\) as the stable division. The demand game requiring both absolute efficiency and claims boundedness (Section 4.2) will predict \((\frac{25}{20}, \frac{25}{20})\).

In all the formulations of the demand game the agents were restricted to demand no more than \(e\), for obvious reasons. Irrespective of the rules of the demand game, that part of an agent’s claim
which is **sunk** will never figure in the calculations. However, as discussed earlier, when someone is asked to act as a third party his prescriptions will, in all likelihood, satisfy claims boundedness. More importantly, there is nothing to prevent this third party from giving serious thought to the original contribution of the high claimant (0.6), and not just the maximum feasible payoff satisfying claims boundedness \((\min(0.6, 0.5) = 0.5)\). This suggests the following conjecture.

**Conjecture 1**  *An adhoc division rule that uses the initial claims of agents while dividing the estate can not be obtained as the unique stochastically stable equilibrium by any choice of rules for the demand game.*

Most of the the division rules truncate the claims from \(c_j\) to \(\min(c_j, e)\), either directly or indirectly. It is transparent that even if some \(c_j > e\), the proportional rule still uses the original contributions \((c_h, c_l)\) to divide the remaining estate. This implies that the proportional rule does take into account the sunk claims which in turn makes it impossible to obtain it as the stochastically stable equilibrium in the evolutionary framework of this paper. It can not be denied that proportional division possesses certain properties that make it very attractive. For example, using proportional division ensures that there is no benefit to an agent from splitting her claim into several smaller claims, or merging her claim with the claims of other agents (Moulin 2003). However, this transfer-proofness of proportional division rule is vacuous when there are only two claimants.

### 5 Conclusion

The main question the paper tries to address is how to divide up scarce resources when the involved parties have claims over it. A simple example is the question of dividing up a bankrupt estate among the creditors. The existing literature has tried to come up with adhoc rules from the perspective of a neutral third party. Proportional division is the most prominent of the several adhoc rules. The adhoc rules differ from each other because of the axioms that are imposed *in addition to* efficiency and claims boundedness. Efficiency requires that the estate be completely
divided between the claimants, and claims boundedness requires that no claimant be awarded more than her initial contribution. This paper tries to explore if a rule will emerge in the long run if agents are asked to bargain amongst themselves. It thus deals with bargaining problems with verifiable initial claims of both parties. The surplus over which bargaining takes place is assumed to be insufficient to honor all the claims completely.

It is shown that an adhoc rule can be rationalized as the unique self-enforcing long run outcome of Young’s (1993) evolutionary bargaining model by adding certain intuitive rules to the usual Nash demand game. If the agents bargain in the framework of the usual demand game, the long run stochastically stable equilibrium turns out to be equal division of the estate. If, in addition to the usual rules, demanding more than one’s initial claim leads to a zero payoff (claims boundedness) then the long run equilibrium corresponds to the constrained equal awards rule. If the rules capture both claims boundedness and efficiency, then the long run divisions are those prescribed by the truncated claims proportional rule.

Proportional division of a bankrupt estate among the creditors seems so just and obvious that it is rarely debated. If we ask a person to act as an arbiter in such a case, the answer will most likely be to divide the estate proportionally to initial contributions. However, the inability of the framework to account for sunk claims stops us short of obtaining exact proportionality.

References


