We establish the existence of pure strategy Nash equilibrium for a class of games we call shared resource games and show that certain games of fiscal competition are shared resource games. In the few studies on the existence of equilibrium in models of fiscal competition, the authors assume unit taxes and allow the aggregate amount of the mobile factor to fall if the net return would otherwise be negative. We adopt an alternative framework and assume ad valorem taxes and a fixed aggregate amount of mobile factor. This makes it easier to show that payoff functions are quasiconcave in own strategies, but it creates discontinuities in payoff functions and the nonexistence of best replies in some cases. We solve a general version of this problem using a recent existence result by Reny (1999). This leads directly to new existence results for games of fiscal competition, including cases in which the strategy spaces are more than one dimensional or there are more than two players.

Keywords: Fiscal competition; discontinuous games; Nash equilibrium, existence.
JEL classification: H7; C72; R10.
1. Introduction

There is a large literature in local public finance that studies taxing and spending by regional governments when a mobile factor of production is part of each region’s tax base. These papers examine the ways that the objectives of local governments, the tax instruments that are permitted, and the pattern of factor ownership determine the efficiency and distributional properties of equilibria. These models are also relevant to the study of fiscal federalism and certain aspects of globalization, like the creation of common factor markets. Thus, this work is relevant to many branches of public sector economics in addition to local public finance.

Two common assumptions in this literature are that only unit taxes are used and the aggregate amount of the mobile factor is fixed. Unit taxes are generally simpler to work with than ad valorem taxes, since the base for a unit tax does not directly depend on the price of the item. The assumption that the total amount of the mobile factor is fixed simplifies, but does not trivialize, the analysis of the equilibrium allocation of the factor across regions.

Unfortunately, these two assumptions are not always tenable. A unit tax that exceeds the marginal product of the mobile factor implies a negative net return. It is not feasible for tax payments to exceed income. Either tax payments must be directly constrained to be less than income, the tax rates themselves must somehow be prevented from being too large, or the total amount of the mobile factor must fall.

This problem goes unnoticed in most of the literature. It is always assumed in applied work that an equilibrium at moderate tax rates exists. This makes it possible to set aside
part of the strategy space, incompletely characterize the relationship between taxes and the
amount of the mobile factor in each region, and only partly specify the payoff functions of
the players. This approach has produced a wealth of important results, but it cannot be
used to study the existence of equilibrium.

There are two papers that explore the existence of equilibrium in models of fiscal com-
petition at a general level, Bucovetsky (1991) and Laussel and Le Breton (1998). These
authors keep the unit tax assumption, but they allow the aggregate amount of the mobile
factor (capital) to shrink just enough so that the net return is zero if a tax would otherwise
cause the net return to be negative.

This approach overcomes the problem of a negative net return. It amplifies, however, the
problem of showing that payoffs are quasi concave in own strategies. The result is that little
is known about existence in these models. Bucovetsky and Laussel-Le Breton assume there
are just two regions with identical technologies. Bucovetsky assumes preferences are general
but technology is quadratic. Laussel and Le Breton assume technology is general but public
and private goods are perfect substitutes.

Here we explore a different approach. We keep the assumption that the total amount of
the mobile factor is fixed, but we assume that taxes are ad valorem. Thus, the tax rate on
each unit is a percentage of its income. Not only is this assumption more realistic, but it also
avoids the problem of a negative net return, since this is impossible under the (reasonable)
assumption that ad valorem tax rates lie in the unit interval.

The shift to ad valorem taxation in itself resolves the problem of a negative net return. A
new problem arises, however, when we use it with the assumption that the total amount of the mobile factor is fixed. If all governments tax the mobile factor at confiscatory rates, then one region may be able to attract all of the factor with a small tax reduction. A discontinuous shift in the quantity of mobile factor in any region is likely to create a discontinuity in the payoff for the owner of the fixed factors there. A similar problem occurs at the opposite extreme, when all tax rates are zero and local public goods are essential. One region may then be able to attract all of the mobile factor with a small tax increase since it would be the only one providing any local public good.

The central claim of this paper is that discontinuities are the lesser problem. That is to say, a model of fiscal competition in which taxes are ad valorem, payoffs are discontinuous, and quasiconcavity is easier to show is a useful framework for studying the existence of equilibrium. We support this claim with new existence results for models with mobile capital and mobile labor. These results are themselves derived as applications of a general result that we establish for a class of games that we call shared resource games. The proof of the existence of pure strategy Nash equilibrium for shared resource games uses a recent theorem by Reny (1999) for games with discontinuous payoffs.

While we could in principle apply Reny’s theorem directly in each of our applications, it makes much more sense to show existence for a general class of games and then show that particular games belong to this class. Showing that particular games are shared resource games is straightforward while showing the existence of equilibrium is not. The logic of the existence result is clearer in the general setting. Finally, this result should be applicable to
other models of fiscal competition and may be useful in other contexts as well.

We begin, therefore, by defining and exploring shared resource games. In these games, the payoffs of each player depend on the strategies of the other players through their impact on the amount of a valuable resource that they share (divide). The total amount of the resource is generally fixed, although for greater plausibility we allow (but do not require) the total amount of the shared resource to fall when all players use extreme strategies. Regardless of whether the total is fixed or not, our basic assumptions imply that the amount of the resource that some player obtains will not be upper semi-continuous in her own strategies. As a result, a best reply does not exist in some cases.

Our primary theoretical result is that pure strategy Nash equilibria exist for shared resource games. Results prior to Reny (1999) for discontinuous games are difficult or impossible to apply because of the nonexistence of best replies. We also show that some players must use moderate strategies in equilibrium. This result can simplify the numerical search for equilibria in some shared resource games.

We then apply this result to the model of fiscal competition for mobile capital (Zodrow and Mieszkowski (1986), Wilson (1986), Wildasin (1988, 1991a)). As noted above, the existence of Nash equilibrium is explored in Bucovetsky (1991) and Laussel and Le Breton (1998). We consider four functional forms for technology, but we do not require the technologies to be identical across regions. Preferences are completely general. Most important, we provide a result for \( N \) regions. No such result exists in the literature, and it could not be derived by analyzing reaction functions as in Bucovetsky and Laussel-Le Breton.
After this, we turn to the somewhat richer model of fiscal competition for a mobile workforce (Flatters, Henderson and Mieszkowski (1974), Boadway and Flatters (1982), Mieszkowski and Zodrow (1989), Myers (1990), Wildasin (1991b)). In the version we consider, there are two regions with fixed and mobile factors (land and labor). The migration of labor has different direct effects on the incomes of residents depending on their type (owner or worker). Agents derive utility from an essential local public good as well as a private good, and there are taxes on both land and labor income. The fact that each region has two taxes makes the analysis of reaction functions intractable. The presence of an essential local public good creates a rich set of extreme strategies and payoff discontinuities.

In the next section we develop the shared resource game and in section 3 prove the existence of pure strategy Nash equilibrium. Section 4 briefly explains why other results for discontinuous games can not be used. Sections 5 and 6 develop the applications to fiscal competition for mobile capital and mobile labor, respectively. In section 7, we briefly summarize our main results and consider directions for further research.

2. The Shared Resource Game

There are $N \geq 2$ players each with pure strategy space $X_i$, $i = 1, 2, ..., N$. We call a particular subset of $X_i$ the degenerate strategies for $i$, denoted $D_i$, and the complement of this set in $X_i$ the nondegenerate strategies for $i$, $X_i \setminus D_i$. 
Assumption 1 (strategy spaces).

For all $i$,

i. $X_i$ contains more than one point and is a compact and convex subset of $\mathbb{R}^k$, $k \geq 1$.\(^1\)

ii. The set of degenerate strategies for $i$ is a nonempty subset of the relative boundary of $X_i$.

We note two implications for future reference. First, all points in the relative interior of $X_i$ are nondegenerate:

$$\text{ri}(X_i) \subset X_i \setminus D_i$$

(1)

Second, every open neighborhood of every point in $X_i$ contains nondegenerate points:

$$(\forall x_i \in X_i)(\forall \mathcal{N}(x_i)), \mathcal{N}(x_i) \cap (X_i \setminus D_i) \neq \emptyset$$

(2)

where the open neighborhood $\mathcal{N}(x_i)$ is the intersection of $X_i$ with some set that contains $x_i$ and is open in $R^k$.

Define $X \equiv \times_i^N X_i$ as the *joint strategy space*. The subset of $X$ in which every agent uses a degenerate strategy is the space of *degenerate joint strategies*, denoted $\mathcal{D} \equiv \times_{i=1}^N D_i$. The

\(^1\)Note that we do not assume that $X_i$ has a nonempty interior in $\mathbb{R}^k$. This allows the dimension of $X_i$ to vary across agents, which is essential for applications in which the number of tax bases varies across regions (we touch on this in section 6). It also makes it necessary to define a few relative topological concepts (see Rockafellar (1970)).

Given a nonempty convex set $C \subset \mathbb{R}^k$, the affine hull of $C$, written aff$(C)$, consists of all points in $\mathbb{R}^k$ that can be written as weighted sums of finitely many points in $C$ with weights (positive or nonpositive) summing to 1. A point $y$ is in the relative interior of $C$, written $y \in \text{ri}(C)$, if $y \in \text{aff}(C)$ and there is some $\epsilon > 0$ such that all points in the affine hull of $C$ with distance $\epsilon$ or less from $y$ are also in $C$ (Rockafellar, p. 44). The relative boundary of $C$, rb$(C)$, consists of the (standard) closure of $C$ less all points in ri$(C)$. The relative interior of $C$ is necessarily nonempty (Rockafellar, Theorem 6.2). If $C$ is bounded and contains more than one point then the relative boundary is nonempty (but if $C$ were a singleton then ri$(C) = C$ and rb$(C) = \emptyset$).
complement of this set in $X$ is the set of joint strategies in which at least one agent uses a nondegenerate strategy. We call this the space of \textit{nondegenerate joint strategies}, denoted $X \setminus D$.\footnote{That is to say, $X \setminus D \equiv \times_{i=1}^{N} X_i \setminus \times_{i=1}^{N} D_i$. We do not use the set $\times_{i=1}^{N} (X_i \setminus D_i)$, which is all joint strategies in which every strategy is nondegenerate, but the distinction between this and $X \setminus D$ should be kept in mind.} The symbol $-i$ denotes “all players but $i$.” It follows that at any element of $D_{-i}$ these $N - 1$ players use degenerate strategies, and at any element of $X_{-i} \setminus D_{-i}$, at least one of these players uses a nondegenerate strategy.

We define two constants, $\bar{s}$ and $s$, that denote the total amount of a resource that will somehow be “shared” among agents at nondegenerate and degenerate joint strategies (respectively). Let $0 \leq s \leq \bar{s}$ and $0 < \bar{s} < \infty$.

We now state our assumptions on the sharing rule.

\textit{Assumption 2} (sharing rule for $i$).

For all $i$ there exists a function $S_i : X \rightarrow [0, \bar{s}]$ with the following properties:

\begin{itemize}
  \item[i.] Aggregate resource constraint at nondegenerate joint strategies: for all $x \in X \setminus D$,
  \[ \sum_{i=1}^{N} S_i(x) = \bar{s}. \]
  \item[ii.] Aggregate resource constraint at degenerate joint strategies: for all $x \in D$,
  \[ \sum_{i=1}^{N} S_i(x) = s. \]
  \item[iii.] $S_i$ is continuous on $X \setminus D$.
  \item[iv.] For each player $i$, there exists a constant $\bar{s}_i \in \left( \frac{s}{N}, \bar{s} \right]$ such that for all $x \in D$ and all $x'_i \in X \setminus D_i$,
  \[ S_i(x'_i, x_{-i}) \geq \bar{s}_i \geq S_i(x_i, x_{-i}). \]
\end{itemize}
Assumption A.2(ii) permits (but does not require) the aggregate amount of the shared resource to fall discontinuously at degenerate joint strategies. This adds a bit of realism without creating any additional complexities. Even without this (so \( \bar{s} = \bar{s} \)), we show below that the sharing rule for some \( i \) must be discontinuous at any degenerate joint strategy.

A.2(iv) is the key substitute for continuity. It says that nondegenerate strategies both absolutely and relatively attract the shared resource when everyone else plays a degenerate strategy. The amount of shared resource exceeds both \( \bar{s}/N \) and the amount obtained from any degenerate strategy.

Assumption 2 permits sharing rules with a wide range of properties.

**Example 1.** \( N = 2, X_i = [0, 1], D_i = \{0, 1\}, 0 \leq s^{jk} \leq \bar{s}, j = 0, 1, k = 0, 1, \) and:

\[
S_1(x_1, x_2) = \begin{cases} 
\frac{x_2^2(1-x_1)^2}{x_1^2(1-x_1)^2 + x_2^2(1-x_2)^2} \bar{s}, & (x_1, x_2) \in X \setminus D \\
\bar{s}^0, & x_1 = 0 \text{ and } x_2 = 0 \\
\bar{s}^{01}, & x_1 = 0 \text{ and } x_2 = 1 \\
\bar{s}^{10}, & x_1 = 1 \text{ and } x_2 = 0 \\
\bar{s}^{11}, & x_1 = 1 \text{ and } x_2 = 1
\end{cases}
\]

\[
S_2(x_1, x_2) = \begin{cases} 
\bar{s} - S_1(x_1, x_2), & (x_1, x_2) \in X \setminus D \\
\bar{s} - S_1(x_1, x_2), & (x_1, x_2) \in D
\end{cases}
\]

A.2(iv) holds with \( \bar{s}_i = \bar{s} \), since any joint strategy in which one player’s strategy is degenerate and the other player’s is nondegenerate provides the latter with all of the shared resource. Notice that the set of degenerate joint strategies is not convex and \( S_i \) need not be constant on this set (in particular, we do not require equal division of the shared resource on \( D \)). \( S_i \) is not monotone in \( x_i \) at a fixed nondegenerate strategy for the other player.
Finally, regardless of how the $s^{jk}$ are defined, at least one of the sharing rules fails to be upper semi-continuous in own strategies at the four degenerate joint strategies. For example, $S_1(x) = s$ at all points of the form $\left( \frac{1}{2j}, 0 \right)$, $j \geq 1$. This converges to $0, 0)$, so if $S_1$ is upper semi-continuous in $x_1$ at $(0, 0)$ then $S_1(0, 0) = s$. This forces $S_2(0, 0) = 0$, but $S_2(x) = s$ at all points of the form $\left( 0, \frac{1}{2j} \right)$, so $S_2$ can not be upper semi-continuous in $x_2$ at $(0, 0)$.

This last result is completely general and important enough to summarize in a proposition.

**Proposition 1.** Under Assumptions 1 and 2, the sharing rule for at least one player must fail to be upper semi-continuous in her own strategy at any degenerate joint strategy.

**Proof.** See the Appendix.

**Example 2.** $N = 3$, $X_i = [0, 1]$, $D_i = \{0\}$, $\underline{s} = \bar{s} = 1$, and:

$$S_1(x_1, x_2, x_3) = \frac{2}{5}$$

$$S_2(x_1, x_2, x_3) = \begin{cases} \frac{\frac{x_2(1-x_3)}{x_2(1-x_1)+x_3} \frac{3}{5^5}}{\frac{3}{10}}, & x_2 \neq 0 \text{ or } x_3 \neq 0 \\ x_2 = 0 \text{ and } x_3 = 0 \end{cases}$$

$$S_3(x_1, x_2, x_3) = \begin{cases} \frac{\frac{x_3}{x_2(1-x_3)+x_3} \frac{3}{5^5}}{\frac{3}{10}}, & x_3 \neq 0 \text{ or } x_3 \neq 0 \\ x_2 = 0 \text{ and } x_3 = 0 \end{cases}$$
Example 2 shows that not all $S_i$ must be discontinuous. Furthermore, unlike in example 1, nondegenerate strategies do not guarantee that a player will receive some of the shared resource (unless the other players use degenerate strategies). For example, $S_2(1, x_2, 1) = 0$ for any $x_2$.\footnote{To see A.2(iv), choose $\hat{s}_i = 2/5$ for all three players. We have $\hat{s}_i \in \left(\frac{1}{3}, 1\right]$ as required. The joint strategy $x = (0, 0, 0)$ is the only jointly degenerate strategy, and $S_1(x) = 2/5$ and $S_2(x) = S_3(x) = 3/10$. Deviation by player 1 to any nondegenerate strategy provides her with $2/5$; by player 2, $3/5$; and by player 3, $3/5$. Therefore, for any $x'_i$ nondegenerate for $i$, $i = 2, 3$, we have $S_i(x'_i, x_{-i}) = 3/5 > 2/5 = \hat{s}_i \geq 3/10 = S_i(x)$.}

Each player $i$ has a payoff function $u_i : X \rightarrow \mathbb{R}$. We now impose the assumption that player $i$ is affected by the strategies of the other players entirely through the sharing rule.

**Assumption 3** (representation of payoff functions).

For each player $i$ there exists a function $F_i : X_i \times [0, \bar{s}] \rightarrow \mathbb{R}$ such that for all $x \in X$:

$$u_i(x) = F_i[x_i, S_i(x)]$$

Let $\xi_i(s_i)$ be any element of $\arg\max_{x_i \in X_i} F_i(x_i, s_i)$. $\xi_i(s_i)$ is therefore an optimal choice of $x_i$ given a particular quantity of shared resource.

**Assumption 4** (properties of $F_i$).

For all $i$:

i. $F_i$ is continuous.

ii. For all $x_i \in X_i$, $F_i(x_i, \cdot)$ is nondecreasing in $s_i$.\footnote{Given any $s_i \in [0, \bar{s}]$ and $s'_i \in [0, \bar{s}]$, if $s_i \geq s'_i$ then $F_i(x_i, s_i) \geq F_i(x_i, s'_i)$.}
iii. For all $i$, $\xi_i(\bar{s}/N)$ and $\xi_i(\bar{s}_i)$ are unique, and $F_i[\xi_i(\bar{s}_i), \bar{s}_i] > F_i[\xi_i(\bar{s}/N), \bar{s}/N]$.

$A.A(iii)$ captures the idea that one can do better with more shared resource (since $\bar{s}_i > \bar{s}/N$), but that is a stronger property than we need for our main theorem. We use generalizations of the following approach to establish $A.A(iii)$ in our applications.

Example 3. Suppose $X_i = [0,1]$ and $F_i = x_is_i$. Then $\xi_i(\bar{s}_i) = \xi_i(\bar{s}/N) = 1$ and $F_i(1, \bar{s}_i) > F_i(1, \bar{s}/N)$. Therefore $F_i[\xi_i(\bar{s}_i), \bar{s}_i] = F_i(1, \bar{s}_i) > F_i(1, \bar{s}/N) = F_i[\xi_i(\bar{s}/N), \bar{s}/N]$. The fact that $X_i$ is one dimensional is not essential, nor is the fact that $F_i$ is increasing in $x_i$ given $s_i > 0$ (see section 6).

We have not yet imposed conditions assuring that $u_i(\cdot, x_{-i})$ is quasiconcave on $X_i$. No single sufficient condition holds in all of our applications. It is therefore convenient to just assume quasiconcavity and return to the issue below.

Assumption 5

For all $i$ and all $x_{-i}$, $u_i(\cdot, x_{-i})$ is quasiconcave on $X_i$.

Any set of $N$ players together with strategy spaces and payoffs satisfying assumptions 1-5 is a shared resource game.\footnote{Note that in a shared resource game, each agent takes the sharing rule as given. It is part of the rules of the game. Wildasin (1991a) states that games of fiscal competition (as well as the standard Cournot}
In all but one of our applications, we use the following result as a starting point for showing quasiconcavity: if $F_i(x_i, s_i)$ is quasiconcave in both arguments and nondecreasing in $s_i$, and if $S_i$ is concave in $x_i$, then $F_i[x_i, S_i(x)]$ is quasiconcave in $x_i$. We weaken these premises in assumption 6, which is used in theorem 2 and in section 6 below:

**Assumption 6**

For all $i$,

i. $F_i$ is quasiconcave on $\text{ri}(X_i) \times (0, \bar{s})$.

ii. Given any $x_{-i} \in X_{-i} \setminus D_{-i}$, there exists a real valued, nondecreasing and continuous function $T$ with domain $(0, \bar{s})$, written $T(s_i) = \hat{s}_i$, and with range denoted $\hat{S}$, such that:

(a) $T[S_i(\cdot, x_{-i})]$ is concave on $\text{ri}(X_i)$.

(b) $F_i[x_i, T^{-1}(\hat{s}_i)]$ is quasiconcave on $\text{ri}(X_i) \times \hat{S}$.

iii. Given any $x_{-i} \in D_{-i}$, $S_i(\cdot, x_{-i})$ is concave on $X_i$.

$A.6(i)$ only requires $F_i$ to be quasiconcave on the relative interior of its domain, but this condition with $A.4(i)$ (continuity) implies quasiconcavity on the entire domain. The weaker and Bertrand oligopoly games), “do not fit the standard Nash framework” because of the presence of side constraints like our restriction on the total quantity of shared resource. While this is to some extent a matter of semantics, the standard framework can at a technical level incorporate these constraints by building them into allocation functions that are then assumed to be part of the rules of the game. For a different approach, see Laussel and Le Breton (1998).
condition is easier to check in applications when \( F_i \) is differentiable. Similarly, \( A.6(ii) \) only requires \( T[S_i(\cdot, x_{-i})] \) to be concave on the relative interior of \( X_i \). We do not allow \( s_i \) to be 0 or \( \bar{s} \) and we do not allow \( T \) to depend on \( x_{-i} \), but this is for ease of exposition only.

**Example 4.** Assume \( S_i \) from example 1 and \( F_i \) from example 3. \( A.6(i) \) is obvious. For \( A.6(ii) \), fix \( i = 1 \) and any \( x_2 \in X_2 \backslash D_2 = (0, 1) \). \( S_1 \) is locally convex in \( x_1 \) at low and high values, so part (a) is not immediate. Given \( x_1 \in (0, 1) \) we have \( 0 < S_i(x) < \bar{s} \). Choose \( T(s_1) = \ln(s_1) \). It is straightforward to verify that \( T[S_i(\cdot, x_2)] = \ln[S_i(\cdot, x_2)] \) is concave in \( x_1 \) on \((0, 1)\). For part (b), we need to verify that \( F_1[x_1, T^{-1}(\hat{s}_1)] = x_1 \exp(\hat{s}_1) \) is quasiconcave on \((0, 1) \times \hat{S} \), where \( \hat{S} = (-\infty, \ln(\bar{s})) \). Taking logs gives \( \ln(x_1) + \hat{s}_1 \) which is the sum of concave functions, so (b) follows. \( A.6(iii) \) holds because \( S_i = \bar{s} \) on the interior of the unit interval, so it can only jump down at the endpoints.\(^6\)

We conclude by showing that payoffs need not be upper semi-continuous in own strategies and that best replies need not exist in shared resource games.

**Example 5.** Assume \( S_i \) from example 1 and \( F_i \) from example 3. We further stipulate \( s - s^{01} < \bar{s} \) (so either \( s < \bar{s} \) or \( s^{01} > 0 \)). We have already verified that all assumptions hold for both players. We now show that player 2 has no best reply to \( x_1 = 0 \). We have \( u_i(0, 0) = (0)S_i(0, 0) = 0 \). For \( 0 < x_2 < 1 \) we have \( S_2(0, x_2) = \bar{s} \), so \( u_2(0, x_2) = x_2 \bar{s} \). This

\(^6\)If the strategy spaces were more than one dimensional, we would have to consider the cases in which a chord lies entirely within the relative boundary. We consider this case in section 6 and provide a general result (Proposition 2).
approaches \( \bar{s} \) from below as \( x_2 \) approaches 1, but \( u_2(0, 1) = s - s^{01} < \bar{s} \), so the payoff function jumps down at \((0, 1)\).

3. Existence of Pure Strategy Nash Equilibrium

To show existence in shared resource games, we need to use the most general result in Reny (1999). To state the result, we need a few definitions. Denote the vector of the players’ payoff functions by \( u : X \rightarrow \mathbb{R}^N \) where \( u(x) = (u_1(x), ..., u_N(x)) \) for every \( x \in X \). The graph of the vector payoff function is the subset of \( X \times \mathbb{R}^N \) given by \( \{(x, u) \in X \times \mathbb{R}^N | u = u(x)\} \).

Definition. Player \( i \) can secure a payoff \( \alpha \in \mathbb{R} \) at \( x \in X \) if there exists \( \bar{x}_i \in X_i \), such that \( u_i(\bar{x}_i, x'_{-i}) \geq \alpha \) for all \( x'_{-i} \) in some open neighborhood of \( x_{-i} \).

Definition. A game \( G = (X_i, u_i)_{i=1}^N \) is better-reply secure if whenever \((x^*, u^*)\) is in the closure of the graph of its vector payoff function and \( x^* \) is not an equilibrium, some player \( i \) can secure a payoff strictly above \( u^*_i \) at \( x^* \).

Definition. A game \( G = (X_i, u_i)_{i=1}^N \) is quasiconcave if each \( X_i \) is convex and for each \( i \) and every \( x_{-i} \in X_{-i} \), \( u(\cdot, x_{-i}) \) is quasiconcave on \( X_i \).

Definition. A game \( G = (X_i, u_i)_{i=1}^N \) is compact if each \( X_i \) is a nonempty compact subset of a topological vector space and each payoff function is bounded.
Theorem (Theorem 3.1, Reny (1999)). If $G = (X_i, u_i)_{i=1}^N$ is better-reply secure, quasi-concave and compact, then it possesses a pure strategy Nash equilibrium.

Our first result is:

**Theorem 1.** There exists a pure strategy Nash equilibrium for the shared resource game.

**Proof.** To see that the game is compact, note that the restrictions on $X_i$ follow from $A.1(i)$ and $u_i$ is bounded since $F_i$ is continuous and real valued with compact domain. The game is quasiconcave by assumptions 1 and 5. Thus, the only issue is better-reply security.

We need two preliminary results. The first states a relationship between payoffs and values of $F_i$.

**Lemma 1.** Fix any $x_i \in X_i$, $0 \leq s_i \leq \bar{s}_i$, and $\alpha \in \mathbb{R}$ such that $F_i(x_i, s_i) > \alpha$. Then for all $x_{-i} \in D_{-i}$ there exists $\bar{x}_i \in X_i \setminus D_i$ such that $u_i(\bar{x}_i, x_{-i}) > \alpha$ and $u_i$ is continuous at $(\bar{x}_i, x_{-i})$.

**Proof of Lemma 1.** If $x_i \in X_i \setminus D_i$ then just choose $\bar{x}_i = x_i$. $u_i$ is then continuous at $x$ and $S_i(x) \geq \bar{s}_i$ by $A.2(iv)$, so $u_i(x) = F_i[x_i, S_i(x)] \geq F_i(x_i, s_i) \geq F_i(x_i, s_i)$ by $A.4(ii)$ and $u_i(x) > \alpha$.

Suppose $x_i \in D_i$. $F_i$ is continuous in its first argument, so there exists a neighborhood
of $x_i$, say $N(x_i)$, such that:

$$F_i[x'_i, S_i(x_i, x_{-i})] > \alpha, \ \forall x'_i \in N(x_i)$$

In particular, there exists $\tilde{x}_i \in N(x_i)$ that is also nondegenerate for $i$. Since $x \in D$, it follows that $S_i(\tilde{x}_i, x_{-i}) \geq S_i(x)$ by A.2(iv). Therefore:

$$u_i(\tilde{x}_i, x_{-i}) \equiv F_i[\tilde{x}_i, S_i(\tilde{x}_i, x_{-i})] \geq F_i[\tilde{x}_i, S_i(x_i, x_{-i})] > \alpha$$

Therefore $u_i(\tilde{x}_i, x_{-i}) > \alpha$ and $u_i$ is continuous there.

Our second result shows that if a player achieves a payoff at $x$ that is strictly greater than some payoff $\alpha$, then she can also secure a payoff at $x$ that is strictly greater than $\alpha$. Formally:

**Lemma 2.** Fix any $x \in X$, $\alpha \in \mathbb{R}$, and individual $i$ such that $u_i(x) > \alpha$. Then $i$ can secure a payoff at $x$ that is strictly above $\alpha$.

**Proof of Lemma 2.** Suppose first that $u_i$ is continuous at $x$. Define $\epsilon \equiv \frac{u_i(x) - \alpha}{2} > 0$. There is a neighborhood of $x_{-i}$, say $N(x_{-i})$, such that $x'_{-i} \in N(x_{-i})$ implies $\alpha - u_i(x_i, x'_{-i}) < \epsilon = \frac{u_i(x) - \alpha}{2}$. Rearranging gives:

$$u_i(x_i, x'_{-i}) > \frac{u_i(x) + \alpha}{2} > \alpha, \ \forall x'_{-i} \in N(x_{-i})$$

Player $i$ therefore secures $\frac{u_i(x) + \alpha}{2}$ at $x$, which is strictly above $\alpha$.

Suppose instead that $u_i$ is discontinuous at $x$, so $x \in D$. If we define $s_i \equiv S_i(x)$ then $F_i(x_i, s_i) = u_i(x) > \alpha$ and furthermore $s_i \leq \tilde{s}_i$ by A.2(iv). Lemma 1 therefore applies, giving
such that \( u_i(\bar{x}_i, x_{-i}) > \alpha \) and \( u_i \) continuous at \((\bar{x}_i, x_{-i})\). If we now repeat the first part of this Lemma, we conclude that \( i \) secures \( \frac{u_i(\bar{x}_i, x_{-i}) + \alpha}{2} \) at \((\bar{x}_i, x_{-i})\). Player \( i \) therefore secures this value at \( x \) as well (recall the definition), and this value strictly exceeds \( \alpha \).

**Lemma 3.** The shared resource game is better-reply secure.

**Proof of Lemma 3.** Fix \((x^*, u^*)\) in the closure of the graph of the vector payoff function. By definition there is a sequence lying entirely in the graph of the vector payoff function converging to this point. Since payoffs at any point in the sequence are defined by the payoff functions, this sequence can be written:

\[
\{(x_1^j, ..., x_N^j, u_1(x^j), ..., u_N(x^j))\}_{j=1}^\infty
\]

Each component converges to its respective component in \((x^*, u^*)\), so:

\[
u_i^* = \lim_{j=1}^\infty u_i(x^j), \forall i
\]

Since \( X \) is compact we have \( x^* \in X \) and \( u_i(x^*) \) is well defined for all \( i \). Possible discontinuities in the payoff functions make it possible that \( \lim_{j=1}^\infty u_i(x^j) \neq u_i(x^*) \).

Following the definition of better-reply security, assume \( x^* \) is not an equilibrium point. Then there is a player, say \( k \), with better reply \( \bar{x}_k \), so \( u_k(\bar{x}_k, x_{-k}^*) > u_k(x^*) \). If \( u_k(x^*) \geq u_k^* \) then \( u_k(\bar{x}_k, x_{-k}^*) > u_k^* \). In this case \( k \) can secure a payoff at \( x^* \) that is strictly above \( u_k^* \), by Lemma 2.

Suppose instead \( u_k(x^*) < u_k^* \). In this case the payoff function and sharing rule for \( k \)
are discontinuous, so $x^* \in \mathcal{D}$. Suppose for the moment that we can show that for some individual the limit of payoffs can not be too high. Specifically, for some individual $l$:

$$F_l[\xi_l(\bar{s}/N), \bar{s}/N] \geq u^*_l$$

(5)

If this were true, then for this person $F_l[\xi_l(\bar{s}_l), \bar{s}_l] > F_l[\xi_l(\bar{s}/N), \bar{s}/N]$ by A.4(iii). Lemma 1 with $x_l = \xi_l(\bar{s}_l)$, $s_l = \bar{s}_l$, and $\alpha = F_l[\xi_l(\bar{s}/N), \bar{s}/N]$ would give $\bar{x}_l \in X_l$ such that $u_l(\bar{x}_l, x^*_{-l}) > u^*_l$. Player $l$ could then secure a payoff at $(\bar{x}_l, x^*_{-l})$, and therefore at $x^*$, that was strictly above $u^*_k$, by Lemma 2.

So, suppose (5) does not hold:

$$\lim_{j \to \infty} u_i(x^j) \equiv u^*_i > F_l[\xi_l(\bar{s}/N), \bar{s}/N], \forall i$$

(6)

The key to showing a contradiction is that if the limit of payoffs were this high for each individual, then at some point in the sequence the aggregate amount of shared resource must exceed the total amount available.

For each individual $i$ there is an integer, say $j_i$, such that at all terms in (4) with $j \geq j_i$,

$$u_i(x^j) > F_i[\xi_i(\bar{s}/N), \bar{s}/N]$$

(7)

Suppose in addition that for some $i$ and some $j \geq j_i$ we have $S_i(x^j) \leq \bar{s}/N$. Then $F_i[\xi_i(\bar{s}/N), \bar{s}/N] \geq F_i(x^j_{-i}, \bar{s}/N) \geq F_i[x^j_i, S_i(x^j)] = u_i(x^j)$, contradicting (7). Therefore

$$S_i(x^j) > \bar{s}/N, \forall i, \forall j \geq j_i$$

(8)

Define $\tilde{j} = \max\{j_1, \ldots, j_N\}$. At the $\tilde{j}$ term in the sequence of joint strategies in (3), denoted $x^{\tilde{j}}$, the inequality in (8) holds for all $i$. Therefore:

$$\sum_{i=1}^{N} S_i(x^{\tilde{j}}) > N \frac{\bar{s}}{N} = \bar{s}$$
This violates both aggregate resource constraints, yet one of them must hold at every term in the sequence of joint strategies. Therefore (6) does not hold, so (5) holds.

**Theorem 2.** Assumptions 1-4 and 6 define a shared resource game.

**Proof.** All remaining proofs are in the appendix.

Our last general theorem shows that some agents use nondegenerate strategies in equilibrium. This simplifies the numerical search for equilibria in these games, especially when the payoff functions have closed forms at nondegenerate joint strategies.

**Theorem 3.** In any pure strategy Nash equilibrium for the shared resource game, some agents use nondegenerate strategies.

It is not true, however, that all agents must use nondegenerate strategies in equilibrium. Nondegenerate strategies are effective in attracting shared resource when every other agent uses a degenerate strategy, but not necessarily otherwise (example 2). Payoffs increase with shared resource, but only weakly. These conditions would have to be strengthened to guarantee that every agent uses a nondegenerate strategy in equilibrium.
Example 6. Assume \( S_i \) from example 2 and \( F_i \) from example 3. We show that \((a, 0, 1)\) is a Nash equilibrium (where \(a\) is any value in the strategy space), so player 2 uses a degenerate strategy. \( x_1 = a \) is obviously a best reply. We have \( S_2(a, x_2, 1) = 0 \) for all \( x_2 \in X_2 \), so \( u_2(a, x_2, 1) = (x_2)(0) = 0 \) and \( x_2 = 0 \) is a best reply. Finally, \( u_3(a, 0, 0) = 0 \), but for all \( 0 < x_3 \leq 1 \) we have \( S_3(a, 0, x_3) = 3/5 \) and \( u_3(a, 0, x_3) = (x_3)(3/5) \), so \( x_3 = 1 \) is a best reply.

4. Other Approaches to Existence

Dasgupta and Maskin (1986) provide an existence theorem for pure strategy Nash equilibrium that requires upper semi-continuity of each payoff function in joint strategies. We have already seen that some payoff functions may not be upper semi-continuous in shared resource games (example 5). Thus, the Dasgupta-Maskin theorem could not have been used to prove Theorem 1.

As noted in Reny, the methods for establishing existence in Milgrom and Roberts (1990) do not require continuity. Best replies must exist, however, and we have seen that best replies need not exist in shared resource games (example 5).\(^7\)

Reny defines two conditions on games, \textit{reciprocal upper semi-continuity} ("reciprocal u.s.c.") and \textit{payoff security}, that together imply better-reply security but are easier to check in some circumstances. We now show that shared resource games need not satisfy reciprocal u.s.c., so Reny’s secondary theorem could not have been used to prove Theorem 1. The example is also interesting because all games in Reny (1999) that fail reciprocal u.s.c. and satisfy

\(^7\)In principle, the theorems developed in Baye et al. (1993) are applicable. They proved difficult to apply in our framework. The work of Simon and Zame (1990) is directly applicable, but only to establish the existence of a Nash equilibrium in mixed strategies.
better-reply security have payoff functions that fail quasiconcavity, while payoffs here are quasiconcave. An analogous but less transparent example can be constructed for the economic model given in section 6.

A game is reciprocally upper semicontinuous if, whenever \((x^*, u^*)\) is in the closure of the graph of its vector payoff function and \(u_i(x^*) \leq u_i^*\) for every player \(i\), then \(u_i(x^*) = u_i^*\) for every player \(i\). That is to say, if for anyone \(u_i(x^*) \neq u_i^*\) then there must be someone for whom \(u_i(x^*) > u_i^*\) (the payoff function jumps up). This is equivalent to the requirement that if some player’s payoff jumps down then some other player’s payoff jumps up.

**Example 7.** Suppose \(N = 2\), \(X_i = [0, 1]\), \(S_i\) from example 1 with \(\bar{s} - s^{01} < \bar{s}\) (recall example 5) and \(F_i\) from example 3. Consider the sequence of joint strategies \(\{(0, 1 - (1/2j))\}_{j=1}^{\infty}\). All points in the sequence are degenerate for player 1 and nondegenerate for 2, so \(S_1(0, 1 - (1/2j)) = 0\) and \(S_2(0, 1 - (1/2j)) = \bar{s}\). The limit is \((0, 1)\), and we have \(S_1(0, 1) = s^{01}\) and \(S_2(0, 1) = \bar{s} - s^{01} < \bar{s}\). There is a jump down in the payoff for 2 but no jump in the payoff for 1:

\[
u_1(0, 1) = (0)(s^{01}) = 0 = \lim_{j=1}^{\infty} S_1\left(0, 1 - \frac{1}{2j}\right) = u_1^*
\]

\[
u_2(0, 1) = (1)(\bar{s} - s^{01}) < \bar{s} = \lim_{j=1}^{\infty} S_2\left(1 - \frac{1}{2j}\right) S_2\left(0, 1 - \frac{1}{2j}\right) = u_2^*
\]
5. Fiscal Competition for Mobile Capital

We now consider the existence of equilibrium in a standard model of fiscal competition for mobile capital (Wildasin (1988, 1991a)), modified for ad valorem taxation. We give a general result and four specific results. One of the latter is for a model with any finite number of regions. No such result exists in the literature, and it could not be derived by analyzing reaction functions as in Wildasin (1991a), Bucovetsky (1991) and Laussel and Le Breton (1998).

There are \( N \geq 2 \) regions and players, and each player owns the fixed factor in her region. There are at most \( s \) units of capital available. Fixed factors and mobile capital combine to produce output \( f_i(s_i) \), where \( f_i : [0, s] \to \mathbb{R}_+ \) where (+) means nonnegative. Capital in region \( i \) is paid its marginal product which is then taxed at rate \( t_i \in [0, 1] = X_i \). Output less payments to capital becomes private consumption good \( c_i = f_i(s_i) - s_if_i'(s_i) \). A portion of the payments to capital becomes local public good \( z_i = t_is_if_i'(s_i) \).

Each player has preferences over private and public good, \( U_i(c_i, z_i) \), where \( U_i : \mathbb{R}_+^2 \to \mathbb{R} \). Substituting the previous expressions into the utility function gives preferences over tax rates for a fixed amount of shared resource:

\[
F_i(t_i, s_i) = U_i[f_i(s_i) - s_if_i'(s_i), t_is_if_i'(s_i)]
\]

This is well defined for all \( (t_i, s_i) \in X_i \times [0, s] \) under the assumptions below.

In the game of fiscal competition for mobile capital, each player chooses \( t_i \in [0, 1] = X_i \) taking into account that capital will be allocated so that no unit of capital could achieve a higher net return in any other location. We must therefore define the allocation of capital
to regions before we define payoffs. The first step in doing this is to consider where in the joint strategy space we want to permit the allocation to be discontinuous. Intuitively, we risk losing a great deal of generality if we require it to be continuous everywhere. If \( t_i = 1 \), then the net return in \( i \) is zero regardless of how much capital is present. If all regions use these extreme rates, then any reasonable production function should imply a discontinuous increase in the amount of mobile capital to any region that lowers its rate. This motivates the choice of \( \mathcal{D}_i = \{1\} \), so \( \mathcal{D} = (1, ..., 1) \).

Given any \( t \in X \), an equilibrium allocation of capital is any \((s_1, ..., s_N)\) such that \( s_i \geq 0 \) for all \( i \), \( \sum_{i=1}^{N} s_i = \bar{s} \) when \( t \in X \setminus \mathcal{D} \), \( \sum_{i=1}^{N} s_i = \bar{s} \) when \( t \in \mathcal{D} \), and:

\[
s_i > 0 \Rightarrow (1 - t_i) f_i'(s_i) \geq (1 - t_j) f_j'(s_j), \quad \forall i, \forall j
\]  

(9)

Given an equilibrium allocation of capital, we define \( S_i(t) \) to be the \( i \)th component of one such allocation at \( t \). This is well defined for all \( t \in X \) under the assumptions below. We define the payoff function for player \( i \) to be \( u_i(t) = F_i[t_i, S_i(t)] \).

The structure just defined is the model of fiscal competition for mobile capital. The following theorem provides general conditions under which the associated game of fiscal competition for mobile capital is well defined and has a pure strategy Nash equilibrium.

**Theorem 4.** Assume for all \( i \):

i. \( f_i(0) = 0, \ f_i > 0 \) for all \( s_i > 0 \), and \( f_i \) is continuous on \([0, \bar{s}]\) and \( C^3 \) on \((0, \bar{s})\).

ii. \( \infty \geq f_i'(0) > 0, \ \infty > f_i'' > 0 \) for all \( s_i > 0 \), and \( s_i f_i'' \) is continuous on \([0, \bar{s}]\).
$iii. \ -\infty < f''_i < 0$ for all $s_i > 0.$

$iv. \ f'_i + s_if''_i \geq 0$ for all $s_i > 0$

$v. \ U_i$ is continuous on the nonnegative orthant and $C^2$ with strictly positive first derivatives on the positive orthant.

Then Assumptions 1-4 in the definition of a shared resource game hold. If we further suppose:

$vi. \ U_i$ is quasiconcave.

$vii. \ s_if'_i f''_i - f''_i(f'_i + 2s_if''_i) \geq 0$ and $2(f'_i + s_if''_i)^2 - (s_if'_i)(2f''_i + s_if''_i) \geq 0$ for all $s_i > 0.$

$viii. \ \text{For all } t_{-i} \in X_{-i} \setminus D_{-i}, \ \text{either } S_i(\cdot, x_{-i}) \text{ is concave on } [0, 1] \text{ or there exists } 0 < t^*_i < 1 \text{ such that } S_i(\cdot, x_{-i}) \text{ is concave on } [0, t^*_i) \text{ and zero on } [t^*_i, 1].$

Then Assumption 5 holds as well. The game of fiscal competition for mobile capital is then a shared resource game and a pure strategy Nash equilibrium exists.

Theorem 4 provides some perspective on the assumptions that define shared resource games. Property A.2(iv) seems to be restrictive, but it is innocuous in this context. The standard properties $f_i(0) \geq 0$ and $f'_i > 0$ imply that a region with a tax rate less than one will obtain all of the capital when all other regions have a tax rate equal to one. This gives A.2(iv) (with $\tilde{s}_i = \tilde{s}$). A.4(ii) and A.4(iii) follow from standard properties together with assumption (iv) of the theorem. The latter states that the tax base (capital income under ad valorem taxation, $s_if'_i$) is nondecreasing in the quantity of capital in the region. This is
an entirely reasonable restriction. Of the assumptions that guarantee \( u_i \) is quasiconcave in \( t_i \), only \((viii)\) proves to be restrictive, and it holds in all but the first application below.

Our main result for this section is:

**Theorem 5.** Assume the model of fiscal competition for mobile capital and the assumptions on \( U_i \) stated in Theorem 4 above. A pure strategy Nash equilibrium exists if:

\[
\begin{align*}
  i. & \quad f_i(s_i) = \phi_i s_i^\beta \text{ provided } 2 \leq N < \infty, \phi_i > 0 \text{ and } 0 < \beta < 1. \\
  ii. & \quad f_i(s_i) = \left( \phi_i - \frac{\beta s_i}{2} \right) s_i \text{ provided } N = 2, \phi_i > 0 \text{ and } 0 < \beta < \frac{\phi_i}{3}. \\
  iii. & \quad f_i(s_i) = \phi_i \ln(1 + \beta s_i) \text{ provided } N = 2, \phi_i > 0 \text{ and } \beta_i > 0. \\
  iv. & \quad f_i(s_i) = \phi_i [1 - \exp(-\beta_i s_i)] \text{ provided } N = 2, \phi_i > 0 \text{ and } 0 < \beta_i < \frac{1}{e}.
\end{align*}
\]

Part \((i)\) is new, and in this case the direct analysis of reaction functions is not tractable because of the number of regions. Part \((ii)\) is slightly more general than the result in Bucovetsky (1991). He also assumes general preferences but requires the quadratic technologies to be identical up to a scale factor. Parts \((ii)-(iv)\) are different from the results in Laussel and Le Breton (1998). They assume \( c_i \) and \( z_i \) are perfect substitutes and the technologies in the two regions are identical, but the form of technology is general. Wildasin (1991a) assumes two regions and uses particular functional forms for both preferences and technology. The restriction to two regions in parts \((ii)-(iv)\) is due to the difficulty of establishing assumption \((viii)\) of Theorem 4. Further work should consider this problem more deeply.
Overall, we have shown that a model of fiscal competition for mobile capital with ad
valorem taxes and a fixed aggregate quantity of capital gives rise to a shared resource game.
This leads to useful alternative framework for studying existence and new existence results.

6. Fiscal Competition for a Mobile Labor Force

Games of fiscal competition for a mobile labor force have been used to study a wide
variety of questions in local public sector economics. The game we present here is useful for
studying the political economy of local public finance, but that is outside our scope. The
game we present extends the model of the previous section in two critical ways: in each
region there are multiple tax rates and an essential local public good. Multiple tax rates
make a direct analysis of existence intractable. More interesting, however, is the rich set of
degenerate strategies created by the local public good. This creates greater problems with
continuity and quasiconcavity than we addressed so far. Nevertheless, a pure strategy Nash
equilibrium exists.

We assume there are just two regions, denoted $A$ and $B$.\footnote{This notation is conventional but creates a minor deviation from the earlier notation. The restriction to two regions is not essential given the functional forms we assume for both technology and preferences.} In both regions there is a
single immobile owner of the fixed factor in her region. There are a finite number of workers
(male) who are mobile and provide a single unit of labor wherever they reside. There are at
most $\bar{s}$ units of labor available. Fixed factors and mobile labor produce an all purpose good
with technology:

\[
f_i(s_i) = \phi_i s_i^\beta, \quad \phi_i > 0, \ 0 < \beta < 1
\]  

(10)
Let $c_{iw}$ ($c_{ir}$) denote private good consumption by a worker (owner) in region $i$. Labor is paid its marginal product which is then taxed at rate $w_i \in [0,1]$, so $c_{iw} = (1 - w_i)f_i'$. The owner receives the remaining income which is then taxed at rate $r_i \in [0,1]$, so $c_{ir} = (1 - r_i)[f_i - s_if_i']$. Finally, let $z_i$ denote local public good in region $i$. All purpose good can be transformed into public good and private good at a constant rate, so $z_i = w_is_if_i' + r_i(f_i - s_if_i')$.

Preferences for all agents are assumed Cobb-Douglas, so $U(c_{ij}, z_i) = c_{ij}^\alpha z_i$, $j = w, r, i = A, B$, with $0 < \alpha < \infty$.

For owners, substituting the previous expressions into the utility function gives:

$$F_i(w_i, r_i, s_i) = H_i(w_i, r_i)s_i^{\beta(\alpha+1)}$$

where $H_i(w_i, r_i) = (1 - r_i)^\alpha[\beta w_i + (1 - \beta)r_i]^{\alpha+1}(1 - \beta)^\alpha$, $i = A, B$. Some level sets of $F_i$ (for given $s_i > 0$) are graphed in Figure 1. $F_i$ is maximized on the unit square with the wage tax equal to 1 and the rent tax equal to $\rho$, which depends only on the exogenous parameters.\(^9\)

**Figure 1**

In the game of fiscal competition for mobile labor, each owner chooses $(w_i, r_i) \in X_i = [0,1]^2$ taking into account that workers will be allocated so that no individual could achieve higher utility in the other location. We leave the formal derivation of worker preferences over tax rates for the appendix. It is clear from the previous expressions, however, that if region $i$ uses tax rates that would leave workers there with no private good ($w_i = 1$) or no

\(^9\)We have $\rho = \max \left\{ 0, \frac{1-\beta(1+\alpha)}{(1-\beta)(1+\alpha)} \right\}$. Necessarily $0 \leq \rho < 1$.  

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public good \((w_i = 0 \text{ and } r_i = 0)\) then workers in that region are at the lower bound of utility. If both regions use these extreme rates, the possibility arises of discontinuous shifts in the quantity of the mobile factor if one region switches to less extreme rates. This motivates the choice of \(D_i = D_{i0} \cup D_{i1}\), where:

\[
D_{i0} = \{(0, 0)\}, \quad i = A, B
\]

\[
D_{i1} = \{(w_i, r_i) | w_i = 1, 0 \leq r_i \leq 1\}, \quad i = A, B
\]

These are indicated in Figure 1.

The definition of a migration equilibrium, \((s_1, \ldots, s_N)\), is formally identical to an equilibrium allocation of capital (see the appendix). Given an equilibrium allocation for all \((w_A, r_A, w_B, r_B) \in X\), we define \(S_i(w_A, r_A, w_B, r_B)\) to be the \(i\)th component of one such allocation at this tax vector. This is well defined for all \(t \in X\) under the previous assumptions. We define the payoff function for player \(i\) to be \(u_i(w_A, r_A, w_B, r_B) = F_i[w_i, r_i, S_i(w_A, r_A, w_B, r_B)]\).

We now have the following:

**Theorem 6.** There is a pure strategy Nash equilibrium for the game of fiscal competition for mobile labor defined above.

A few points are worth noting. First, the sharing rule for this game (see the appendix) is fundamentally similar to that in example 1, and \(F_i\) is similar to the function in example 3. It is straightforward to show that there may be no best replies to certain strategies (as in
example 5) and with certain parameter values reciprocal upper semi-continuity may fail (as in example 7).

Second, we establish quasiconcavity by establishing Assumption 6. In doing this, the case \( x_{-i} \in D_{-i} \) raises new issues since \( X_i \) has two dimensions. In Figure 1, however, we see that the degenerate strategies for \( i \) consist of a vertex and an entire edge of \( X_i \). \( S_i \) is constant at \( \bar{s} \) at all other points, it is necessarily less than or equal to \( \bar{s} \) at the vertex and edge, and it is constant on the edge. This is sufficient for \( S_i \) to be concave on \( X_i \).

The previous argument works because the set of degenerate strategies has a particular structure. This structure is restrictive, but it is not quite as restrictive as it at first appears. The following proposition generalizes the result.

**Proposition 2.** Suppose assumptions 1-4 hold. Suppose \( \bar{s}_i = \bar{s} \) for all \( i \), \( D_i \) can be written as a union of sets \( D_{im}, m = 1, \ldots, M, M \geq 1 \), where each set \( D_{im} \) is a face of \( X_i \), and for all \( x_{-i} \in D_{-i}, x_i \in D_{im}, \) and \( x'_{i} \in D_{im}, \) we have \( S_i(x_i, x_{-i}) = S_i(x'_{i}, x_{-i}) \). Then A.6(iii) holds.

Finally, it is possible to extend the main theorem to include the case in which a representative worker chooses the tax rates in one region while the owner chooses the rates in the other. Worker payoffs do not satisfy assumption 3. Their payoffs are continuous, however (even though their location is not), and at any nondegenerate strategy they achieve a payoff.

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See Rockafellar (1970), p. 162. The key property for our purposes is that if a (nontrivial) convex combination of points in \( X_i \) lies in a face then the endpoints do as well. Given any \( X_i \), two of its faces are the set itself and the empty set. If \( X_i \) is the unit interval, then each endpoint is also a face. If \( X_i \) is the unit square, then each of the four edges and each of the four vertices is also a face.
that is strictly above what they receive at any degenerate joint strategy. They can therefore secure a payoff that is strictly above what they receive at any degenerate joint strategy. The game is therefore better-reply secure.

7. Conclusion

We establish the existence of pure strategy Nash equilibrium for a class of games we call shared resource games and show that certain games of fiscal competition are shared resource games. Our approach differs from the standard one in the fiscal competition literature by adopting an ad valorem tax framework and maintaining the assumption that the aggregate quantity of the mobile factor is fixed. Fundamentally, this makes it easier to show that payoff functions are quasiconcave in own strategies, but it creates discontinuities in payoff functions. The latter problem is overcome using a recent result by Reny (1999).

Our main result provides an alternative to the direct analysis of reaction functions in showing the existence of equilibrium. It is therefore especially useful when a direct analysis would be intractable, as is the case when there are more than two players (considered in section 5) or the strategy spaces are more than one dimensional (considered in section 6). Even for simple games, however, it may be easier to establish the conditions of Theorem 4 (or Theorem 1 for that matter) than to analyze reaction functions.

It is important to note that our main existence theorem places a very weak restriction on the allocation of shared resources at degenerate joint strategies. We do not require equal division of the shared resource. This division, whatever it is, need not be the same at
all degenerate strategies. Neither the existence of equilibrium nor the particular equilibria depend upon the allocation of shared resource at degenerate joint strategies. On the other hand, degenerate strategies may be best replies for some agents, and they may be used in equilibrium by some agents.

A general existence theorem for games of fiscal competition with $N$ regions requires a more general analysis of capital market equilibrium than has been done to date. Future work should consider this question more deeply. Future work should also consider restrictions that would permit some of the agents to have payoffs $u_i(x)$ that can not be represented in the form $F_i[x_i, S_i(x)]$. As mentioned at the end of section 6, an equilibrium exists in some situations in which this condition fails for some players. On the other hand, it is easy to construct examples in which an equilibrium does not exist if all we do is replace this condition with the usual continuity and quasiconcavity conditions for some players.
Appendix

**Proof of Proposition 1.** Fix any $x \in D$ and suppose each $S_i$ is upper semi-continuous in $x_i$ at $x$. For each player we can construct a sequence of nondegenerate points $\bar{x}_i^j$ that converges to $x_i$ (by (2)). Construct a sequence of joint strategies $x^j$ in which $x_i^j = \bar{x}_i^j$ and $x_k^j = x_k$ for all $k \neq i$. Player $i$ is the only agent using a nondegenerate strategy at all points in this sequence, so $S_i(x^j) \geq \tilde{s}_i$ for all $j$ (by A.2(iv)). Upper semi-continuity implies $S_i(x) \geq \lim_{j \to \infty} S_i(x^j) \geq \tilde{s}_i$. Since this is true for all $i$ we have $\sum_{i=1}^N S_i(x) \geq N\tilde{s}_i > s \geq \bar{s}$, contradicting A.2(ii).

**Proof of Theorem 2.** All we need to show is that $u_i$ is quasiconcave on $X_i$. Suppose first that $x_{-i} \in X_{-i} \setminus D_{-i}$. Fix $x_i$ and $x'_i$ in ri$(X_i)$ and $\lambda \in (0, 1)$. Assumption A.6(ii) applies, so define $\tilde{s}_i = T[S_i(x_i)]$ and $\tilde{s}'_i = T[S_i(x'_i)]$. Define the following notation for convex combinations of any points $x$ and $x'$:

$$x\lambda x' \equiv \lambda x + (1 - \lambda)x'$$

We now have:

$$u_i(x_i\lambda x'_i, x_{-i}) = F_i[x_i\lambda x'_i, S_i(x_i\lambda x'_i, x_{-i})]$$

$$= F_i[x_i\lambda x'_i, T^{-1}\{T[S_i(x_i\lambda x'_i, x_{-i})]\}]$$

$$\geq F_i[x_i\lambda x'_i, T^{-1}\{T[S_i(x_i)] \lambda T[S_i(x'_i)]\}]$$

$$= F_i[x_i\lambda x'_i, T^{-1}\{\tilde{s}_i\lambda \tilde{s}'_i\}]$$

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\[ \geq \min \left\{ F_i \left[ x_i, T^{-1} (\tilde{s}_i) \right], F_i \left[ x'_i, T^{-1} (\tilde{s}'_i) \right] \right\} \]
\[ = \min \left\{ F_i \left[ x_i, S_i(x_i) \right], F_i \left[ x'_i, S_i(x'_i) \right] \right\} \]

To extend this to any \( x_i \) and \( x'_i \) in \( X_i \), note that \( F_i \) by assumption is continuous on its entire domain and \( S_i \) is continuous on \( X_i \) given that \( x_{-i} \in X_{-i} \setminus D_{-i} \).

Now suppose \( x_{-i} \in D_{-i} \). By assumption \( S_i \) is concave on \( X_i \) and \( F_i \) is quasiconcave and nondecreasing in \( s_i \) on its entire domain (again using continuity of \( F_i \)). \( u_i \) is therefore quasiconcave on \( X_i \).

**Proof of Theorem 3.** Let \( x^* \) be an equilibrium joint strategy. Suppose by way of contradiction that each \( x^*_i \) is degenerate. Some individual \( i \) must have \( \tilde{s}/N \) or less of the shared resource (not everyone can have strictly more than an equal share), so \( S_i(x^*) \leq \tilde{s}/N \).

Therefore:

\[ u_i(x^*) = F_i[x^*_i, S_i(x^*)] \leq F_i[x^*_i, \tilde{s}/N] \leq F_i[\xi_i(\tilde{s}/N), \tilde{s}/N] < F_i[\xi_i(\tilde{s}_i), \tilde{s}_i] \]

where the last step uses \( A.4(iii) \). Lemma 1 with \( x_i = \xi_i(\tilde{s}_i) \), \( s_i = \tilde{s}_i \), and \( \alpha = u_i(x^*) \) gives \( \bar{x}_i \in X_i \) such that \( u_i(\bar{x}_i, x^*_{-i}) > u_i(x^*) \). Player \( i \) has a better reply, a contradiction.

**Proof of Theorem 4.** We first show there is an equilibrium allocation of capital for any \( t \in X \). If \( t \in D \) then any allocation of capital satisfying \( s_i \geq 0 \) and \( \sum_{i=1}^{N} s_i = \bar{s} \) is an equilibrium. Choose one arbitrarily. Now suppose \( t \in X \setminus D \). Consider the following optimum problem: maximize \( \sum_{i=1}^{n}(1 - t_i)f_i(s_i) \) subject to \( s_i \geq 0 \), \( i = 1, \ldots, n \), and \( \sum_{i=1}^{n} s_i = \bar{s} \). The problem has a unique solution, since the objective function is strictly concave and continuous.
and the constraint set is compact and convex. The solution must satisfy the Kuhn-Tucker conditions, which are necessary at the optimum. The Kuhn-Tucker conditions guarantee that the solution is an equilibrium: all regions with capital provide the same net return, regions without capital do not offer a higher net return, and all capital is allocated. We use the solution to define $S_i(t), i = 1, \ldots, N$. Most important, each $S_i$ is continuous on $t \in X \setminus D$ by the maximum theorem under (strict) convexity; see Sundaram (1996), Theorem 9.17, part 2.

$F_i$ is well defined for all $(t_i, s_i) \in [0, 1] \times [0, \bar{s}]$ given $T.4(i) - T.4(ii)$.\(^{11}\)

We now establish Assumptions 1-4 of Theorem 1. Assumption 1 is immediate. $A.2(i) - A.2(iii)$ follow from the construction of $S_i$ above. For $A.2(iv)$, fix $t = (1, 1, \ldots, 1)$ (this is the only degenerate joint strategy) and any allocation of capital $(s_1, \ldots, s_N)$. Capital has a zero net return in all regions. Given any $t'_j < 1$ we have $(1 - t'_j)f'(s_j) > 0$ by $T.4(ii)$, so $j$ is the only region offering a positive net return. The only way (9) can be satisfied is if region $j$ now has all of the capital. Therefore $S_j(t'_j, t_{-i}) = \bar{s} \geq S_j(t)$ and $A.2(iv)$ holds with $\bar{s}_j = \bar{s}$. Assumption 3 is established in the text.

Regarding Assumption 4, $F_i$ is continuous given $T.4(i) - T.4(ii)$, so $A.4(i)$ holds. Fix any $(t_i, s_i) \in (0, 1] \times (0, \bar{s}]$. The first partials of $F_i$ (with the obvious notation) are $F_i^1 = (U_i^1)(s_i f'_i) > 0$ and $F_i^2 = (U_i^1)(-s_i f''_i) + (U_i^2)[t_i(f'_i + s_i f''_i)] > 0$. $F_i^2 > 0$ and $F_i$ continuous give $F_i$ nondecreasing in $s_i$ at all $t_i$, so $A.4(ii)$ holds. $F_i^1 > 0$ and $F_i^2 > 0$ give $A.4(iii)$.

Assumptions 1-4 of Theorem 1 therefore hold.

The remaining issue is quasiconcavity of $u_i$ in $t_i$, for which we use $T.4(vi) - T.4(viii)$.

\(^{11}\)We use this notation to refer to the premises of Theorem 4.
We first show $F_i$ is quasiconcave. Fix any $i$ and any $(t_i, s_i) \in (0, 1) \times (0, \bar{s})$. T.4(vii) with $U_i$ quasiconcave give $2F_1F_2F_{12} - (F_1)^2F_{22} - (F_2)^2F_{11} \geq 0$ on this set (the proof is straightforward but tedious). $F_i$ is then quasiconcave by continuity.

Consider the case $t_{-i} \in X_{-i} \setminus D_{-i}$. If $S_i(\cdot, t_{-i})$ is concave on $[0, 1]$ we are done, since $F_i$ is quasiconcave and nondecreasing in $s_i$. So, suppose not. By assumption T.4(viii), there exists $0 < t_i^* < 1$ such that $S_i(\cdot, x_{-i})$ is concave on $[0, t_i^*)$. It follows that $u_i(\cdot, t_{-i})$ is quasiconcave on $[0, t_i^*)$. On the interval $[t_i^*, 1)$ we have $S_i(\cdot, t_{-i}) = 0$ by T.4(viii), so $u_i(\cdot, t_{-i}) = U_i(0, 0)$. The flat extension of a quasiconcave function is quasiconcave (the function remains monotone or single-peaked), so $u_i$ is quasiconcave on $[0, 1)$. $F_i$ is continuous and so is $S_i$ (since $t_{-i} \in X_{-i} \setminus D_{-i}$), so $u_i$ is quasiconcave on $[0, 1]$.

Now fix $t_{-i} \in D_{-i}$. If $t_i < 1$, we saw above that $S_i(t) = \bar{s}$. Therefore $u_i(t) = F_i(t_i, \bar{s}) = U_i[f_i(\bar{s}) - \bar{s}f'_i(\bar{s}), t_i\bar{s}f'_i(\bar{s})]$. This increases on $[0, 1)$, so regardless of the jump at $t_i = 1$ (in fact it must be down), $u_i$ is quasiconcave on $X_i$.

**Proof of Theorem 5.** In all cases, showing T.4(i)–T.4(iv) is straightforward. T.4(v)–T.4(vi) hold by assumption. To establish quasiconcavity, we derive the equilibrium allocation of capital explicitly and then complete the proof.

*Cobb-Douglas.* Fix any $t \in X \setminus D$. The net return is positive in any region with $t_i < 1$, so in equilibrium $t_i = 1 \Rightarrow s_i = 0$. Since the Inada condition $f'(0) = \infty$ holds in this case, we also have the converse. Thus, $t_i = 1 \Leftrightarrow s_i = 0$. This means that we can identify the regions that have capital in equilibrium from the vector of tax rates.
Let $N'$ denote the number of regions with nondegenerate taxes. We know $N' \geq 1$ by $t \in X \setminus \mathcal{D}$. If $N' = 1$ then all of the capital resides in that region, by the previous result. If $N' \geq 2$, the equivalence established above and equation (9) give $N' - 1$ equations that together with the adding-up condition determine the equilibrium allocation of capital:

$$S_i(t) = \frac{s_i[(1 - t_i)\phi_i]^{1-\beta}}{\sum_{j=1}^{N'}[(1 - t_k)\phi_k]^{1-\beta}}, \quad i = 1, \ldots, n$$

Now fix any $t_j < 1$. At any $0 < t_i < 1$, $\frac{\partial u_i}{\partial t_i} = F^1_i + F^2_i S'_i$ and $\frac{\partial^2 u_i}{\partial t_i^2} = F^1_{11} + 2F^1_{12}S'_i + F^1_{22}(S''_i)^2 + F^1_{22}(S''_i)$. We would be done if $S_i$ were concave in $t_i$, but it is not. $u_i$ is quasiconcave on $0 < t_i < 1$ if, for all points at which $\frac{\partial u_i}{\partial t_i} = 0$, we also have $\frac{\partial^2 u_i}{\partial t_i^2} < 0$. In other words, thinking of $u_i$ as a function of the single variable $t_i$, if every stationary point of $u_i$ is a strict local maximum then it must be quasiconcave. We use properties of $S''_i$ at stationary points of $u_i$ to simplify $\frac{\partial^2 u_i}{\partial t_i^2}$ and show that the latter is negative where $\frac{\partial u_i}{\partial t_i} = 0$. The details are available on request. With this done, $u_i$ is quasiconcave on $X_i$ from the continuity of $F_i$ and $S_i$.

If $t_j = 1$ then $S_i = s$ for all $t_i < 1$, giving (as before) $u_i(t_i, 1) = F_i(t_i, s) = U_i(f_i(s) - \bar{s}f'_i(s), t_i\bar{s}f'_i(s)]$. This increases on $[0, 1)$, so regardless of the jump at $t_i = 1$ (in fact it must be down), $u_i$ is quasiconcave on $X_i$.

Quadratic, Log, Exponential. Because the Inada condition does not hold in these cases, it is possible to have $t_i < 1$ and yet $s_i = 0$ in equilibrium. We can no longer identify the regions that have capital in equilibrium from the vector of tax rates alone. This makes it more difficult to solve for the equilibrium.\footnote{Note that we do not need to solve for the equilibrium to establish the existence of $S_i$ and continuity on $t \in X \setminus \mathcal{D}$. This follows from the maximum theorem under strict convexity, used in Theorem 4.} The first step is to solve for the solution to the
equation $(1 - t_i)f'_i(\sigma_i) = (1 - t_j)f'_j(\bar{s} - \sigma_i)$, where $\sigma_j(t) \equiv \bar{s} - \sigma_i(t)$.

For the quadratic:

$$\sigma_i(t) = \frac{(1 - t_i)\phi_i - (1 - t_j)(\phi_j - \beta_j \bar{s})}{(1 - t_i)\beta_i + (1 - t_j)\beta_j}, \quad t \in X \setminus D$$

For log:

$$\sigma_i(t) = \frac{(1 - t_i)\phi_i \beta_i (1 + \beta_j \bar{s}) - (1 - t_j)\phi_j \beta_j}{[(1 - t_i)\phi_i + (1 - t_j)\phi_j]\beta_i \beta_j}, \quad t \in X \setminus D$$

For the exponential:

$$\sigma_i(t) = \begin{cases} \frac{1}{\beta_i + \beta_j} \left[ \ln \left( \frac{1 - t_i}{1 - t_j} \right) + \ln \left( \frac{\phi_i \beta_i}{\phi_j \beta_j} \right) + \bar{s} \beta_j \right] & 0 \leq t_i < 1, \ 0 \leq t_j < 1 \\ +\infty & 0 \leq t_i < 1, \ t_j = 1 \\ -\infty & t_i = 1, \ 0 \leq t_j < 1 \end{cases}$$

which covers all of $t \in X \setminus D$.

It is straightforward to show that the following $S_i(t), i = 1, \ldots, n$, define an equilibrium allocation of capital in each case:

$$S_i(t) = \begin{cases} \bar{s} & \text{if } \sigma_i(t) > \bar{s} \\ \sigma_i(t) & \text{if } 0 \leq \sigma_i(t) \leq \bar{s} \\ 0 & \text{if } \sigma_i(t) < 0 \end{cases}$$

Now fix any $t_j < 1$. In all cases we have $\frac{\partial \sigma_i}{\partial t_i} < 0$ and $\frac{\partial^2 \sigma_i}{\partial t_i^2} < 0$ on $[0, 1)$. Define $t^*_i$ by $\sigma_i(t^*_i, t_j) = 0$ (this can be found explicitly in each case). If $t^*_i \leq 0$ then $\sigma_i(\cdot, t_j) \leq 0$ on $[0, 1]$ since $\sigma_i$ is decreasing on $[0, 1]$. From the characterization above, $S_i(\cdot, t_j) = 0$ on $[0, 1]$. If $0 < t^*_i < 1$ then $S_i(\cdot, t_j)$ is concave on $[0, t^*_i)$ (perhaps flat at $\bar{s}$ for $t_i$ close to zero) and zero on $[t^*_i, 1]$. If $t^*_i \geq 1$ then $\sigma_i(\cdot, t_j) \geq 0$ on $[0, 1]$. $S_i(\cdot, t_j)$ is concave on $[0, 1]$ (perhaps flat at $\bar{s}$ for some or all $t_i$).
Proof of Theorem 6. We first complete the derivation of worker preferences over tax rates. Following the standard approach, we assume agents are small enough so that we can set aside integer issues and the possibility that the migration of an individual has an effect on the destination or source region. Thus, if region $i$ is occupied ($s_i > 0$), we let $V_i$ denote both the utility of a worker in $i$ and the utility any individual worker in $-i$ would obtain after migrating to $i$. If region $i$ is unoccupied ($s_i = 0$), then $V_i$ denotes just the latter. We state $V_i$ and then discuss it:

$$V_i = \begin{cases} 
\beta^\alpha G_i(w_i, r_i) \left( \frac{1}{s_i} \right)^{\alpha - \beta(1 + \alpha)} & \text{if } s_i > 0 \text{ and } (w_i, r_i) \in [0, 1]^2 \\
+\infty & \text{if } s_i = 0 \text{ and } (w_i, r_i) \in [0, 1]^2 \setminus D_i \\
0 & \text{if } s_i = 0 \text{ and } (w_i, r_i) \in D_i 
\end{cases}$$

where $G_i(w_i, r_i) = (1 - w_i)^\alpha [\beta w_i + (1 - \beta) r_i]^{\phi_i^\alpha + 1}$, $i = A, B$.

In the first case, (10) and the equations for $c_{iw}$, $z_i$, and $U$ give $V_i$ directly. Reasonable comparative statics (or "stability") require $\alpha - \beta(1 + \alpha) > 0$, so the utility of resident workers decreases with in-migration.

To see the next expression, note that $G_i(w_i, r_i) = 0$ if and only if the taxes are degenerate in region $i$. Therefore $s_i = 0$ and $(w_i, r_i) \in [0, 1]^2 \setminus D_i$ means that region $i$ is unoccupied and has nondegenerate taxes. Since workers in an occupied region always has finite utility, $V_i = \infty$ means that migration to an unoccupied region with nondegenerate taxes will always increase the migrant’s utility. This is an entirely reasonable requirement. It also follows mechanically if we assume $(G_i)(\infty) = \infty$ when $G_i > 0$.

In the final case, $s_i = 0$ and $(w_i, r_i) \in D_i$, so region $i$ is unoccupied and has degenerate taxes. Therefore $V_i = 0$ means that migration to an unoccupied region with degenerate taxes can not increase the migrant’s utility. This is again reasonable since consumption of private
or public good after migration to \( i \) would be zero. It also follows mechanically if we assume \((G_i)(\infty) = 0\) when \( G_i = 0 \).

Given any \( t = (w_A, r_A, w_B, r_B) \in X \), a migration equilibrium is any \((s_A, s_B)\) such that 
\[
s_i \geq 0 \quad \text{for } i = A, B; \quad s_A + s_B = \bar{s} \quad \text{when } t \in X \setminus \mathcal{D}, \quad s_A + s_B = \underline{s} \quad \text{when } t \in \mathcal{D},\]  
and:
\[
s_i > 0 \Rightarrow V_i[G_i(w_i, r_i), s_i] \geq V_{-i}[G_{-i}(w_{-i}, r_{-i}), s_{-i}], \quad i = A, B \tag{12}
\]

We can now derive the migration equilibrium for any \( t \in X \). If \( t \in \mathcal{D} \) then any allocation of labor satisfying \( s_i \geq 0 \) and \( \sum_{i=1}^{N} s_i = \bar{s} \) is an equilibrium. The need to establish quasiconcavity imposes some restrictions on the choice; see below. Now suppose \( t \in X \setminus \mathcal{D} \).

Clearly \((w_i, r_i) \in X_i \setminus \mathcal{D}_i \Leftrightarrow s_i > 0 \). The argument is the same as for the Cobb-Douglas case in Theorem 5 above. It follows that if only region \( i \) has nondegenerate taxes then all of the capital must reside there. If both regions have nondegenerate taxes then it follows that capital must reside in both regions (12) implies worker utility must be equal. The result is:

\[
S_A(w_A, r_A, w_B, r_B) = \begin{cases} 
\frac{\bar{s}[G_A(w_A, r_A)]^{-\gamma}}{[G_A(w_A, r_A)]^{-\gamma} + [G_B(w_B, r_B)]^{-\gamma}} & \text{if } (w_A, r_A, w_B, r_B) \in X \setminus \mathcal{D} \\
\frac{s^{00}}{s^{00}} & \text{if } (w_A, r_A) \in \mathcal{D}_{A0} \text{ and } (w_B, r_B) \in \mathcal{D}_{B0} \\
\frac{s^{01}}{s^{01}} & \text{if } (w_A, r_A) \in \mathcal{D}_{A0} \text{ and } (w_B, r_B) \in \mathcal{D}_{B1} \\
\frac{s^{10}}{s^{10}} & \text{if } (w_A, r_A) \in \mathcal{D}_{A1} \text{ and } (w_B, r_B) \in \mathcal{D}_{B0} \\
\frac{s^{11}}{s^{11}} & \text{if } (w_A, r_A) \in \mathcal{D}_{A1} \text{ and } (w_B, r_B) \in \mathcal{D}_{B1}
\end{cases}
\]

\[
S_B(w_A, r_A, w_B, r_B) = \begin{cases} 
\bar{s} - S_A(w_A, r_A, w_B, r_B), & (w_A, r_A, w_B, r_B) \in X \setminus \mathcal{D} \\
\bar{s} - S_A(w_A, r_A, w_B, r_B), & (w_A, r_A, w_B, r_B) \in \mathcal{D}
\end{cases}
\]

We now establish Assumptions 1-4 of Theorem 1. Assumptions 1 and \( A.2(i) - A.2(\text{iii}) \) are immediate. \( A.2(\text{iv}) \) holds with \( \bar{s}_i = \bar{s} \): starting from any \( x \in \mathcal{D} \), either player can obtain

\[0 \leq s^{jk} \leq \bar{s}, \quad j = 0, 1, \quad k = 0, 1, \quad \gamma = \frac{1}{\alpha - \beta(1+\alpha)}. \quad \text{The stability condition gives } \gamma > 0.\]

\[39\]
all of the shared resource by deviating to any nondegenerate strategy. Assumption 3 is
established in the text. Regarding Assumption 4, A.4(i) − A.4(ii) are immediate from (11).
A.4(iii) holds since for any \( s_i > 0 \) we have \( \xi_i(s_i) = (1, \rho) \) so (using (11)) \( F_i[\xi_i(s_i), s_i] \) equals
a positive constant multiplied by \( s_i^{\beta(\alpha+1)} \). Obviously then \( F_i[\xi_i(\bar{s}), s] > F_i[\xi_i(\bar{s}/2), \bar{s}/2] \).

We now establish Assumption 6 (instead of Assumption 5) and conclude we have a shared
resource game from Theorem 2 (instead of Theorem 1). We establish A.6(iii) in the text.
For A.6(i), we show \( \ln[F_A(w_A, r_A, s_A)] = \ln[H_A(w_A, r_A)] + \beta(\alpha + 1)\ln(s_A) \) is concave on
\( \tilde{r}(X_A) \times (0, \bar{s}) \). This follows if and only if \( \ln\tilde{H}(w_A, r_A) \equiv \ln((1 - r_A)^\alpha[\beta w_A + (1 - \beta)r_A]) \) is
concave. We have:

\[
\frac{\partial \ln \tilde{H}}{\partial w_A} = \frac{\beta}{\beta w_A + (1 - \beta)r_A}, \quad \frac{\partial \ln \tilde{H}}{\partial r_A} = \frac{1 - \beta}{\beta w_A + (1 - \beta)r_A} - \frac{\alpha}{1 - r_A}.
\]

\[
\frac{\partial^2 \ln \tilde{H}}{\partial w_A^2} = \frac{-\beta^2}{[\beta w_A + (1 - \beta)r_A]^2}, \quad \frac{\partial^2 \ln \tilde{H}}{\partial r_A^2} = -\left\{ \frac{(1 - \beta)^2}{[\beta w_A + (1 - \beta)r_A]^2} + \frac{\alpha}{(1 - r_A)^2} \right\}
\]

\[
\frac{\partial^2 \ln \tilde{H}}{\partial w_A \partial r_A} = \frac{-\beta(1 - \beta)}{[\beta w_A + (1 - \beta)r_A]^2}.
\]

Therefore \( \frac{\partial^2 \ln \tilde{H}}{\partial w_A^2} < 0 \) and:

\[
\frac{\partial^2 \ln \tilde{H}}{\partial w_A^2} \frac{\partial^2 \ln \tilde{H}}{\partial r_A^2} - \left( \frac{\partial^2 \ln \tilde{H}}{\partial w_A \partial r_A} \right)^2 = \frac{\alpha \beta^2}{[\beta w_A + (1 - \beta)r_A]^2(1 - r_A)^2} > 0.
\]

Finally, we need to show A.6(ii). As in example 4, we always have \( 0 < S_i < \bar{s} \) in this
case, so we can choose \( T(s_i) = \ln(s_i) \). We show that \( \ln[S_A(\cdot)] \) can be written as a concave
transformation of a concave function. Define:

\[
Z(y_A, y_B) = \ln(\bar{s}) - \ln \left[ 1 + \frac{\exp(y_B \gamma)}{\exp(y_A \gamma)} \right]
\]

Clearly:

\[
\ln[S_A(w_A, r_A, w_B, r_B)] = Z \{ \ln[G_A(w_A, r_A)], \ln[G_B(w_B, r_B)] \}
\]
Note that $G_A \neq 0$ and $G_B \neq 0$ since $(w_A, r_A)$ and $(w_B, r_B)$ must be nondegenerate in $A, B$ respectively, so these expressions are in fact well defined. The similarity in form between $\ln H$ above and $\ln G_A$ establishes that the latter is concave in $(w_A, r_A)$ on $(0, 1)^2$. The function $Z$ is concave in $y_A$ since:

$$\frac{\partial Z}{\partial y_A} = \frac{\gamma}{1 + \frac{\exp(y_A \gamma)}{\exp(y_B \gamma)}} > 0, \quad \frac{\partial^2 Z}{\partial y_A^2} = -\frac{\gamma^2 \exp(y_A \gamma)}{(1 + \frac{\exp(y_A \gamma)}{\exp(y_B \gamma)})^2} < 0$$

Finally, $F_A[w_A, r_A, \exp(\hat{s}_A)] = H_A(w_A, r_A)[\exp(\hat{s}_A)]^{\beta(\alpha+1)}$. Taking logs gives $\ln[H_A(w_A, r_A)] + \beta(\alpha+1)\hat{s}_A$, which is the sum of concave functions. Therefore $F_A[w_A, r_A, \exp(\hat{s}_A)]$ is quasi-concave in $(w_A, r_A, \hat{s}_A)$ on $(0, 1)^2 \times (-\infty, \ln(s))$.

**Proof of Proposition 2.** Fix any $x_{-i} \in D_{-i}$. Fix any $x_i$ and $x_i'$ in $X_i$ and any $\lambda \in (0, 1)$. Suppose first that $x_i\lambda x_i' \in X_i \setminus D_i$ (the notation was defined in the proof of Theorem 2). We have $\tilde{s}_i = \bar{s}$, so $S_i(x_i\lambda x_i', x_{-i}) = \bar{s}$, and we always have $S_i(x_i, x_{-i}) \leq \bar{s}$ and $S_i(x_i', x_{-i}) \leq \bar{s}$. Therefore $S_i(x_i\lambda x_i', x_{-i}) \geq \lambda S_i(x_i, x_{-i}) + (1 - \lambda)S_i(x_i', x_{-i})$. Suppose instead $x_i\lambda x_i' \in D_{im}$. Then $x_i \in D_{im}$ and $x_i' \in D_{im}$ since this is a face of $X_i$. Furthermore $S_i(x_i\lambda x_i', x_{-i}) = S_i(x_i, x_{-i}) = S_i(x_i', x_{-i})$, since $S_i$ is constant on the face. Again we have $S_i(x_i\lambda x_i', x_{-i}) \geq \lambda S_i(x_i, x_{-i}) + (1 - \lambda)S_i(x_i', x_{-i})$. 


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