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Essays on macroeconomic models with heterogeneous agents

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Abstract

This dissertation consists of three chapters dealing with the topic of heterogeneity in macroeconomics and macroeconomic models.

Chapter 1 contributes to the literature on computational approaches to solving DSGE models with heterogeneous agents. One possible approach, a hybrid method described in Reiter (2009) combines a nonlinear solution with respect to individual state variables and a linearized solution with respect to aggregate shocks. Since linearization has typically been used in representative agent models, a natural question is how well it works in a setting with heterogeneity and whether a higher order approximation is not needed. I compare solutions obtained with linearization and second order perturbation for a benchmark stochastic growth model with idiosyncratic labor income shocks. In terms of accuracy, I find that second order solution does not differ much when aggregate volatility is low (e.g. in case of a typical calibration for productivity shocks in developed economies), but becomes more precise when volatility is higher. Another potential issue is that linearization implies certainty equivalence, which makes it unsuitable for analyzing certain issues. I illustrate potential economic applications of the 2nd order solution by showing how it can be used to easily compute welfare costs of uncertainty conditional on an agent's individual state or to capture effects of time-varying volatility in aggregate shocks.

Chapter 2 studies risk premia in an incomplete-markets economy with households facing idiosyncratic consumption risk. If the dispersion of idiosyncratic risk varies over the business cycle and households have a preference for early resolution of uncertainty, asset prices will be affected not only by movements in current and expected future aggregate consumption (as in models with a representative agent), but also by news about current and future changes in cross-sectional distribution of individual consumption. I investigate whether this additional effect can help to explain high risk premia in a production economy, where the aggregate consumption process is endogenous and thus can potentially be affected by the presence of idiosyncratic risk. Analyzing a neoclassical growth model combined with Epstein-Zin preferences and a tractable form of household heterogeneity, I find that countercyclical idiosyncratic risk increases the risk premium, but also effectively lowers willingness of households for intertemporal substitution and thus changes the dynamics of aggregate consumption. Nevertheless, with the added flexibility of Epstein-Zin preferences, it is possible to both increase risk premia and to maintain the same dynamics of quantities if we allow for higher intertemporal elasticity of substitution at the individual level.

Chapter 3 investigates effects of heightened uncertainty on firms and their owners. An uncertainty shock that increases dispersion in firm-specific productivity will typically lead to a drop in economic activity as firms delay investments due to the higher value of waiting. Given that in the real world, firm ownership is far from perfectly diversified, it is also likely that larger volatility affects firm owners as well. Motivated by empirical evidence showing that more financially developed countries respond less strongly to uncertainty shocks, I use a dynamic model with heterogeneity across both firms and risk-averse firm owners to look at how a degree of diversification affects the response of the economy to such a shock. If a substantial part of an entrepreneur's income comes from a single firm which they control, an increase in uncertainty will cause a further drop in investment and consumption and a greater increase in savings due to entrepreneur's precautionary motive and risk aversion. As a result, the impact of an uncertainty shock is more amplified in economies with lower degrees of diversification.

Abstrakt

Dizertace obsahuje tři kapitoly zabývající se heterogeneitou v makroekonomii a makroekonomických modelech.

Kapitola 1 přispívá k literatuře o výpočetních přístupech k řešení DSGE modelů s heterogenními agenty. Jeden z možných přístupů, hybridní metoda popsaná v práci Reitera (2009), kombinuje nelineární řešení vzhledem k individuálním stavovým proměnným a linearizované řešení vzhledem k agregátnímu šoku. Jelikož linearizace se typicky používá v modelech s reprezentativním agentem, vyvstává otázka, jak dobře funguje v modelech s heterogenitou a jestli není potřeba použít aproximace vyššího řádu. V kapitole porovnávám linearizaci s perturbací druhého řádu pro základní stochastický model ekonomického růstu s idiosynkratickými šoky v příjmech z práce. Co se týče přesnosti, zjišťuji, že řešení druhého řádu se moc neodlišuje od lineárního, pokud je agregátní volatilita nízká (jako například v kalibraci typické pro rozvinuté země), ale umožňuje dosáhnout větší přesnosti při vyšších úrovních volatility. Dalším potenciálním problémem linearizace je vlastnost jistotní ekvivalence, kvůli které je linearizace nevhodná pro analyzování určitých otázek. Ilustruji potenciální ekonomické aplikace řešení druhého řádu ukázkou, jak se dá využít k jednoduchému výpočtu nákladů blahobytu z důvodů nejistoty v závislosti na individuálních stavových proměnných agenta, nebo k zachycení efektů v čase se měníci volatility agregátních šoků.

Kapitola 2 zkoumá rizikovou prémii v ekonomice s nekompletními trhy a domácnostmi čelícími idiosynkratickému riziku ve spotřebě. Pokud je rozptyl idiosynkratického rizika proměnlivý v průběhu hospodářského cyklu a domácnosti preferují dřívější rozřešení nejistoty, pak ceny finančních aktiv budou ovlivněny nejen zprávami o současné a očekávané budoucí spotřebě (jako je tomu v modelech s reprezentativní domácností), ale také zprávami o současných a budoucích změnách distribuce individuální spotřeby napříč domácnostmi. V článku zkoumám, jestli tento dodatečný efekt může pomoci vysvětlit vysokou rizikovou prémii v produkční ekonomice, ve které je proces pro agregátní spotřebu endogenní a potenciálně může být ovlivněn přítomností idiosynkratického rizika. Analýzou neoklasického růstového modelu kombinovaného s Epstein-Zin preferencemi a jednoduše řešitelnou formou heterogenity domácností jsem zjistil, že proticyklické idiosynkratické riziko zvyšuje rizikovou prémii, ale zároveň snižuje efektivní ochotu domácností k intertemporální substituci, čímž se změní dynamika agregátní spotřeby. Pokud umožníme vyšší elasticitu intertemporální substituce na individuální úrovni, pak je díky flexibilitě Epstein-Zin preferencí možné zvýšit rizikovou prémii beze změny dynamiky agregátních

veličin.

Kapitola 3 zkoumá efekt zvýšené nejistoty na podniky a jejich vlastníky. Šok zvyšující nejistotu prostřednictvím zvětšeného rozptylu produktivity mezi podniky vede typicky k poklesu ekonomické aktivity, kdy firmy odsouvají investice z důvodu vyšší hodnoty vyčkávání. Vzhledem k tomu že ve skutečnosti není vlastnictví podniků perfektně diverzifikováno, dá se očekat, že zvýšená volatilita bude mít přímý dopad taky na vlastníky firem. Motivován empirickými odhady, které ukazují že více finančně rozvinuté země reagují méně citlivě na šoky zvyšující nejistotu, za pomoci dynamického modelu s heterogenitou na úrovni firem a také rizikovo-averzních podnikatelů zkoumám do jaké míry ovlivňuje úroveň diverzifikace odezvu ekonomiky na zvýšenou nejistotu. Pokud značná část podnikatelova příjmu pochází z jeho vlastní firmy, větší nejistota způsobí dodatečný pokles investic a spotřeby a větší nárůst úspor kvůli averzi k riziku a preventivnímu spoření podnikatelů. Ve výsledku ekonomiky s menší diverzifikací reagují na šoky zvyšující nejistou intenzivněji.

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Introduction

Macroeconomic theory often proceeds by constructing and studying models, i.e. simplified artificial economies which ignore many real world features. One commonly used simplification is to assume that many different agents in the economy can be captured by a single "representative agent" standing in for the "average" household or firm. Of course, the main feature of any model is precisely that it abstracts away from details that bear little relevance to the question under consideration, and the representative agent approach has been fruitfully used to study many issues over the decades. Nevertheless, for many topics heterogeneity actually plays a key role. A model with a representative agent can hardly offer much insight into the determinants of inequality or about the distributional impacts of different policies, but even for some of the more traditional topics, a large degree of movement and uncertainty at the individual level can affect how the economy behaves in the aggregate. The obvious disadvantage of using models with heterogeneity is that they are, in general, more complicated and less tractable. Still, with advances in methodology, computing power and the availability of large microeconomic datasets, macroeconomists have become increasingly interested in using models that explicitly account for differences between individuals in the economy.

The first chapter in this dissertation deals with the hard methodological issue of how to solve dynamic stochastic equilibrium models that contain a population of agents facing uninsurable shocks and thus differing in their individual outcomes. Agents usually need to forecast future variables such as prices in order to behave optimally, but the future state of the economy will, in general, depend both on present decisions of agents and on their current cross-sectional distribution. For example, two economies with the same average level of capital may evolve differently depending on how the capital is distributed across households. As a result, the distribution becomes a state variable in the model, and thus needs to be approximated with some finite-dimensional representation in order to solve the model numerically. The chapter looks more closely at one particular approach previously proposed in the literature that allows for a relatively rich representation of the distribution (such as a histogram with hundreds of points) by using linearization around the steady state to deal with aggregate shocks. I investigate potential benefits of replacing linearization with second-order approximation, both in terms of solution accuracy and economic applications for which linearization would be too restrictive.

The remaining two chapters study effects of time-varying volatility faced by individual agents in two different contexts. The second chapter considers the risk premium on financial assets in a situation when households face cyclical dispersion in individual consumption due to uninsurable risks, and also have Epstein-Zin preferences for early resolution of uncertainty. According to standard theory, the risk premium of an asset depends on how it covaries with marginal utility, typically related to consumption growth. With preferences for early resolution of uncertainty, changes in expected future consumption further away also become another priced risk factor (so-called long run risk), and with cyclical dispersion in individual consumption, so does the current level of dispersion. When the two features are combined, an additional interaction term representing news about future levels of dispersion will become relevant as well, and can potentially help to explain high risk premia observed in real markets. I investigate this mechanism in a model where aggregate consumption is determined endogenously through production and capital accumulation, while heterogeneity in individual consumption is incorporated on top of it in a tractable manner. Accounting for production turns out to matter because the presence of cyclical individual risk will affect incentives for intertemporal substitution at the aggregate level, which in turn leads, all else being equal, to different dynamics of aggregate quantities.

The last chapter shifts attention to heterogeneity on the firm side of the economy. Empirically, we observe large differences in productivity between firms. When firms face idiosyncratic productivity shocks, they will typically accumulate capital in good times and disinvest in bad times. On the other hand, if they face an overall increase in the volatility of shocks and when investment is irreversible, firms across the board will respond by delaying investment due to the value of waiting becoming higher in more uncertain times. An uncertainty shock, in the sense described, will cause a afall in overall investment and can lead to a drop in economic activity and and, poitentially, recession. If the firms had many owners, each of whom holds a diversified portfolio of many firms, the story ends here. However, in the real world, firm owners are not perfectly diversified and a risk-averse entrepreneur who receives a substantial part of their income from a single firm will be directly affected by higher volatility of the firm's profits. Due to risk aversion and the precauionary saving motive, this in turn would represent an additional incentive to decrease both consumption and investment while increasing savings. Therefore, in this chapter I consider how a lack of diversification in firm ownership can make an economy's response to an uncertainty shock stronger and more persistent, both empirically and by using a dynamic model with heterogeneity that affects both firms and their owners.

Chapter 1

Solving a heterogeneous-agent DSGE model with 2nd-order perturbation

1.1 Introduction

Modern macroeconomic theory is to a large extent based on dynamic, stochastic, general equilibrium models with explicit microfoundations. This work has often relied on simplifying the model by working with a single representative agent, which makes the analysis tractable. In recent years, however, macroeconomists have increasingly focused their attention on models with heterogeneous agents, not only to check the robustness of previous results, but also to study questions which are simply unsuitable for a representative agent framework, such as the distribution of income. Heathcote, Storesletten, and Violante (2009) and Guvenen (2011) provide a recent review of this research, which shows that heterogeneity can affect both the level and the dynamics of aggregate variables, and is relevant for evaluating welfare effects that may differ across different agents. More generally, it allows us to study not only the determination and dynamics of aggregates, but also cross-sectional distributions of variables, which may have direct consequences for both economic theory and policy.

Incorporating heterogeneity naturally raises issues related to solving these types of models using numerical methods, especially when we want to include aggregate uncertainty in the model as well. While in a representative agent model we can, for example, work with a single variable representing aggregate capital, with heterogeneity the whole distribution of capital holdings across agents becomes a relevant state variable, which increases the dimension of the problem tremendously. Thus, research into efficient computational methods for heterogeneous-agent DSGE models has been the subject of a steady stream of attention by macroeconomists, starting with the seminal paper by Krusell and Smith (1998) and continuing today.

In this paper, I contribute to this line of research by extending one particular method, previously described by Reiter (2009a), and evaluate its performance. More specifically, Reiter solves the stochastic growth model enriched by uninsurable idiosyncratic shocks (a standard benchmark in this field) by combining two steps – first, solving for the steady state without an aggregate shock by the projection method; and second, deriving the dynamics of projection coefficients by linearization around this steady state with respect to the aggregate shock. I will extend the second step to obtain a second-order approximation and evaluate its accuracy. I find that accuracy gains in the benchmark model are not large, which suggests that with low volatility of aggregate shocks, a first order solution can work well (at least in models similar to the standard growth model). On the other hand, gains become noticeable in an alternative calibration with larger volatility, indicating that in some situations linearization may not be sufficient.

Moreover, there are additional reasons for going beyond linearization, which has the unfortunate side-effect of certainty equivalence. I illustrate two possible aplications. First, evaluating welfare losses from aggregate uncertainty requires that the solution linking the agent's value function to state variables depends, in some way, on the size of aggregate shocks, which is something that cannot be obtained by linearization. The second extension consists of incorporating time-varying uncertainty into the model. Effects of so-called "uncertainty shocks" on the business cycle have received some attention recently, but so far mostly in the context of models with a representative household, which leave a limited role e.g. for the precautionary savings motive in consumption. I will show how such shocks can be easily accommodated in the second-order approximation using the approach by Benigno, Benigno, and Nistico (2013), and that heterogeneity affects response to a volatility shock significantly more than it affects response to a level shock.

The following section contains a more thorough literature review about computational methods for heterogeneous-agent DSGE models. Section 1.3 briefly reviews the benchmark economic model, and section 1.4 describes the solution method and motivation for its use in the present setting. Section 1.5 discusses accuracy gains from the 2nd-order solution, and section 1.6 illustrates the two applications.

1.2 Literature

Models incorporating heterogeneity are usually based on a standard incomplete markets model (Bewley 1977; Huggett 1993; Aiyagari 1994), in which there is a continuum of households maximizing lifetime utility from consumption, while being subject to borrowing constraints and idiosyncratic income shocks. Agents can save only through a single asset, such as capital, so markets are incomplete. Because of this incompleteness and the borrowing constraint, individual shocks are not fully insurable, and thus different agents will have different amounts of savings depending on the realizations of their idiosyncratic shocks. In the basic version of the model without aggregate uncertainty, finding a solution (i.e. individual policy function and stationary wealth distribution) is relatively simple, since individual decisions depend on prices, which are constant over time.

On the other hand, models with both heterogeneity and aggregate uncertainty are challenging to solve numerically – the distribution of relevant variables across agents itself becomes a state variable. Krusell and Smith (1998) analyze a stochastic growth model with a continuum of agents who face uninsurable idiosyncratic labor endowment shocks (the model is described in more detail later). In a rational expectations equilibrium, agents need to forecast future prices, for which they need to forecast future aggregate capital, which will depend on the whole distribution of capital across agents today (since decision rules will differ across agents with different individual state). Theoretically, distribution over a continuum of agents is an infinite-dimensional object, which of course cannot be stored on a computer, and thus standard methods are not applicable.

Krusell and Smith solve this problem by assuming that agents keep track only of the mean of the distribution (which can be interpreted as bounded rationality on their part). Therefore, aggregate uncertainty will enter an individual's problem through a perceived law of motion for aggregate capital (of some parametric form). At the same time, an agent's resulting policy function will, when aggregated across all agents, imply an "objective" law of motion for aggregate capital that can be obtained, e.g., by simulation. Krusell and Smith then recompute the individual's problem with a new law of motion and iterate this process until both laws converge together.

Since aggregate capital today is not a sufficient statistic for calculating the distribution of aggregate capital tomorrow, strictly speaking this is not a true equilibrium, as the perceived law of motion is misspecified. However, he authors show that in practice this algorithm works very well (in that particular model) because of "approximate aggregation" – the policy functions of most agents are almost linear in their asset holdings except for very poor agents (who have little capital and thus do not influence aggregate dynamics), so the mean is, in itself, (approximately) sufficient for forecasting its future value and thus prices. Having agents take into account higher moments of the distribution and solving for their laws of motion in a similar way doesn't change the results (though, as Young (2005) reports, the mean by itself is not a sufficient statistic for calculating higher moments or other characteristics of the distribution).

Since Krusell and Smith, there has been further research on computational methods to solve these types of models; see the survey by Algan et al. (2014), as well as a recent comparison project in Journal of Economic Dynamics and Control (Den Haan 2010). Other approaches using moments of cross-sectional distribution include Den Haan (1997) and Algan, Allais, and Den Haan (2008), who parametrize the conditional expectation of an individual agent instead of the law of motion for capital. Den Haan and Rendahl (2010) avoid simulation altogether by explicitly integrating over individual decision rules ("explicit aggregation"). While previous algorithms could be considered "projection" methods (Judd 1998), as they are solving for coefficients of parametrized unknown functions that are "close" to the real solution in the global sense, Preston and Roca (2007) find the solution using second order perturbation around the steady state without either idiosyncratic or aggregate shocks. Generally speaking, the idea of using only a few moments or characteristics of the cross-sectional distribution is in fact shared by all methods described so far.

Reiter (2009a) proposes a method (described in more detail later) which combines projection and perturbation. First, in the projection step, policy functions and distributions are approximated by a finite (but typically large) number of coefficients, and we solve for the steady state without aggregate uncertainty (but with idiosyncratic shocks). Then, we make those coefficients themselves vary over time and solve for their dynamics (driven by aggregate shocks) by linearization around their values in the steady state, computed previously. Among the advantages of this method is the fact that it does not depend on approximate aggregation, and also that it captures the dynamics of the whole distribution, and thus of any statistic of interest as well. The solution is globally valid over different individual states in the steady state, although due to the nature of the perturbation algorithm, it is only locally valid for small fluctuations of aggregate shocks around their mean (however, this is standard practice even in simpler, representative agent models).

Reiter uses linearization (also known as first-order perturbation)[1]. There are, however, good reasons to be interested in higher-order approximations. First, they may lead to a more precise solution. Second, and more importantly, linearization implies certainty equivalence, i.e., matrices of the resulting linear state-space system do not depend on the variance of random shocks. This is problematic for capturing precautionary savings, welfare losses from uncertainty or risk premia, all of which should depend on variance of shocks. Linearization is also insufficient for analyzing optimal policy problems (Kim and Kim 2003). Therefore, to use a combination of projection and perturbation for these purposes, one has to go beyond linearization, which motivates the extension proposed in the current paper.

[1] In a later paper Reiter (2010), extends his method to a solution where policy function is quadratic in the aggregate state. However, his approach is somewhat different and more involved than the one considered here.

1.3 Model

The model is close to Krusell and Smith (1998), with a couple of differences: idiosyncratic shocks are iid over time and the aggregate shock is AR(1) instead of a two-state Markov chain. The following subsections summarize the model in more detail.

1.3.1 Households

There are many households indexed by i, which solve the following problem:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \mathbb{E}_t \left[\sum_{t=0}^{\infty} \beta^t U(c_t^i) \right]$$

subject to the borrowing constraint

$$0 \leq c_t^i \leq x_t^i.$$

and

$$x_{t+1}^i = (1 + r_{t+1})(x_t^i - c_t^i) + w_t l_t^i. \tag{1.1}$$

Here, c_t^i is consumption and x_t^i is "cash-on-hand", composed of value of capital k_t^i owned by the household, income from renting the capital and supplying an idiosyncratic labor endowment l_t^i:

$$x_t^i = (1 + r_t)k_t^i + w_t l_t^i,$$

Prices r_t, w_t are functions of aggregate state Θ_t (to be described later), which is taken as given by the household.

Labor endowment is random and has discrete distribution:

$$l_t^i = \mathrm{e}^{\eta_t^i}, \ \eta_t^i \in \{\eta_1, \dots, \eta_{N_\eta}\}, \ \mathrm{Prob}(\eta_t^i = \eta_r) = q_r,$$

and the shock is iid across time and across households. I use standard CRRA utility:

$$U(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}, \tag{1.2}$$

With iid shocks, the relevant state variables for household are x_t^i and Θ_t, which determines prices. The value function satisfies the Bellman equation

$$V(x,\Theta) = \max_{c\in[0,x]} \left(U(c) + \beta\mathbb{E}\left[V(x',\Theta')|x,\Theta\right]\right), \tag{1.3}$$

subject to evolution of x given by (1.1) and law-of-motion for aggregate state (to be described later).

Let the optimal policy be denoted $c(x,\Theta)$, which will typically be nondifferentiable at the point where the liquidity constraint starts to hold. For the purposes of a numerical solution, the maximization problem can alternatively be characterized in terms of the expectation function that satisfies:

$$\mathcal{E}(x_t,\Theta_t) = \beta\mathbb{E}_t\left[(1+r_{t+1})\cdot c_{t+1}^{-\gamma}\right]. \tag{1.4}$$

Given $\mathcal{E}$, the optimal consumption is given by

$$c(x,\Theta) = \min\left\{(\mathcal{E}(x,\Theta))^{-\frac{1}{\gamma}}, x\right\}, \tag{1.5}$$

which simply sets the consumption to the value implied by the Euler equation (given future expectation), or to the maximum possible value if the liquidity constraint binds, i.e. to x.

1.3.2 Firms

There is a representative firm producing output from an aggregate supply of capital and labor, subject to stochastic productivity:

$$Y_t = A\mathrm{e}^{z_t}K_t^{\alpha}L_t^{1-\alpha},$$

where the stochastic component of TFP follows an AR(1) process:

$$z_{t+1} = \rho z_t + \sigma_\epsilon\epsilon_{t+1},\ \epsilon_t \sim \mathcal{N}(0,1). \tag{1.6}$$

First-order conditions determine return on capital (net of depreciation) and wage:

$$\begin{aligned} r_t &= \alpha A\mathrm{e}^{z_t}K_t^{\alpha-1}L_t^{1-\alpha} - \delta \\ w_t &= (1-\alpha)A\mathrm{e}^{z_t}K_t^{\alpha}L_t^{-\alpha}. \end{aligned} \tag{1.7}$$

1.3.3 Market clearing

Aggregate labor is given simply by averaging over individual endowments, and is constant over time:

$$L_t = \bar{L} = \sum_{r=1}^{N_\eta} \exp(\eta_r) q_r. \tag{1.8}$$

Aggregate capital is given by averaging over cross-sectional distribution of individual capital holdings (which is non-degenerate, due to idiosyncratic shocks). More precisely, let $k_t^i \in \mathbb{R}_+$ describe assets held by the household at the beginning of period t before receiving capital and labor income, which is equal to what they have left of their wealth after consuming in the last period:

$$k_t^i = x_{t-1}^i - c_{t-1}^i.$$

Let $\mathcal{B}_+$ denote Borel sets over $\mathbb{R}_+$. Then define $\lambda_t : \mathcal{B}_+ \to [0, 1]$ a measure which describes the distribution of assets, i.e. for $B \in \mathcal{B}_+$ we have $\lambda_t(B)$ a proportion of households for which $k_t^i \in B$. Aggregate capital available for production at period t is then simply the first moment of the distribution:

$$K_t = \int k \, \mathrm{d}\lambda_t(k). \tag{1.9}$$

1.3.4 Equilibrium

The recursive competitive equilibrium can be defined by state variables consisting of the productivity and cross-sectional distribution of capital (because the whole distribution is relevant for forecasting next-period aggregate capital and prices), so $\Theta = (z, \lambda)$. Then, informally, the equilibrium includes:

- expectation function $\mathcal{E}(x, \Theta)$, consumption function $c(x, \Theta)$ and value function $V(x, \Theta)$,
- pricing functions $r(\Theta)$ and $w(\Theta)$,
- and law of motion for the distribution $\lambda' = \Gamma(\Theta)$,[2]

such that the following holds:

[2] We assume that a suitable law of large numbers holds, so that the cross-sectional distribution evolves "deterministically".

- for any $x \geq 0$, the expectation function gives the actual conditional expectation in (1.4) with equlibrium consumption and prices, the consumption function is consistent with (1.5) and the value function satisfies the Bellman equation (1.3),

- pricing functions satisfy firm FOCs (1.7), where aggregate capital and labor are functions of Θ as defined in (1.8) and (1.9),

- and the law of motion Γ is consistent with the process for productivity (1.6) and evolution of the cross-sectional distribution implied by individual policy function, law-of-motion for x in (1.1) and distribution of the labor endowment shock in (1.2).

1.4 Solution

This section describes the hybrid solution method based on Reiter (2009a) and its second-order solution.

1.4.1 Approximate model

In order to solve the model numerically, we must replace functions and cross-sectional distribution with finite-dimensional approximations:

- The cross-sectional distribution at time t is approximated by a vector $\mathbf{p}_t \in \mathbb{R}^M$, which describes a distribution over a discrete grid of capital levels $\{\kappa_1, \ldots, \kappa_M\}$, so that $p_{i,t}$ is the proportion of agents with their beginning-of-period capital at t equal to κ_i. Essentially, we are approximating a continuous distribution using a histogram. Thus, the agregate state in the approximate model will consist of $\mathbf{p}_t$ and z_t.

- Within a single period, I approximate expectation and value functions, conditional on current aggregate state, by a linear combination of univariate basis functions (such as polynomials or splines) $\{\varphi_i^c(x)\}_{i=1}^{N_c}$ and $\{\varphi_i^v(x)\}_{i=1}^{N_v}$ defined over an individual state variable (on some interval $[\underline{x}, \overline{x}]$)[3]. The combination will be parametrized by a

[3] I use c as superscript for approximation of the expectation fuction, as it ultimately determines consumption. The main reason to approximate the expectation instead of the consumption function directly is that the former should be smoother.

finite vectors of coefficients $\mathbf{a}^c \in \mathbb{R}^{N_c}, \mathbf{a}^v \in \mathbb{R}^{N_v}$:

$$\begin{aligned}
\log\left(\mathcal{E}(x;\mathbf{a}^c)\right) &= \sum_{i=1}^{N_c} a_i^c \varphi_i^c(x), \\
\tilde{V}(x;\mathbf{a}^v) &= \sum_{i=1}^{N_V} a_i^v \varphi_i^v(x).
\end{aligned} \tag{1.10}$$

Given the parameterized expectation function (approximated in log, so that it remains positive), consumption $\tilde{c}(x;\mathbf{a_c})$ can be computed directly from (1.5).

Of course, the true functions depend on both individual and aggregate state, which changes over time, so we will capture the latter dependence by making approximation coefficients themselves functions of the aggregate state:

$$\begin{aligned}
\mathbf{a}^c &= g^c(\mathbf{p}, z), \\
\mathbf{a}^v &= g^v(\mathbf{p}, z).
\end{aligned}$$

Thus, the relevant variables in the approximate model are: $\mathbf{a}_t^c = g^c(\mathbf{p}_t, z_t)$ and $\mathbf{a}_t^v = g^v(\mathbf{p}_t, z_t)$, which describe the shape of individual policy and value functions, given the aggregate state at time t (and thus can be understood as "control" variables); $\mathbf{p}_t$, which captures the cross-sectional distribution of wealth; and exogenous productivity z_t. Our goal is to solve for functions g^c, g^v and the law of motion for the distribution h^p, so that a recursive equilibrium of the approximate model can be summarized as:

$$\begin{aligned}
\mathbf{a}_t^c &= g^c(\mathbf{p}_t, z_t), \\
\mathbf{a}_t^v &= g^v(\mathbf{p}_t, z_t), \\
\mathbf{p}_{t+1} &= h^p(\mathbf{p}_t, z_t), \\
z_{t+1} &= \rho z_t + \sigma_\epsilon \epsilon_{t+1}.
\end{aligned} \tag{1.11}$$

1.4.2 Model equations

Our ultimate goal is to solve for functions g^c, g^v, h^p using perturbation methods, which deliver Taylor approximations to these functions around the steady state without aggregate uncertainty. For that, we need to describe equations that characterize the equilibrium – we have $N_c + N_V + M + 1$ variables, so we need as many equations.

Auxiliary functions: Aggregate capital, rental rate and wage rate can be expressed as functions of the aggregate state:

$$K(\mathbf{p}, z) = \sum_{i=1}^{M} p_i \kappa_i,$$
$$r(\mathbf{p}, z) = \alpha A \mathrm{e}^z K(\mathbf{p}, z)^{\alpha-1} \bar{L}^{1-\alpha} - \delta,$$
$$w(\mathbf{p}, z) = (1-\alpha) A \mathrm{e}^z K(\mathbf{p}, z)^{\alpha} \bar{L}^{-\alpha}.$$

Euler equation block: Going back to the defintion of the expectation function (1.4), replace it with its approximated version and c_{t+1} with $\tilde{c}(x_{t+1}, \mathbf{a}^c_{t+1})$, with $\tilde{c}$ defined as above. Expanding the expectation with respect to the idiosyncratic shock into a sum, substituting for the law of motion of x and requiring the resulting equation to hold exactly at a set of collocation points $x_t \in \{x_1, \dots, x_{N_c}\}$, we obtain a system of N_c equations linking the shape of the consumption function at time t and $t+1$:

$$\forall x \in \{x_1, \dots, x_{N_c}\}: \ \mathcal{E}(x, \mathbf{a}^c_t) = \beta \cdot \mathbb{E}_t \Bigg[\Big\{ 1 + r(\mathbf{p}_{t+1}, z_{t+1}) \Big\} \cdot$$
$$\cdot \sum_{r=1}^{N_\eta} \left\{ q_i \cdot \tilde{c} \left(\Big[1 + r(\mathbf{p}_{t+1}, z_{t+1}) \Big] \cdot \Big[x - \tilde{c}(x, \mathbf{a}^c_t) \Big] + w(\mathbf{p}_{t+1}, z_{t+1}) \cdot \mathrm{e}^{\eta_r}, \mathbf{a}^c_{t+1} \right)^{-\gamma} \right\} \Bigg] \quad (1.12)$$

Bellman equation block: Proceeding in the same way as with the Euler equation, we obtain a system of N_v equations linking the shape of the value function at the two time periods:

$$\forall x \in \{x_1, \dots, x_{N_v}\}: \ \tilde{V}(x, \mathbf{a}^v_t) = \frac{\tilde{c}(x, \mathbf{a}^c_t)^{1-\gamma} - 1}{1-\gamma} +$$
$$+ \beta \cdot \mathbb{E}_t \left[\sum_{r=1}^{N_\eta} q_i \cdot \tilde{V} \left(\Big[1 + r(\mathbf{p}_{t+1}, z_{t+1}) \Big] \cdot \Big[x - \tilde{c}(x, \mathbf{a}^c_t) \Big] + w(\mathbf{p}_{t+1}, z_{t+1}) \cdot \mathrm{e}^{\eta_r}, \mathbf{a}^v_{t+1} \right) \right] \quad (1.13)$$

Distribution updating block: Given the current state and consumption function, the distribution of capital at the beginning of the next period is non-stochastic and can be expressed as

$$\mathbf{p_{t+1}} = T\Big(\mathbf{p_t}, z_t, \mathbf{a}^c_t\Big) \mathbf{p}_t, \quad (1.14)$$

where T is $M \times M$ matrix (depending on state and policy in period t) with elements T_{ij} being a conditional probability that a household with capital κ_j ends up with capital κ_i in the next period. Elements of T can be computed by the following algorithm (Young 2010): of all households with some capital level κ_j, those with the same realization of individual shock (say, η_r, for some r) will have the same level of wealth, choose the same consumption and end up with the same level of capital in the next period, e.g. κ_i for some i. Since their share (of those with the same initial capital) will be q_r, we increment T_{ij} by q_r. If, as will be the typical case, next-period capital doesn't fall exactly on the grid, we divide the probability mass proportionally between the closest nodes. Then we simply repeat the process for other realizations of individual shock and other initial levels of capital – see algorithm 1 for a more precise description.

Algorithm 1 Computing the transition matrix

```
initialize T_ij = 0, ∀i, ∀j
for j = 1 to M do {loop over initial levels of capital}
  for r = 1 to N_η do {loop over idiosyncratic shock realizations}
    compute wealth x = (1 + r(p_t, z_t)) · κ_j + w(p_t, z_t) · exp(η_r)
    compute consumption c = c̃(x, a_t^c) and next-period capital k' = x − c
    if k' < κ_1 then
      set T_1j = T_1j + q_r
    if k' ≥ κ_M then
      set T_Mj = T_Mj + q_r
    if κ_1 ≤ k' < κ_M then
      find index i such that κ_i ≤ k' < κ_{i+1}
      set T_ij = T_ij + (κ_{i+1} − k')/(κ_{i+1} − κ_i) q_r, T_{i+1,j} = T_{i+1,j} + (k' − κ_i)/(κ_{i+1} − κ_i) q_r
```

Exogenous shock block: The final equation simply expresses the law of motion for the exogenous productivity process:

$$\mathbb{E}_t\Big[z_{t+1}\Big] = \rho z_t. \tag{1.15}$$

1.4.3 Solution

Now, denote $y_t = \begin{bmatrix} a_t^c \\ a_t^v \end{bmatrix}$, $\chi_t = \begin{bmatrix} p_t \\ z_t \end{bmatrix}$ and collect equations (1.12) - (1.15) into a single system that can be written as[4]:

$$\mathbb{E}_t\left[F(y_{t+1}, \chi_{t+1}, y_t, \chi_t)\right] = 0. \tag{1.16}$$

Then, the solution method proceeds in two steps:

- First, solve for a steady state without aggregate uncertainty, i.e. solve for $\bar{y}$, $\bar{\chi}$ such that

$$F(\bar{y}, \bar{\chi}, \bar{y}, \bar{\chi}) = 0.$$

[4]In practice (and to avoid invertibility problems), it is enough to keep track of first $M-1$ elements of p, since probabilities must sum to one.

In this particular case, the steady state corresponds to the well-known Aiyagari model, and its solution can thus be reduced to a univariate problem of finding a root to excess demand for capital as a function of the interest rate. To evaluate excess demand for given $r \in (-\delta, \frac{1}{\beta} - 1)$, demand for capital (and for the implied wage rate) is given by firm FOC, while the supply of capital is determined by ergodic distribution of household asset holdings. The latter can be found by solving for the household policy function (which is, with constant prices, relatively straightforward, e.g. through time iteration on Euler equation), computing a discretized transition matrix (see previous subsection) and solving for its eigenvector corresponding to the unit eigenvalue.

- Next, we use pertubation methods to solve for the dynamics of the approximate model in the following form:

$$y_t = g(\chi_t, \mu)$$
$$\chi_{t+1} = h(\chi_t, \mu) + \mu \begin{bmatrix} 0 \\ \sigma_\epsilon \end{bmatrix} \epsilon_{t+1},$$

where we let the unknown function formally depend on μ, the perturbation parameter that scales uncertainty (where $\mu = 0$ corresponds to no aggregate uncertainty, $\mu = 1$ corresponds to the original problem as written in (1.11)). Perturbation finds Taylor approximations to g, h around the deterministic steady state with $\mu = 0$ by applying the implicit function theorem to the system (1.16) (Jin and Judd 2002; Schmitt-Grohé and Uribe 2004; Gomme and Klein 2011).

1.4.4 Linearization vs. second order approximation

In the notation of Gomme and Klein (2011), second order approximation to g, h can be written as:

$$g(\chi.\mu) \approx \bar{y} + F\tilde{\chi} + \frac{1}{2}(I_{n_y} \otimes \tilde{\chi}')E\tilde{\chi} + \mu^2 k_y$$
$$h(\chi, \mu) \approx \bar{\chi} + P\tilde{\chi} + \frac{1}{2}(I_{n_w} \otimes \tilde{\chi}')G\tilde{\chi} + \mu^2 k_\chi,$$

where $\tilde{\chi} = \chi - \bar{\chi}$ is the deviation of state from its steady state, n_χ, n_y are dimensions of χ, y, F is $n_y \times n_\chi$, P is $n_\chi \times n_\chi$, E is $n_y n_\chi \times n_\chi$, G is $n_\chi^2 \times n_\chi$, k_y is $n_y \times 1$ and k_χ is $n_\chi \times 1$. In linear approximation (where E, G, k_y, k_χ are zero), the solution does not depend on volatility of exogenous shocks due to certainty equivalence. In second-order approximation, cross-terms between μ and χ are zero (Schmitt-Grohé and Uribe 2004), however, the level of uncertainty enters through constants k_y, k_χ. These constants represent shifts in policy rules and laws-of-motion that move the distribution of variables away from the steady state and can capture economically relevant effects of uncertainty, such as precautionary savings or welfare effects.

In practice, to obtain a second-order approximation, the perturbation method requires the first and second derivatives of model equations (1.16) evaluated at the steady state, and specifying second moments of exogenous shocks (normalized to unity here) as inputs. I obtain derivatives through numerical differentiation. The solution is then obtained by using `solab` and `solab2` routines by Gomme & Klein.

1.5 Accuracy

This section presents some results from the solution to the model, as well as accuracy checks that allow comparison of first and second order solutions.

1.5.1 Calibration

Economic parameters: I use the same calibration as in Reiter (2009a): $\beta = 0.95$, $\gamma = 1$ (log utility), $\alpha = 0.33$, $\delta = 0.1$, $\rho_z = 0.8$, $\sigma_z = 0.014$ and A is set so that steady-state capital in the corresponding representative-agent model is 1. Idiosyncratic labor endowment shock is obtained by discretizing lognormal distribution, with log-labor endowment $\sim \mathcal{N}(\mu_\eta, \sigma_\eta^2)$, with $\sigma_\eta = 0.2$ and $\mu_\eta = -\frac{1}{2}\sigma_\eta^2$. I use 10-point Gauss-Hermite quadrature nodes and weights obtained from `qnwnorm` function in CompEcon toolbox (Miranda and Fackler 2004). To evaluate the performance in a setting where nonlinearities may be more important, I also consider a case with large aggregate uncertainty $\sigma_z = 0.1$.

Numerical parameters: The distribution is approximated by an equidistant grid with 30 points between 0 and 5 (steady state aggregate capital is about 1.13). Expectation and value functions are approximated by Chebyshev polynomials up to order 9, both over interval $[0.1, 7]$.

Computational considerations: All computations were done in MATLAB. Evaluating the hessian of model equations through numerical derivatives is the most time-consuming step. Total computation time for the examples presented here is on the order

	K	C	Y	I	r	w
steady state	1.1097	0.3940	0.5050	0.1110	0.0502	0.3124
$\sigma_\epsilon = 0.014$						
order 1	1.1116	0.3946	0.5059	0.1112	0.0502	0.3129
order 2	1.1127	0.3948	0.5060	0.1113	0.0501	0.3130
$\sigma_\epsilon = 0.1$						
order 1	1.1229	0.4039	0.5193	0.1154	0.0559	0.3212
order 2	1.1789	0.4093	0.5281	0.1188	0.0496	0.3266

Table 1.1: Simulated data - first moments.

of dozens of minutes, most of which are spent on obtaining the hessian (though this part could be presumably sped up by using lower-level language or parallelization).

1.5.2 Results

Steady state: Figure 1.1 plots the steady-state consumption function and capital distribution. We can see that the borrowing constraint binds for small values of wealth and for approximately 6% of households. The first row of table 1.1 contains the values of aggregate macroeconomic variables. Aggregate capital stock is about 10% higher than in the corresponding representative agent model.

First moments: Further rows of table 1.1 give the averages of macroecnomic aggregates from the simulations with aggregate uncertainty. We can see that the means for calibration with low volatility are quite close to the steady state, in both first and second order solutions. However, with larger volatility we can see a difference – the second order solution clearly displays higher capital stock and lower return on capital than in the steady state.

1.5.3 Accuracy

Law of motion for capital: A common, though not necessarily sufficient way (Den Haan 2010) to evaluate accuracy in models with heterogeneous agents is to look at the fit of the approximate law of motion for aggregate capital. Since the method described here does not rely on such an approximate law of motion and instead solves for the dynamics of the whole distribution (albeit approximated by a histogram), such a test is less relevant. We can however compare simulated paths for aggregate capital obtained from the model solution and from simulating a panel of individual agents within the model. With 1000

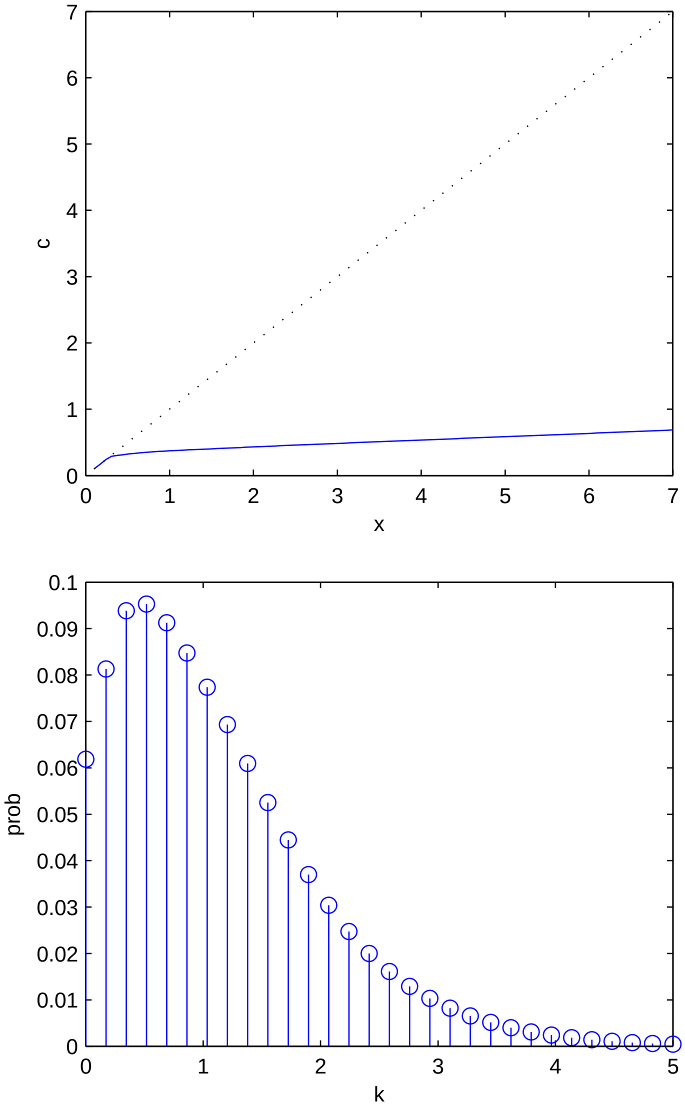

Figure 1.1: Steady state: consumption function (upper panel) and cross-sectional distribution of capital (lower panel).

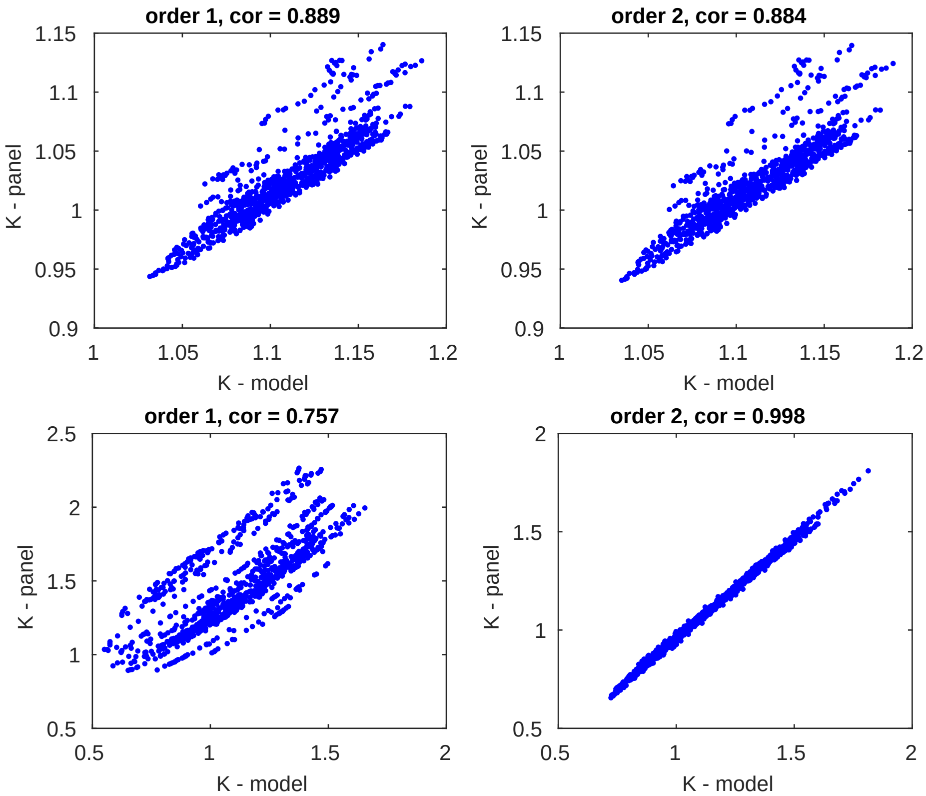

Figure 1.2: Aggregate capital realizations: model-implied vs. panel simulation from 1st (left) and 2nd order solution, in case of low(up) or high (down) volatility. Conditional on volatility, a more precise solution should yield values closer to the 45-degree line and higher correlation.

periods and 10000 individuals, resulting scatter-plots are shown in figure 1.2. With small volatility, solutions from the first and second orders yield very similar results; with larger volatility, first order solution appears to be surprisingly imprecise[5].

Euler equation errors: Next, we may be interested in the accuracy of the individual policy function. A common metric is to evaluate Euler equation errors (usually expressed in consumption units). At any point in state space, we can compute (unconstrained) consumption c^{appr} implied by the approximated expectation function, as well as consumption c^{true} implied by "true" expectation evaluated from the model solution (using quadrature

[5] This is somewhat puzzling, since the difference between the paths of aggregate capital and prices implied by 1st and 2nd order perturbation solutions is relatively small, but aggregate capital and average capital from a panel simulation diverge in the linear model. Upon closer inspection, the linearized model implies higher return to capital in a couple of periods (compared to the 2nd order model), leading to a relatively large increase in wealth for the richest households, which then persists over the rest of the simulation. This development is however not captured by the linearized dynamics of aggregate capital from the model solution. One possible explanation is that the two inaccuracies in the linear model (one leading to imprecise prices, the other to imprecise tracking of aggregate capital) effectively cancel each other out, but it might be useful to take closer look at this issue in the future.

	$\sigma_\epsilon = 0.014$		$\sigma_\epsilon = 0.1$	
	order 1	order 2	order 1	order 2
Euler errors				
mean (log10)	-2.47	-2.48	-2.19	-2.49
DHM test (inst.: constant)				
single run, χ^2_1 stat.	4.161	4.099	7.366	2.148
mult. runs, KS stat.	0.460	0.453	0.744	0.249
DHM test (inst.: constant, x, K, z)				
single run, χ^2_4 stat.	7.168	7.309	17.89	5.491
mult. runs, KS stat.	0.184	0.181	0.776	0.149

Table 1.2: Accuracy checks (smaller values indicate more accurate solution).

for next-period aggregate shock). The error is then defined as

$$\frac{\left|c^{\text{appr}} - c^{\text{true}}\right|}{c^{\text{true}}}$$

The corresponding row of 1.2 contains average error from a simulation with 1000 periods and one individual (i.e. error is evaluated at points from the simulated trajectory). For calibration with small volatility, there is almost no difference, while for larger volatility second-order solution leads to slightly lower errors.

Den Haan & Marcet test: Another way to check accuracy of the individual policy function is to formally test whether restrictions implied by the model hold in simulated data (Den Haan and Marcet 1994). If we simulate a time series for aggregate return on capital r_t, together with individual wealth x_t, consumption c_t and expectation $\mathcal{E}_t$ (obtained from approximation to the individual expectation function, see (1.4)), it should be the case that

$$\mathbb{E}_t \left[\mathcal{E}_t - \beta(1 + r_{t+1})c_{t+1}^{-\gamma}\right] = 0,$$

which for any vector of instruments I_t observable in time t implies moment conditions

$$\mathbb{E}\left[\left(\mathcal{E}_t - \beta(1 + r_{t+1})c_{t+1}^{-\gamma}\right) I_t\right] = 0.$$

A formal test, similar to the OIR test in GMM estimation, constructs a test statistic with χ^2_q distribution, where q is the number of instruments. Table 1.2 presents results for two cases: first, I_t includes only a constant (thus essentially testing whether expectation errors are zero on average); second, I_t includes constant, individual wealth x_t, aggregate

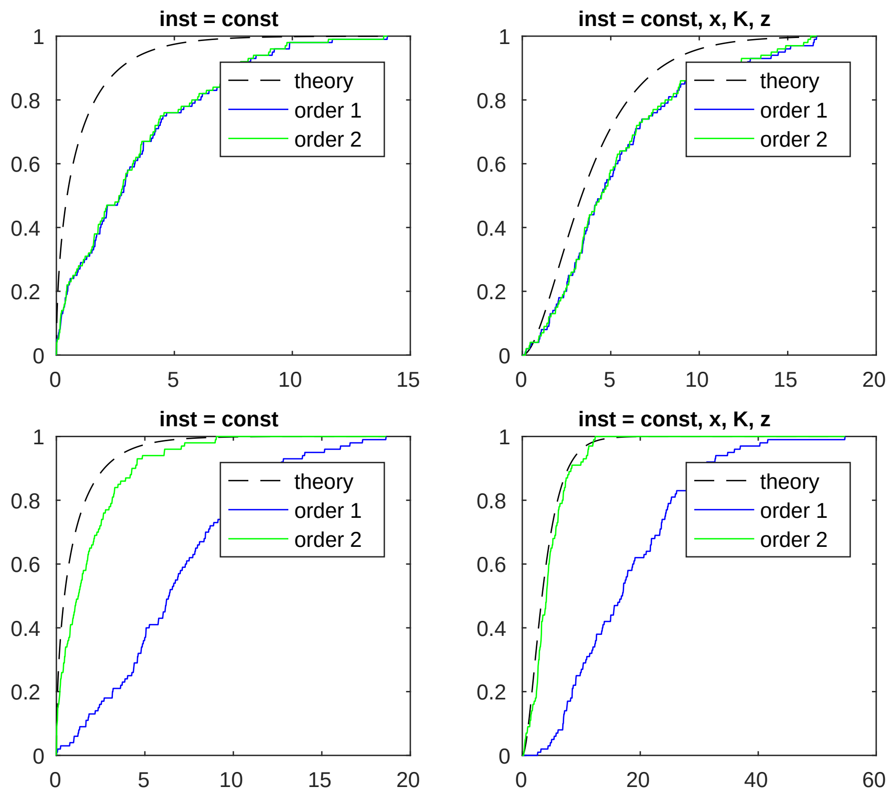

Figure 1.3: Cumulative distribution function of DHM statistics in case of small (upper panels) or large (lower panels) volatility. A more precise solution should yield distribution closer to the theoretical benchmark (black dashed line). "Instruments" for evaluating orthogonality of forecast errors include the constant term on the left and a constant with individual and aggregate states on the right.

capital K_t and productivity z_t (thus also jointly testing whether errors are orthogonal to information available at time t). Alternatively, we can simulate the model many times, construct a distribution of observed statistics and compare it with the theoretical benchmark, graphically (shown in figure 1.3, 100 simulations) or by a Kolmogorov-Smirnov test statistic (table 1.2).

The results indicate that when aggregate volatility is low, there is not much difference between first and second order solutions. On the other hand, the second order solution is clearly more accurate when volatility is larger.

1.6 Applications

1.6.1 Welfare cost of fluctuations

One of advantages of models with heterogeneity is the possibility to make welfare comparisons conditional on an agent's individual state. A well known economic question involving welfare comparisons considers the cost of business cycle fluctuations. In his influential contribution, Lucas (1987) tried to estimate costs of business cycles using postwar US data and concluded that they are very small, on the order of no more than one tenth of one percent of annual consumption.Naturally, this result turned out to be somewhat controversial and motivated further research into relaxing the assumptions made above, see, e.g., reviews by Barlevy (2004) and Lucas (2003). One strand of this literature uses models with heterogeneous agents who face uninsurable idiosyncratic income or employment shocks. This allows us to evaluate welfare impacts for different agents, as those can differ from impacts for a representative agent, e.g., when the consumption of individual agents is more volatile than aggregate consumption. Eliminating aggregate uncertainty can change the prices that agents face, as well as their individual income process.

One of the first contributions in this direction was by Imrohoroglu (1989), who finds that in a model with idiosyncratic risk, welfare gains from stabilization can be quite high (more than 1%). Atkeson and Phelan (1994), on the other hand, present an example in which stabilization leads to very small gains. Krusell and Smith (1999) and Krusell et al. (2009) compute welfare costs in a version of their model discussed previously, and find that although the effects can be small on average, they differ across agents depending on their individual state. Other authors have found larger impacts: Storesletten (2001) argues that idiosyncratic risk is countercyclical, so recessions are accompanied by greater volatility of individual income; Krebs (2003) adds a permanent component to idiosyncratic shocks, which are hard to self-insure against through savings; Beaudry and Pages (2001) investigate persistance in wages, when workers laid off during recessions will reenter employment with lower wages; Chatterjee and Corbae (2007) allow for the possibility of a particularly bad aggregate state ("depression").

From a methodological point of view, the papers cited above usually either assume fixed prices, in which case it is enough to solve an individual agent's problem, make special assumptions to obtain closed-form solutions, or to use the standard Krusell

& Smith algorithm and work with welfare computed numerically. The last case is of particular interest from a computational point of view since aggregate uncertainty enters the individual's problem (and thus his utility) essentially only through the perceived law of motion for aggregate capital. However, although this is an approximation which may work well for describing the dynamics of aggregate capital, it is not obvious whether it also leads to precise computation of welfare.

There are some previous results which indicate that this may be problematic. Preston and Roca (2007) use a different perturbation-based solution method, and find that the law of motion for aggregate (mean) capital also depends slightly on second moments of capital distribution, and more importantly, those second moments may also influence welfare (however, their algorithm provides an approximation only around the deterministic representative agent steady state). Another possible issue is that the Krusell & Smith algorithm typically requires one to solve an individual's dynamic programming problem over grids for both individual and aggregate capital, where the second grid is usually sparse to conserve on computational resources. Horvath (2012) reports that results may not be robust with respect to the choice of aggregate capital grid, as different choices result in significantly different properties of cross-sectional distribution.

Therefore it might be useful to revisit these calculations using a computational method which incorporates time-varying prices, allows us to derive a measure of welfare conditional on the individual state (so we can distinguish impacts on different agents), does not rely on the approximate aggregation property and avoids problems associated with grid choice for aggregate capital. Reiter's method combined with second-order perturbation is likely to satisfy these conditions. In the following, I describe how such costs can be computed in the simple benchmark model studied here.[6]

From the model solution, we obtain an approximation to value function that depends on individual wealth x, aggregate state χ and perturbation parameter μ:

$$V^{\text{unc}}(x,\chi,\mu) = \sum_{i=1}^{N_V} a_i^v(\chi,\mu)\varphi_i^v(x),$$
$$a_i^v(\chi,\mu) = g_i^v(\chi,\mu) \approx \bar{a}_i^v + \mathrm{D}_\chi g_i^v \tilde{\chi} + \frac{1}{2}\tilde{\chi}'\mathrm{H}_{\chi\chi} g_i^v \tilde{\chi} + k_{a_i^v}\mu^2,\ i = 1,\dots,N_v$$

where $\mathrm{D}_\chi g_i^v$, $\mathrm{H}_{\chi\chi} g_i^v$ are jacobians and hessians of the function g_i^v at $\chi = \bar{\chi}, \mu = 0$ and

[6] The results are thus to be understood more as illustrations of the methodological approach, since serious quantitative analysis would require enrichment of the model by at least by some elements referred to above.

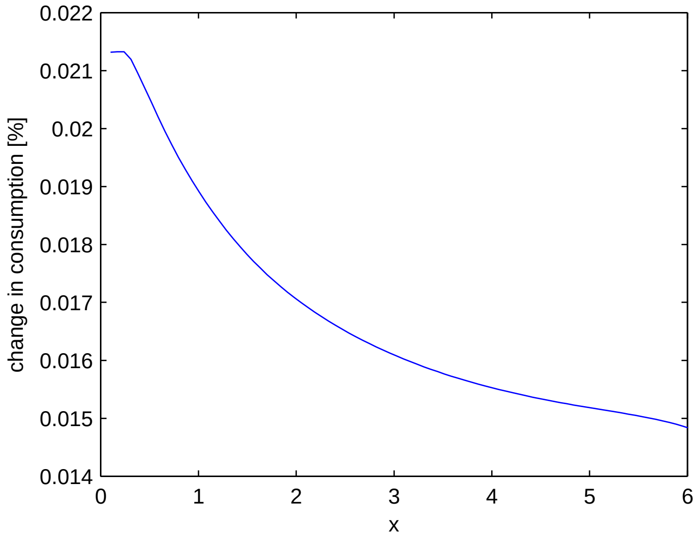

Figure 1.4: Welfare gain from eliminating TFP fluctuations vs. individual wealth, measured in terms of compensating relative increase in consumption a household would require to bear aggregate shocks.

$\tilde{\chi}$ is deviations from the steady state $\chi - \bar{\chi}$. These value fuctions can be then used to compare welfare between situations with and without aggregate fluctuations simply by setting $\mu = 1$ or $\mu = 0$.

Figure 1.4 shows the welfare comparison, in terms of an equivalent permanent relative change in consumption[7], for calibration with small volatility and when the aggregate state corresponds to the steady state. To avoid conflating the results with trivial effects of Jensen inequality, I modify equation (1.6) so that the shock enters level, not log, of productivity.

The results indicate that gains from stabilization are positive, but quantitatively small (cross-sectional average gain is 0.018% in consumption units). The reason is that in a given calibration and with iid idiosyncratic shocks, households can self-insure relatively easily through savings (the same calculation for the representative-agent version of the model yields a gain of 0.017%). On the other hand, we can see that the gain varies across households and decreases with individual wealth, with the poorest agents experiencing the largest costs due to the difficulty of smoothing shocks when they are near the borrowing constraint.

[7]A value of, e.g., 0.1% would mean that a household facing aggregate fluctuations would require permanent increase in consumption by one tenth of a percentage point in order to attain the same utility as the same household in the steady state. Positive numbers thus indicate gains from stabilization, negative numbers, losses.

1.6.2 Time-varying volatility

Shocks to aggregate productivity in the model presented above are homoscedastic. However, there is some evidence to suggest that volatility of macroeconomic shocks varies over time (Sims and Zha 2006; Justiniano and Primiceri 2008). Moreover, these changes in volatility have also received attention as a distinct source of economic fluctuations, or so-called "uncertainty shocks" (Bloom 2009a). Existing literature has proposed several channels, through which an increase in uncertainty can influence economic outcomes, see, e.g., survey by Fernandez-Villaverde and Rubio-Ramirez (2010). For example, if there are nonconvex adjustment costs or irreversibilities in investment, an increase in uncertainty may depress investment by inducing firms to postpone decisions (Bernanke 1983; Bloom 2009a). Other papers have studied effects of time-varying volatility in an open economy setting (Benigno, Benigno, and Nistico 2011; Fernandez-Villaverde et al. 2011), in models with financial frictions (Gilchrist, Sim, and Zakrajsek 2010; Christiano, Motto, and Rostagno 2013) or preferences for robustness (Bidder and Smith 2012). However, existing work usually relies on the representative agent assumption, so any idiosyncratic risks are implicitly assumed to be perfectly shared among households.

Once we allow for incomplete markets, it is likely that changes in aggregate uncertainty will also have impact on idiosyncratic risks facing the household and thus affect its consumption decisions through the precautionary savings motive. From an empirical point of view, Parker and Preston (2005) provide evidence that changes in precautionary saving explain a nontrivial part of variation in average consumption growth. In a general equilibrum setting, precautionary saving may interact with other channels of uncertainty, or even counteract them (e.g., while firms may prefer to delay investment, households want to increase their savings, so the overall effect of an uncertainty shock may be ambiguous).

Of course, general equilibrum models wth heterogeneity and time-varying uncertainty are challenging to solve and estimate. As a step in this direction, we can incorporate stochastic volatiity in the model discussed above. The presence of idiosyncratic risk will likely increase sensitivity of aggregate consumption to uncertainty shock and thus generate a distinct role for such shocks in explaining the business cycle.

Benigno, Benigno, and Nistico (2013) show how to incorporate distinct effects of time-varying volatility in a second-order perturbation solution, and their extension is easily applicable in this setting.

The model used is the same as described in previous chapter. The deviation consists of introducing stochastic volatility into the TFP process:

$$z_{t+1} = \rho_z z_t + \sigma_z \sqrt{1+v_t}\epsilon_{z,t+1}$$
$$v_{t+1} = \rho_v v_t + \sigma_v \epsilon_{v,t+1},$$

where $\epsilon_{z,t+1}$ and $\epsilon_{v,t+1}$ are iid uncorrelated shocks with zero mean and unit variance. Thus the conditional variance $\text{Var}_t[z_{t+1}] = (1+v_t)\sigma_z^2$ changes over time, and v_t follows an AR(1) process.

This particular way of modeling stochastic volatility is motivated by Benigno, Benigno, and Nistico (2013), who describe how such a formulation can be acccommodated wthin a second-order perturbation approximation[8]. Recall that such a solution consists of a function linking state variables w_t (composed of a representation of cross-sectional distribution and z_t) to control variables y_t (consisting of coefficients describing individual policy functions), and the law of motion for w_t, which will in this case also depend on v_t as an additional state:

$$y_t = g(\chi_t, v_t)$$
$$\chi_{t+1} = h(\chi_t, v_t) + \begin{bmatrix} 0 \\ \sigma_z \end{bmatrix} \epsilon_{z,t+1}.$$

In a second order solution obtained with the BBN method, functions g, h will be quadratic in χ and linear in v.

We are interested in impulse responses to a volatility shock ϵ_v, and more specifically, whether such response is stronger in model with idiosyncratic risk than in corresponding representative-agent model. Figure 1.5 plots impulse responses to both level and volatility TFP shocks (where we use calibration with $\sigma_z = 0.014$ and $the \sigma_v = 0.1$, i.e. low average aggregate volatility, and one s.d. volatility shock increases the variance of next-period z by 10%). We see that responses to a level shock are very similar, whereas response to a volatility shock is somewhat stronger (by one third to one half) in a model with heterogeneity. Although the magnitude of response to a volatility shock is very small (which is, however, at least partially due to the calibration chosen), the qualitative difference does suggest that precautionary saving due to idiosyncratic risk plays a nontrivial role.

[8]Whereas simply including the equation for s.v. among other model equations and applying a standard perturbation method would require a third-order solution in order for volatility to play a distinct role.

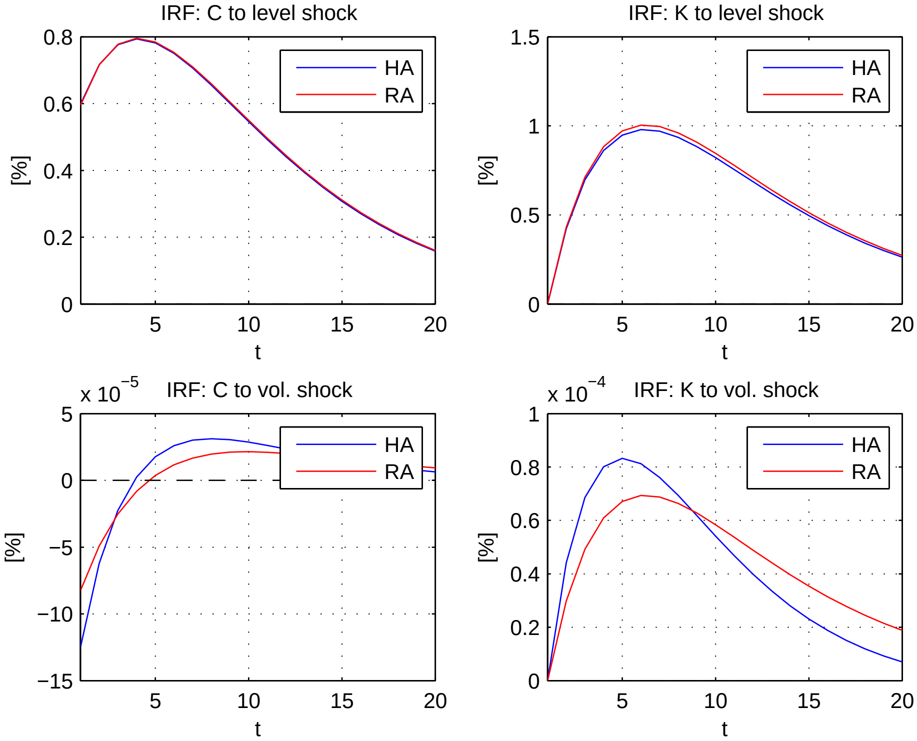

Figure 1.5: Impulse responses of consumption (left panels) and capital (right) to a level (upper) and volatility (lower) shock in productivity for a representative agent model (RA) and model wth idiosyncratic risk (HA).

1.7 Conclusion

This paper has investigated the application of second-order perturbation to solving DSGE models with heterogeneity in the context of the hybrid projection/perturbation approach. In the simple benchmark model considered here, accuracy gains have been shown to be rather modest in parametrization with small volatility of aggregate shocks, indicating that the presence of heterogeneity does not necessarily introduce strong nonlinearities in aggregate dynamics, at least in models similar to the standard growth model. Nevertheless, results of a model with larger volatility suggest that going beyond linearization may be desirable in some situations. Moreover, as we have seen, a second-order solution can be easily extended to study questions about welfare and impacts of time-varying volatility of aggregate shocks, which might be difficult to answer otherwise. In the future, it would be interesting to study those effects in more complicated models where heterogeneity plays a larger role, and hopefully the approach described in this paper would constitute a relevant addition to the macreconomists' toolbox for dealing with such questions.

Chapter 2

Asset prices in a production economy with long run and idiosyncratic risk

2.1 Introduction

Explaining joint dynamics of both macroeconomic quantities and asset prices within the context of a microfounded general equilibrium model remains an active area of economic research. This paper contributes to that effort by constructing a tractable model of a production economy that combines recursive utility with preference for early resolution of uncertainty and time-varying uninsurable idiosyncratic risk, and investigates its macroeconomic and asset pricing properties.

Individually, each of these elements have been studied previously as a possible solution to the well-known failures of a standard representative-agent model with power utility in explaining observed equity premium and interest rate[1]. When households have recursive preferences (Kreps and Porteus 1978; Epstein and Zin 1989), which break the link between risk aversion and elasticity of intertemporal substitution and allow for preference for early resolution of uncertainty, their marginal utility depends not only on current consumption, but also on the continuation value which encodes expectations about future consumption. News regarding the level or volatility of future consumption thus becomes an additional priced factor, as in the long-run risk model of Bansal and Yaron (2004) and in the production economy[2] of Kaltenbrunner and Lochstoer (2010). Another line of research has shown that when agents face incomplete markets and uninsurable shocks, the amount of risk they face can also affect asset prices if it changes over time, as in Constantinides and Duffie (1996) and Krusell and Smith (1997)[3].

Therefore, if agents have preference for early resolution of uncertainty and at the same time face idiosyncratic risk and incomplete markets, it follows that both current change in the amount of idiosyncratic risk, and also news about future such changes enter the continuation value and thus affect asset prices. This presents the potential for interaction between the two mechanisms, studied in the context of an endowment

[1] See e.g. Mehra and Prescott (1985), Weil (1989) and Hansen and Singleton (1982). A review of the literature is provided in e.g. Cochrane (2008) and Ludvigson (2013).

[2] Regarding asset pricing in production/DSGE models, see, e.g., survey by Kogan and Papanikolaou (2012). Among papers that study asset prices in production economies with recursive preferences are Tallarini (2000), Kaltenbrunner and Lochstoer (2010), Croce (2014), Rudebusch and Swanson (2012), van Binsbergen et al. (2012) and Campanale, Castro, and Clementi (2010).

[3] See also Mankiw (1986), Telmer (1993), Heaton and Lucas (1996), Krebs and Wilson (2004), Storesletten, Telmer, and Yaron (2007) and Pijoan-Mas (2007). Gomes and Michaelides (2008) also study a model with heterogeneity, production and recursive preferences, but their focus is primarily on the effects of limited participation and they do not model variation in either individual or aggregate risk over time. Empirical evidence is analyzed, e.g., by Cogley (2002, Brav, Constantinides, and Geczy (2002) and Balduzzi and Yao (2007), with somewhat mixed results.

economy in recent work by Constantinides and Ghosh (2017), Herskovic et al. (2016) and Schmidt (2014). However, matching asset prices in a production economy is harder than in endowment economies due to endogenous consumption process and the need to simultaneously match properties of quantities and prices. The main focus of this paper is therefore to look more closely at the interaction between the effects of varying idiosyncratic risk on macroeconomic dynamics and asset prices.

To illustrate the mechanism, I first construct a simple AK model with households having access to linear production technologies subject to heterogeneous rates of return on capital with time-varying variance. Assuming unit intertemporal elasticity of substitution, the model can be solved analytically and asset returns can be characterized by their exposure to news about current and future aggregate consumption and variance of idiosyncratic risk. A quantitative illustration suggests that omitting the last term could nontrivially underestimate the importance of overall long run risk for determining risk premia.

Next, I contruct a tractable model that embeds the Constantinides-Duffie framework within an otherwise standard real business cycle (RBC) model[4]. Individual household consumption growth is determined, in a reduced-form way, by aggregate consumption growth and idiosyncratic shock. With homothetic preferences and random walk in individual consumption, the model has a no-trade equilibrium in which each household consumes its income. The aggregate stochastic discount factor is determined by the cross-sectional average of individual intertemporal marginal rates of substitution, and is used by a representative firm to make choices about investment and dividends, which in turn determines aggregate consumption growth. Distribution of idiosyncratic shocks varies over time, possibly allowing for countercyclical variance (Storesletten, Telmer, and Yaron 2004) or procyclical skewness (Guvenen, Ozkan, and Song 2014).

The fact that there is no trade between households is somewhat unappealing (and thus resulting allocations should perhaps be interpreted rather as post-trade outcomes after households have smoothed out transitory shocks), yet it allows us to solve the model without keeping track of the distribution over individual savings, and thus avoid the need for numerically intensive computation. The model can be solved by standard perturbation methods and its linearized dynamics can be characterized semi-analytically. I find that the countercyclical idiosyncratic risk can raise risk premia, but also affects aggregate dynamics

[4] A similar approach is used to analyze monetary policy in New-Keynesian models in recent papers by Braun and Nakajima (2012), Werning (2015) and Takahashi et al. (2016). In these setups, variation in idiosyncratic risk manifests itself in a similar way as discount rate shocks after aggregation. In a related study, Albuquerque et al. (2016) study the role of discount rate shocks in asset pricing.

through its impact on saving and intertemporal smoothing incentives of households. The introduction of idiosyncratic risk leads to lower "effective" intertemporal elasticity of substitution on the aggregate level, resulting in more volatile and less predictable aggregate consumption growth. Inspecting the linearized solution suggests that the strength of this feedback depends on the cyclicality of idiosyncratic risk and household risk aversion.

On the other hand, thanks to the flexibility of Epstein-Zin preferences, it is, in principle, possible to recalibrate the discount rate and intertemporal elasticity of substitution (IES) parameters (to make households more willing to substitute consumption over time) in a way that compensates for the effect described above, while risk premia remain higher. After suitable recalibration of the model, I find that introducing heterogeneity raises the price of risk (Sharpe ratio) by about a third. Decomposing the price of risk by its source (aggregate consumption or dispersion of individual shocks) and channel (short-run or long-run risk) shows that the long run idiosyncratic dispersion accounts for about 30 percent of the overall long run channel, which in turn accounts for more than half of the overall Sharpe ratio. The results are quite similar regardless of whether the variation in individual risk unfolds through cyclical variance or skewness.

The paper is organized as follows: section 2 presents a simple example to motivate introduction of recursive preferences, section 3 describes the model, while section 4 discusses calibration and results and section 5 concludes.

2.2 Simple Model

Standard consumption-based asset pricing models explain the existence of risk premia by comovement of returns with consumption. Assets that pay off more in good times (i.e. states of the world with high consumption and low marginal utility) than in bad times (states with low consumption and high marginal utility), are less attractive for households wishing to smooth their consumption, and thus must offer higher returns to be held in equilibrium. However, it is well established that the standard model with representative household and power utility has problems matching the observed level of risk premia quantitatively. This paper considers two modifications of the baseline model that have been previously studied as possible explanations of high risk premia.

First, a richer specification for the household utility function, which includes the preference for earlier resolution of uncertainty, implies that "bad times" happen not only when current consumption is low, but also when the household receives bad news about future consumption. This amplifies the sensitivity of the household to small but persistent changes in consumption, which helps to increase the price of risk through the so-called long run risk chanel. Second, households face not only aggregate risk, but also a large amount of individual variation in their consumption arising from idiosyncratic shocks and incomplete markets. If the amount of this idiosyncratic risk is larger when the aggregate consumption is already low, households will be again more sensitive to aggregate fluctuations and will require higher returns to hold assets with procyclical payoffs.

This paper considers these two features together. If households care about both the volatility of individual shocks and news about the future, it follows that persistent cyclical variation in idiosyncratic risk will also be amplified by the long run risk mechanism, and this interaction can potentially imply higher resk premia with smaller values of risk aversion. On the other hand, it is also important to consider whether such a story is consistent with the supply side of the economy, since the consumption process is, in the end, an endogeous outcome affected by the saving behavior of households. I will therefore study a production economy with idiosyncratic shocks and long run risk in the subsequent section. First, however, it may be useful to flesh out the intuition discussed above more formally in a setting where the consumption process is still effectively exogenous.

This section thus presents a simple AK-like model[5] in which the output is produced using a linear technology with capital as the only input. Each household operates such technology independently, subject to aggregate and individual productivity shocks with time-varying dispersion, and can spend the output on consumption, investment or a risk-free asset. If we assume that households have a unit intertemporal elasticity of substitution, the model has an analytical solution. Subsequently, the price of risk can be cleanly decomposed into four contributions, from short run and long run risk in aggregate productivity and level of idiosyncratic risk. I look into how these contributions depend on the parameters of the model, and argue that they can be quantitatively relevant.

2.2.1 Setup

Time t is discrete and there is a continuum of agents indexed by i. Each agent enters the period with some stock of capital $K_{i,t}$ which is used for production according to $Y_{i,t} = A_{i,t}K_{i,t}$, subject to exogenous productivity process $A_{i,t}$ (which will have an idiosyncratic

[5] Previous literature using AK models to analyze asset prices in the presence of idiosyncratic risk includes Krebs and Wilson (2004), who focused on the case of log utility, and Toda (2014), who provides theoretical analysis for a class of similar models.

component and is thus indexed by i). Agents can also trade in risk-free one-period bonds, although the overall net supply of bonds is zero. Income obtained from production and bond holdings $B_{i,t}$ can be used for consumption $C_{i,t}$, stored as capital for the next period (for simplicity we shall assume full depreciation) or spent on new bonds. The budget constraint thus reads

$$C_{i,t} + K_{i,t+1} + P_t^b B_{i,t+1} = A_{i,t} K_{i,t} + B_{i,t},$$

where P_t^b is the bond price.

Agents have identical Epstein-Zin preferences with unit intertemporal elasticity of subtitution, so that their value function satisfies

$$V_{i,t} = C_{i,t}^{1-\beta} \left(E_t [V_{i,t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}} \right)^{\beta}.$$

Here parameter β controls time preference and γ is the coefficient of relative risk aversion. In the following, we shall focus on the empirically relevant case $\gamma > 1$, so that agents have preference for early resolution of uncertainty. Given the process for productivity, bond price and initial capital, each household will make its consumption-savings and portfolio choice to maximize the value function defined above.

We shall assume that the productivity has aggregate and idiosyncratic component:

$$\log(A_{i,t}) = \log(A_t) + \sqrt{x_t}\eta_{i,t} - \frac{x_t}{2}, \;\; \eta_{i,t} \sim N(0,1)$$

where idiosyncratic shocks $\eta_{i,t}$ are independent both across time and across households. Another exogenous process x_t denotes the cross-sectional variance of log productivity, which will fluctuate over time, and the last term ensures that the normalization $A_t = \tilde{E}[A_{i,t}]$ holds ($\tilde{E}[\,]$ will denote cross-sectional averages, conditional on realizations of aggregate variables).

2.2.2 Equilibrium

The equilibrium of this economy turns out to be particularly simple:

- Since preferences are homothetic and the value function is linear in wealth, there is a separation between the consumption-saving decision and portfolio choices. Since idiosyncratic shocks are uncorrelated over time, the only source of heterogeneity is in differing levels of wealth, so that all households make the same portfolio choice. Given the zero net supply of bonds, the equilibrium must thus involve no trade in them, so that $\forall i, \forall t : B_{i,t} = 0$.

- Without bonds, all wealth comes from current production. With unit IES, the consumption choice will be a constant linear function of wealth, so that $C_{i,t} = \kappa Y_{i,t}$ and $K_{i,t} = (1-\kappa)Y_{i,t}$, where $\kappa = 1 - \beta$.

Defining aggregates straightforwardly as cross-sectional averages (e.g. $K_t = \tilde{E}[K_{i,t}]$, etc.), aggregate dynamics can be summarized easily:

$$\begin{aligned} Y_t &= A_t K_t, \\ C_t &= \kappa Y_t, \\ K_{t+1} &= (1-\kappa)Y_t. \end{aligned}$$

Note that aggregate dynamics of quantities depends only on the aggregate productivity process A_t, not on the cross-sectional variance process x_t. If we denote logs in lowercase, we can also derive aggregate and individual consumption growth as

$$\begin{aligned} \Delta c_t &= \log(C_t/C_{t-1}) = \log((1-\kappa)A_t) = \log(1-\kappa) + a_t, \\ \Delta c_{i,t} &= \log(C_{i,t}/C_{i,t-1}) = \log((1-\kappa)A_{i,t}) = \log(1-\kappa) + a_t + \sqrt{x_t}\eta_{i,t} - \frac{x_t}{2}. \end{aligned}$$

The process for individual consumption thus has a similar form as in Constantinides and Duffie (1996).

2.2.3 Asset prices

Moving on to asset prices, although strictly speaking there is no aggregate capital, we can naturally define aggregate return to capital as an average payoff at time $t+1$ to one unit of good invested at time t, so that $R^k_{t+1} = A_{t+1}$. Return on bonds is then defined as $R^b_{t+1} = \frac{1}{P^b_t}$, and the difference between the two returns will be the equity premium. In this case, the return to capital is entirely determined by the linear technology, so the premium will be driven by adjusting the risk-free rate in accordance with the intertemporal marginal rate of substitution of households in the no-trade equilibrium, to which we turn next.

The intertemporal marginal rate of substitution (IMRS) of i-th household is given by

$$M_{i,t+1} = \beta \left(\frac{C_{i,t+1}}{C_{i,t}} \right)^{-1} \left(\frac{V_{i,t+1}}{E_t[V_{i,t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}} \right)^{1-\gamma}$$

and includes the usual consumption growth term as well as deviation of the next-period value function from its certainty-equivalent that would capture news about future consumption. In the equilibrium, each household's IMRS is a valid stochastic discount factor, and so will be their cross-sectional average $M_{t+1} = \tilde{E}[M_{i,t+1}]$. Returns to capital and bonds must satisfy the following equations:

$$1 = E_t[M_{t+1} R_{t+1}^k], \; 1 = E_t[M_{t+1} R_{t+1}^b].$$

Assuming (conditional) lognormality, we can express the conditional equity premium in terms of logarithm of stochastic discount factor (SDF) and log returns as

$$E_t[r_{t+1}^k] + \frac{1}{2}\text{Var}_t[r_{t+1}^k] - r_{t+1}^b = -\text{Cov}_t[m_{t+1}, r_{t+1}^k]. \tag{2.1}$$

Since the capital return is exogenous, asset pricing properties will mainly depend on conditional distribution of the stochastic discount factor and its sensitivity to aggregate shocks.

To explicitly characterize the innovation to the logarithm of SDF, we need to find the innovation to the value function. To this purpose, define the logarithm of normalized value function $v_{i,t} = \log(V_{i,t}/C_{i,t})$ and rewrite the value function recursion as

$$\begin{aligned} v_{i,t} &= \beta \frac{1}{1-\gamma} \log E_t \left[\exp\left((1-\gamma)(v_{i,t+1} + \Delta c_{i,t+1}) \right) \right] \\ &= \beta \frac{1}{1-\gamma} \log E_t \left[\exp\left((1-\gamma)(v_{i,t+1} + \Delta c_{t+1} - \frac{1}{2}\gamma x_{t+1}) \right) \right], \end{aligned}$$

where the second line follows from substituting for individual consumption growth and integrating out idiosyncratic shock. Since the above expression involves only aggregate variables, clearly the normalized value function will be equalized across households: $v_{i,t} = v_t$. If we furthermore assume that a_t (and thus Δc_t) and x_t jointly follow Gaussian homoscedastic process, we get

$$v_t = \beta \left(E_t \left[v_{t+1} + \Delta c_{t+1} - \frac{1}{2}\gamma x_{t+1} \right] + \frac{1-\gamma}{2}\Sigma \right)$$

with $\Sigma = \text{Var}_t\left[v_{t+1} + \Delta c_{t+1} - \frac{1}{2}\gamma x_{t+1}\right]$ being a (constant) conditional variance. Iterating forward and imposing proper terminal condition, value function can be expressed as

$$v_t = \frac{\beta}{1-\beta}\frac{1}{2}(1-\gamma)\Sigma + \sum_{i=1}^{\infty}\beta^i\left(\mathbb{E}_t\left[\Delta c_{t+i} - \frac{1}{2}\gamma x_{t+i}\right]\right).$$

The log of aggregate SDF in terms of v_{t+1} has the form of

$$m_{t+1} = \log(\beta) - \gamma\Delta c_{t+1} + (1-\gamma)(v_{t+1} - v_t/\beta) + \frac{1}{2}\gamma(1+\gamma)x_{t+1},$$

where the last term arises from integrating over cross-sectional consumption growth.

2.2.4 Price of risk

The innovation to m_{t+1} can subsequently be shown to equal

$$m_{t+1} - E_t[m_{t+1}] = -\gamma\epsilon^c_{t+1} + \frac{1}{2}\gamma(1+\gamma)\epsilon^x_{t+1} - (\gamma-1)\eta^c_{t+1} + \frac{1}{2}\gamma(\gamma-1)\eta^x_{t+1}$$

where $\epsilon^c_{t+1} = \Delta c_{t+1} - E_t[\Delta c_{t+1}]$ is a short-run innovation to consumption growth, $\epsilon^x_{t+1} = x_{t+1} - E_t[x_{t+1}]$ is a short-run innovation to cross-sectional consumption growth variance, $\eta^c_{t+1} = (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty}\beta^i\Delta c_{t+1+j}\right]$ is an innovation to long-run expected consumption growth, and $\eta^x_{t+1} = (E_{t+1} - E_t)\left[\sum_{j=1}^{\infty}\beta^i x_{t+1+j}\right]$ is an innovation to long-run expected cross-sectional variance. Increases in current or future consumption growth decrease marginal utility and thus carry a positive market price of risk, whereas increases in current or future cross-sectional variance enter with the opposite sign and thus carry a negative price of risk. In other words, assets which pay well in those states of the world in which a household receives bad news about current or *future* cross-sectional risk are less attractive and must offer higher returns.

In the above expression, the first term is standard and captures aggregate consumption growth. The second term is the same as in the Constantinides & Duffie model and captures contemporaneous effects of idiosyncratic risk. The third term describes news about future consumption, and has been studied in long run risk literature. The final term then captures news about future idiosyncratic risk, and is present only with preference for early resolution of uncertainty ($\gamma > 1$) and in a non-iid environment. The presence of this last term can potentially increase the equity premium if bad news about current and future consumption growth are accompanied by bad news about future levels of

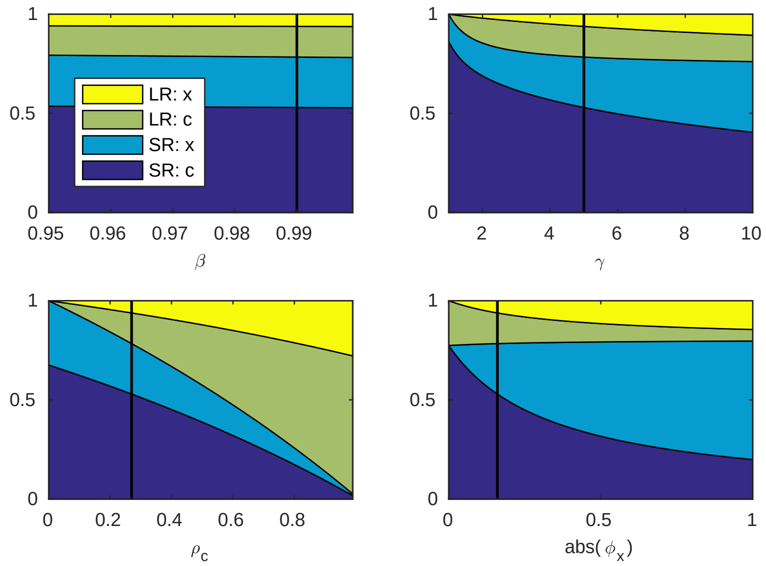

Figure 2.1: Comparative static of conditional Sharpe ratio decomposition according to equation (2.2). Filled areas show the relative contribution of each channel (long or short run, aggregate consumption or idiosyncratic risk). While varying each parameter, others are kept fixed ($\beta = 0.99, \gamma = 5, \rho_c = 0.27, \phi_x = -0.16$, see also black lines in corresponding subplots).

idiosyncratic risk.

As a more specific example, consider the following joint process for $\Delta c_t, x_t$:

$$\Delta c_t = (1 - \rho_c)\mu_c + \rho_c \Delta c_{t-1} + \epsilon_t, \;\; \epsilon_t \sim N(0, \sigma_\epsilon^2)$$
$$x_t = \mu_x + \phi_x(\Delta c_t - \mu_c).$$

so that aggregate consumption growth follows the AR(1) process and the idiosyncratic risk level is its affine function. Setting $\phi_x < 0$ corresponds to the countercyclical cross-sectional variance emphasized by Constantinides & Duffie. Since there is just one aggregate shock, we can obtain the following expression for log SDF innovation:

$$m_{t+1} - E_t[m_{t+1}] = \left(-\gamma + \frac{1}{2}\gamma(1+\gamma)\phi_x - (\gamma - 1)\frac{\beta\rho_c}{1 - \beta\rho_c} + \frac{1}{2}\gamma(\gamma - 1)\phi_x \frac{\beta\rho_c}{1 - \beta\rho_c} \right) \epsilon_{t+1}. \tag{2.2}$$

When $\gamma > 1$ and $\phi_x < 0$, all terms inside the paretheses have the same sign and their magnitude can be interpreted as the contribution of individual channels to the overall price of risk.

For a quantitative illustration, choose $\beta = 0.99$, $\gamma = 5$ (standard values), $\rho_c = 0.27$

(autocorrelation of quarterly US consumption growth) and $\phi_x = -0.16$ (see section 2.4.1). Following the above expression, we obtain that short-run consumption risk contributes 53.0%, short-run idiosyncratic risk 25.4%, long-run consumption risk contributes 15.5% and long-run idiosyncratic risk 6.2%. In relative terms, news about future idiosyncratic risk constitute 40% of the overall long-run risk. Figure 2.1 shows the sensitivity of this decomposition to each parameter. Varying the discount rate should in principle affect the weight households put on future events and thus also the relative importance of long run risk, but for the range of values usually considered it does not seem to play a large role. Higher risk aversion raises the share of both long run and idiosyncratic risk. Autocorrelation of consumption growth has a similar, although even stronger, effect, as with more predictability, a current shock to consumption causes greater revision of expectations about future. Finally, the degree of countercyclicality (plotted using its absolute value) makes the role of idiosyncratic risk larger.

The model presented in this section is too simplified in certain aspects. In a more standard production economy, the aggregate consumption process is endogenous and thus introduction of idiosyncratic risk may affect asset pricing results via general equilibrium effects. In addition, equity returns are also endogenous in the sense that the presence of idiosyncratic risk can affect the sensitivity of price-dividend ratios (and thus of returns themselves) to aggregate shocks, which might affect the predicted equity premium (although not the Sharpe ratio). For these reasons, in the next section I embed idiosyncratic risk into a version of a real business cycle model which will allow for both of these additional effects.

2.3 Full Model

This section describes the main model of a production economy with households facing idiosyncratic shocks. The model could be described as a variant of standard stochastic growth model, similar to Kaltenbrunner and Lochstoer (2010), modified with a tractable form of heterogeneity on the household side, modelled according to Constantinides and Duffie (1996).

2.3.1 Production

On the production side, there is a representative firm with standard Cobb-Douglas technology, producing output from capital K_t and labor H_t:

$$Y_t = K_t^{\alpha}(Z_t H_t)^{1-\alpha}, \tag{2.3}$$

where Z_t is labor-augmenting productivity and its log growth rate $\Delta z_t = \log(Z_t) - \log(Z_{t-1})$ is a given exogenous stochastic process. The firm hires labor on a competitive market at wage rate W_t to the point where wage equals the marginal product of labor:

$$W_t = (1-\alpha)\frac{Y_t}{H_t}. \tag{2.4}$$

The household labor supply is inelastic and fixed at unity, so in equilibrium

$$H_t = 1 \tag{2.5}$$

The firm owns its capital stock, uses part of its profits for investment I_t into the capital stock and pays the residual as dividend D_t:

$$Y_t = W_t H_t + I_t + D_t. \tag{2.6}$$

Capital accumulation is standard:

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{2.7}$$

Since the firm faces an intertemporal choice, it is necessary to discuss its objective. We shall assume the firm will choose an investment policy to maximize the present value of its dividends evaluated with a one-period stochastic discount factor M_{t+1} (to be discussed later), which is taken as given by the firm. Multi-period SDF is then defined as $M_{t\to t+j} = \prod_{i=1}^{j} M_{t+i}$, and the firm's objective is to maximize the sum of current dividend and (ex-dividend) stock price P_t^s, with the latter equal to the present discounted value of future dividends:

$$\max D_t + \underbrace{\mathbb{E}_t\left[\sum_{j=1}^{\infty} M_{t\to t+j} D_{t+j}\right]}_{P_t^s}.$$

Under constant returns to scale, return to the claim to firm's equity (priced with the SDF referred to above) will be equal to return on physical capital (Restoy and Rockinger 1994), in this case given by

$$R_{t+1}^K = \alpha \frac{Y_{t+1}}{K_{t+1}} + 1 - \delta \tag{2.8}$$

and by standard variational arguments, firm's first order condition is

$$1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^K \right]. \tag{2.9}$$

Finally, resources left for aggregate consumption consist of wages and dividend payments, or, equivalently, of output less investment:

$$C_t = D_t + W_t H_t = Y_t - I_t. \tag{2.10}$$

Note that the production side of the model determines the dynamics of macroeconomic aggregates such as capital, output and consumption once the stochastic discount factor is specified. Of course, in equilibrium the SDF process captures the attitudes of households toward intertemporal choice and risk, so we shall discuss the household side of the model next.

2.3.2 Households

There is a continuum of households indexed by i, with each having (the same) Epstein-Zin preferences over its own consumption stream $\{C_{i,t}\}$, summarized by a recursion for the value function

$$V_{i,t} = \left\{ (1-\beta) C_{i,t}^{1-\rho} + \beta \mathbb{E}_t \left[V_{i,t+1}^{1-\gamma} \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}, \tag{2.11}$$

where β captures time preference, ρ is the inverse of intertemporal elasticity of substitution and γ is relative risk aversion. Each household also inelastically supplies one unit of labor.

The main object of interest on the household side of the model is the stochastic discount factor, which enters into the firm's intertemporal decision. In a model with a representative household, we could drop the i subscript and the relevant SDF would be directly determined by the representative household's intertemporal marginal rate of substitution, the expression for which is known to be

$$M_{t+1}^{\mathrm{RA}} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\mathbb{E}_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma}.$$

On the other hand, if households face idiosyncratic risks and markets are incomplete, so the risk cannot be insured away, we will observe dispersion in individual consumption growth rates. In principle, individual consumption is an endogenous outcome, depending on the household's optimal decisions, which are themselves functions of individual and aggregate state variables. Generally, the aggregate state would include a cross-sectional distribution of wealth, necessitating the use of complex solution methods, such as those used in Krusell and Smith (1998). Instead, I will follow Constantinides and Duffie (1996) and assume directly[6] that the resulting dispersion of consumption growth rates can be described by a multiplicative shock to the aggregate consumption growth:

$$\frac{C_{i,t+1}}{C_{i,t}} = \frac{C_{t+1}}{C_t} \exp(\eta_{i,t+1}) \tag{2.12}$$

where innovations $\eta_{i,t+1}$ are uncorrelated across households and across time. However, since we are interested in idiosyncratic risk with varying severity over the business cycle, we shall allow the distribution of $\eta_{i,t}$ to vary according to an exogenous parameter process x_t. It will turn out advantageous to summarize this dependence via a moment-generating function

$$G(\tau; x) = \mathbb{E} \left[\mathrm{e}^{\tau \eta} | x \right] \tag{2.13}$$

and to assume that the parametrization satisfies the property $G(1, x) = 1$ for all possible x, ensuring that average consumption equals the aggregate consumption. For example, if $\eta_{i,t}$ is normal with variance x_t and mean $-x_t/2$, the MGF would be $G(\tau; x) = \mathrm{e}^{(x/2)(\tau^2-\tau)}$.

The main advantage of the above approach is that it allows us to define the aggregate stochastic discount factor as a cross-sectional average of individual marginal rates of subtitution in a tractable way, so that the resulting expression depends only on aggregate variables. For this purpose, define the logarithm of value function scaled by individual consumption $v_{i,t} = \log(V_{i,t}/C_{i,t})$, as well as the logarithm of scaled certainty equivalent

[6] See section 2.3.4 for a discussion of how such a result could be derived as a particular equilibrium outcome.

$\psi_{i,t} = \log\left(\mathbb{E}_t\left[V_{i,t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}/C_{i,t}\right)$, which satisfy the following:

$$v_{i,t} = \frac{1}{1-\rho}\log\left((1-\beta) + \beta\exp((1-\rho)\psi_{i,t})\right)$$
$$\psi_{i,t} = \frac{1}{1-\gamma}\log\left(\mathbb{E}_t\left[\exp((1-\gamma)(v_{i,t+1} + \Delta c_{i,t+1}))\right]\right)$$

Under the maintained assumption on individual consumption growth, we have $\Delta c_{i,t+1} = \Delta c_t + \eta_{i,t+1}$, and the distribution of $\eta_{i,t+1}$ is the same for each household from the point of view of period t. Using the law of iterated expectation to integrate over $\eta_{i,t+1}$ (conditional on the next-period parameters of its distribution x_{t+1}), we can rewrite the scaled value function recursion in terms of aggregates only, implying that these variables are equalized across households (thus we can drop the i subscript):

$$\begin{aligned} v_t &= \frac{1}{1-\rho}\log\left((1-\beta) + \beta\exp((1-\rho)\psi_t)\right) \\ \psi_t &= \frac{1}{1-\gamma}\log\left(\mathbb{E}_t\left[\exp((1-\gamma)(v_{t+1} + \Delta c_{t+1}))\cdot G(1-\gamma, x_{t+1})\right]\right) \end{aligned} \tag{2.14}$$

Note the MGF term $G(1-\gamma, x_{t+1}) = \mathbb{E}[\exp((1-\gamma)\eta_{i,t+1})|x_{t+1}]$, which arises from integrating over individual shock in the next period, conditional on its distribution which depends on aggregate variables x_{t+1}.

The individual household's intertemporal marginal rate of substitution is

$$M_{i,t+1} = \beta\left(\frac{C_{i,t+1}}{C_{i,t}}\right)^{-\rho}\left(\frac{V_{i,t+1}}{\mathbb{E}_t\left[V_{i,t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma} \tag{2.15}$$

which can be equivalently expressed as

$$M_{i,t+1} = \beta\exp\left(-\gamma\Delta c_{i,t+1} + (\rho-\gamma)(v_{t+1} - \psi_t)\right), \tag{2.16}$$

and subsequently the aggregate SDF is obtained by averaging over individual $M_{i,t+1}$ conditional on aggregate variables up to and including in period $t+1$:

$$M_{t+1} = \beta\exp\left(-\gamma\Delta c_{t+1} + (\rho-\gamma)(v_{t+1} - \psi_t)\right)\cdot G(-\gamma, x_{t+1}). \tag{2.17}$$

where again the term $G(-\gamma, x_{t+1})$ appears due to integration over individual shock.

Although defining aggregate SDF by averaging individual rates of substitution may seem arbitrary, if we grant that individual consumption allocations are outcomes of some (still unspecified) equilibrium, and abstracting from binding portfolio constraints, each household intertemporal rate of substitution would in fact be a valid SDF in the sense that it would be compatible with asset prices in the economy. Taking a cross-sectional average of these will result in a SDF which is valid too, but does not depend directly on any individual-level variables.

The presence of idiosyncratic risk thus affects the resulting discount factor through the properties of its distribution: specifically, through the $G(1-\gamma, x_{t+1})$ term in the value function recursion, provided that $\rho \neq \gamma$, as well as through $G(-\gamma, x_{t+1})$ term in the SDF. Since the modifications are expressed in terms of moment generating functions, all the higher moments of idiosyncratic risk could, in principle, affect the economy, although in the most commonly studied case of normal shocks, only the variance will matter. It is also clear that if the distribution of idiosyncratic shocks were time-invariant (i.e. x_t were constant), the only effect would be to introduce constant offsets into the value function and discount factor, while risk premia would not be affected directly. Finally, making the distribution of η collapse to a constant would yield expressions identical to those of a representative-agent version of the model, which can thus be considered a special case of the setup presented above.

2.3.3 Quantity dynamics and asset prices

To close the model, we need to further specify the exogenous process for productivity Z_t and the evolution of parameters x_t controlling the distribution of individual shocks (these could be functions of other aggregate variables, or follow their own exogenous process).

Productivity is assumed to be a random walk, so that

$$\Delta z_t = \mu_z + \sigma_z \epsilon_t, \epsilon_t \sim \mathcal{N}(0, 1) \tag{2.18}$$

Regarding the form of individual risk, I will assume that the individual element of consumption growth is lognormal, so that

$$\eta_{i,t} \sim \mathcal{N}\left(-\frac{x_t}{2}, x_t\right)$$

and x_t representes its variance, which is exogenously given as an affine function of consumption growth

$$x_t = \mu_x + \phi_x(\Delta c_t - \mu_z). \tag{2.19}$$

Equations (2.18) and (2.19) together with equations (2.3), (2.5), (2.7), (2.8), (2.9), (2.10), (2.14), (2.17) and the functional form for $G(\tau, x)$ mean that we have a sufficient number of relationships for solving the model. Since there is no need to track cross-sectional distribution of assets, the model can be solved by standard perturbation methods after detrending.

In terms of asset prices, unlevered return to capital has been defined in (2.8), and its logarithm will be denoted $r^k_{t+1} = \log R^k_{t+1}$. We will define the price of a one-period riskless bond that pays one unit in the following period in a standard way:

$$P^b_t = \mathbb{E}_t\left[M_{t+1} \cdot 1\right] \tag{2.20}$$

and define log-return on the bond as $r^b_{t+1} = \log(1/P^b_t)$. The excess return is the difference between return to capital and return to bonds: $r^x_{t+1} = r^k_{t+1} - r^b_{t+1}$. The conditional equity premium and Sharpe ratio are then defined as:

$$\begin{aligned} \text{EP}_t &= \mathbb{E}_t[r^x_{t+1}] \\ \text{SR}_t &= \frac{\mathbb{E}_t[r^x_{t+1}]}{\sqrt{\text{Var}_t[r^x_{t+1}]}} \end{aligned} \tag{2.21}$$

and their unconditional averages are $\text{EP} = \mathbb{E}[\text{EP}_t]$, $\text{SR} = \mathbb{E}[\text{SR}_t]$.

Recall the expression for conditional equity premium in a lognormal setting (adjusted for Jensen inequality) from equation (2.1):

$$\mathbb{E}_t[r^k_{t+1}] + \frac{1}{2}\text{Var}_t[r^k_{t+1}] - r^b_{t+1} = -\text{Cov}_t[m_{t+1}, r^k_{t+1}].$$

In case of just one aggregate shock, so that $m_{t+1} - \mathbb{E}_t[m_{t+1}] = \eta_{m\epsilon}\epsilon_{t+1}$, the conditional Sharpe ratio and equity premium is approximately

$$\text{SR}_t \approx |\eta_{m\epsilon}|\sigma_z, \;\; \text{EP}_t = \text{SR}_t\text{Var}_t[r^x_{t+1}].$$

In the model, all conditional volatility of returns arises from fluctuations in the marginal product of capital, which is not volatile enough to match the observed variation in stock returns. This issue could in principle be fixed by introducing capital adjustment costs or leveraged equity, although in this paper I will focus mainly on the price rather than quantity of risk, i.e. on the Sharpe ratio.

2.3.4 No trade equilibrium

The model presented so far relies on a reduced-form way to incorporate idiosyncratic consumption risk. It is possible to support such an outcome as a no-trade equilibrium[7] of a model with households facing particularly defined idiosyncratic additive shocks to their budget constraints, which could represent unexpected expenditures, gains or redistitributive payments (which, however, cancel out in the aggregate) that cannot be insured against due to incomplete markets. Intuitively, given that a household's utility function is homothetic and in the proposed equilibrium the deviation of individual consumption from the aggregate is a geometric random walk with shocks uncorrelated in time, all the households behave essentially symmetrically in their consumption/saving and portfolio decisions, thus implying no trade in assets. No trade, together with symmetric initial portfolios, in turn lead to individual consumption heterogeneity of the form described in previous sections. For completeness, this section will present such an equilibrium in more detail.

The individual household receives labor income and can trade firm shares and bonds. Its budget constraint reads:

$$C_{i,t} + P_t^s A_{i,t+1} + P_t^b B_{i,t+1} = W_t + (P_t^s + D_t)A_{i,t} + B_{i,t} + \Upsilon_{i,t} C_t,$$

where P_t^s, P_t^b are prices of firm equity and a risk-free one-period bond respectively, $A_{i,t}$, $B_{i,t}$ are the household's beginning-of-period portfolio positions, and other variables are as defined previously. The household also faces an additive shock $\Upsilon_{i,t}$ to its wealth, scaled by the current level of aggregate consumption. We will require that the cross-sectional average of $\Upsilon_{i,t}$ equals zero, so that individual shocks do not add or subtract resources to the economy.

The evolution of idiosyncratic shock is specified as:

$$\Upsilon_{i,t} = (1 + \Upsilon_{i,t-1}) \exp(\eta_{i,t}) - 1$$

[7] The discussion here adapts the no-trade equilibrium setup of Constantinides and Duffie (1996) from endowment to a production economy with EZ preferences. A close, although not identical aggregation approach is offered in Braun and Nakajima (2012), who allow for elastic labor supply, but also consider only time-separable utility function.

where $\eta_{i,t}$ are the same shocks which were previously characterized in equation (2.13). Since we assumed $\int \exp(\eta_{i,t})\mathrm{d}i = 1$, the above law of motion maintains a zero cross-sectional mean of $\Upsilon_{i,t}$. For example, if $\eta_{i,t}$ is normally distributed, $\Upsilon_{i,t}$ will have a lognormal distribution shifted by a negative constant.

The household takes asset prices, wages, dividends, aggregate consumption and idiosyncratic shocks as given, and chooses its consumption and portfolio positions to maximize its value function (2.11). Given the allocation of consumption across households, the rest of the model functions as previously described, although we will also require that stock and bond prices are consistent with market clearing in financial markets, so that, in the aggregate, households own the whole firm ($\int A_{i,t}\mathrm{d}i = 1$) and bonds are in zero net supply ($\int B_{i,t}\mathrm{d}i = 0$). Given the specification of exogenous shocks $Z_t, \Upsilon_{i,t}$, the equilibrium of the economy can be thus defined as:

- stochastic process for aggregate output Y_t, consumption C_t, investment I_t, capital K_t, wage W_t, return to capital R_t^k and dividend D_t,
- firm equity price P_t^s and bond price P_t^b
- individual household consumption $C_{i,t}$, portfolio positions $A_{i,t}, B_{i,t}$, value function $V_{i,t}$ and IMRS $M_{i,t+1}$
- aggregate SDF M_{t+1}

such that

- given the aggregate SDF, $Y_t, I_t, K_t, C_t, D_t, R_t^k, W_t$ are consistent with firm optimality condition (2.9), production function (2.3), capital accumulation (2.7), resource constraints (2.6), (2.10) and marginal products (2.4), (2.8).
- markets for financial assets clear.
- $C_{i,t}$, $A_{i,t}, B_{i,t}$, $V_{i,t}$ and $M_{i,t+1}$ are consistent with optimal decisions by a household.
- M_t is consistent with cross-sectional aggregation of household intertemporal rates of substitution $M_{i,t}$ as described in (2.17).

Next, notice that if households held symmetric market-clearing portfolios, i.e. $\forall t, \forall i$: $A_{i,t} = 1, B_{i,t} = 0$, their consumption growth would be in fact described by (2.12), since in such case their consumption is $C_{i,t} = W_t + D_t + \Upsilon_{i,t} C_t = (1 + \Upsilon_{i,t}) C_t$ and their consumption growth thus satisfies

$$\frac{C_{i,t+1}}{C_{i,t}} = \frac{C_{t+1}}{C_t} \frac{1 + \Upsilon_{i,t+1}}{1 + \Upsilon_{i,t}} = \frac{C_{t+1}}{C_t} \exp(\eta_{i,t+1})$$

The following result shows that an outcome where households hold symmetric portfolios at all times, embedded within the rest of the model described previously, is in fact an equilibrium:

Claim: Consider an allocation where

- firm stock price is given by $P_t^s = K_{t+1}$ and bond price is determined by aggregate SDF as in (2.20),
- households hold symmetric portfolios $A_{i,t} = 1, B_{i,t} = 0$,
- and rest of the model functions as described previously;

then such an allocation is an equilibrium. Moreover, households are in agreement in terms of the firm's investment policy.

To see why the above holds, we need to check whether first-order conditions of individual households are satisfied. The intertemporal rate of substitution of household i between two consecutive periods (implicitly, taking as given current aggregate state of the economy; I also supress time indices for clarity) can be generally written as a function of some first-period individual state s_i and second-period individual shock η_i' and aggregate shock ϵ': $M_i(s_i, \eta_i', \epsilon')$. In our case, however, individual IMRS given by (2.16) depends on the individual state only through the household's consumption growth, which is assumed to be uncorrelated over time and determined by future idiosyncratic shock η_i'. Therefore individual IMRS does not depend on the initial individual state and can be written as $M(\eta_i', \epsilon')$. Intuitively, if individual consumption behaves like a multiplicative random walk and households have homothetic preferences, any differences in wealth are simply a matter of scale.

The aggregate stochastic discount factor is obtained by averaging over individual shocks: $M(\epsilon') = \mathbb{E}\left[M_i(\eta_i', \epsilon') \mid \epsilon'\right]$ (since distribution of shocks is symmetric across households, this does not actually depend on i). We can then show that the aggregate optimality condition $\mathbb{E}\left[M(\epsilon')R(\epsilon')\right]$ for some return R also implies individual optimality $\mathbb{E}\left[M_i(\eta_i', \epsilon')R(\epsilon')\right]$, since here this follows directly from the law of iterated expectations. The aggregate optimality is satisfied by return on bonds by assumption, and it is easy to show that it also holds for return on stocks held by households[8]. It then follows that the household individual optimality conditions are also satisfied and that a no-trade euqilibrium is consistent with optimal consumption and portfolio choice by households.

The same argument also ensures that households do not differ in their preferred investment policy (see also Carceles-Poveda and Coen-Pirani (2009) for a more general discussion of when this is true): in equilibrium, each household receives the stream of dividends from the firm, so its preferred policy is to maximize the present value of future dividends, using its own IMRS as a discount factor. This would lead to a first order condition for investment $1 = \mathbb{E}[M_i(\eta_i', \epsilon')R^K(\epsilon')]$, but by the same logic of iterated expectations, this is equivalent to the assumed firm's condition (2.9). Another possible question is whether a different choice of weights across households when defining the aggregate SDF might affect the results. In general this is possible in models with incomplete markets (Carceles-Poveda 2009), but it turns out that in the current model weighting does not matter. Any weights corresponding to some reasonable corporate governance mechanism should depend only on current states of firm owners, not on realizations of next-period shocks. A weighted SDF $\tilde{M}(\epsilon') = \mathbb{E}\left[w(s)M_i(s, \eta_i', \epsilon') \mid \epsilon'\right]$ will not make a difference when M_i is independent of s.

2.4 Results

To evaluate how the addition of idiosyncratic risk affects the behavior of the neoclassical growth model, I first calibrate most of the parameters based on a representative-agent version of the model, then solve the model with and without idiosyncratic risk, and inspect its properties. In the second part of this section, I proceed by describing a log-linear approximate solution to the model, which is helpful to illustrate the interplay between idiosyncratic risk and dynamics of macroeconomic aggregates in the model. Finally, I will also consider an alternative way to model cyclical variation in the distribution of idiosyncratic risk by way of cyclical skewness rather than variance.

[8] This can be verified by plugging in the proposed expression for stock price into the definition of return and using the fact that $D_{t+1} = Y_{t+1} - W_{t+1} - I_{t+1} = \alpha Y_{t+1} - I_{t+1}$. After some rearranging, we obtain that the stock return is equal to the return to capital defined in (2.8), and thus satisfies the condition due to the firm's optimality condition (2.9).

Parameter	Value	Description
β	0.988	discount factor
ρ	0.7	inverse of IES
γ	5	risk aversion
α	0.33	capital share
δ	0.025	depreciation rate
μ_z	0.005	mean productivity growth
σ_z	0.015	volatility of productivity shock
μ_x	0.0036	mean level of ind. risk
ϕ_x	-0.16	cyclicality of ind. risk

Table 2.1: Parameter values.

2.4.1 Calibration

Model calibration is summarized in table 2.1. Frequency is quarterly. Starting with a representative-agent version of the model, most parameters are chosen close to standard values in the literature, as in, e.g., Campbell (1994). α is set to match the capital share of income of one third, δ implies annual depreciation rate of 10%. Discount rate β and the inverse of IES ρ are set so as to match the steady state return to capital of 6% per annum and output growth being twice as volatile as consumption growth. Trend productivity growth is set at 2% per year. The volatility of productivity shocks matches standard deviation of quarterly output growth of 1%, roughly corresponding to postwar US data. Finally, risk aversion is set to 5, a relatively standard value.

Following Storesletten, Telmer, and Yaron (2007), who use a process for variance of idiosyncratic shocks of the same form, I set $\mu_x = 0.0036$ (i.e. their value 0.014 rescaled to quarterly setting) and $\phi_x = -0.16$. The average level μ_x corresponds to annualized standard deviation of individual consumption growth of about 12%. The value of sensitivity ϕ_x captures the sensitivity of idiosyncratic risk to the business cycle, with negative values representing counter-cyclical variation. Given that quarterly (non-annualized) standard deviation of consumption growth will be approximately half a percent and assuming a normal distribution, the chosen value implies that fluctuations in x_t correspond to the annualized standard deviation of individual consumption growth ranging from approximately 9% to 15% with 95% probability (in terms of the ergodic distribution).

After detrending by productivity (a list of detrended equations can be found in the

	data	model: RA	model: HA1	model: HA2
moments:				
$\sigma[\Delta y_t]$	1.90%	2.02%	2.01%	2.02%
$\sigma[\Delta c_t]/\sigma[\Delta y_t]$	0.56	0.50	0.74	0.49
$\sigma[\Delta i_t]/\sigma[\Delta y_t]$	2.58	2.65	1.81	2.63
$\text{cor}(\Delta y_t, \Delta y_{t-1})$	0.37	0.03	0.02	0.03
$\text{cor}(\Delta c_t, \Delta c_{t-1})$	0.27	0.21	0.06	0.21
Sharpe ratio	0.39	0.121	0.163	0.161
risk price decomposition:				
short run, Δc	-	39.1%	45.8%	29.7%
short run, x	-	0.0%	22.0%	14.2%
long run, Δc	-	60.9%	23.0%	40.1%
long run, x	-	0.0%	9.2%	16.0%

Table 2.2: Comparison of model-implied annualized moments. Data: US quarterly series 1947-2016; see appendix for definitions. Model RA: calibrated as in table 2.1, but setting $\mu_x = \phi_x = 0$. Model HA1: as in table 2.1, but setting $\beta = 0.973$ to match RA model steady state. Model HA2: as in table 2.1, but setting $\beta = 0.975, \rho = 0.214$ to match RA model steady state and quantity dynamics. Standard deviations and Sharpe ratio are annualized by doubling from quarterly values. The bottom section shows relative contributions to the price of risk based on loglinear approximation.

appendix), I solve the model by a 3rd-order perturbation method using Dynare (Adjemian et al. 2011), as higher-order approximation is necessary to obtain non-zero risk premia when the perturbation approach is used for numerical solution. Model-implied moments for various variables are then computed from a pruned representation of the system, using the approach and code presented by Andreasen, Fernandez-Villaverde, and Rubio-Ramirez (2013). In a recent work, Pohl, Schmedders, and Wilms (2018) argue that models with long-run risk can exhibit nonlinearities that make local approximations potentially unreliable, and suggest using global solution methods. It turns out that in the model presented here, nonlinearities are quite mild, so that local and global solutions yield very similar results, as documented in the appendix.

2.4.2 Quantitative results

Table 2.2 displays selected unconditional moments from three versions of the model, as well as from US quarterly macroeconomic data. A representative agent variant of the model (RA column) matches variances of output and consumption growth (which, of

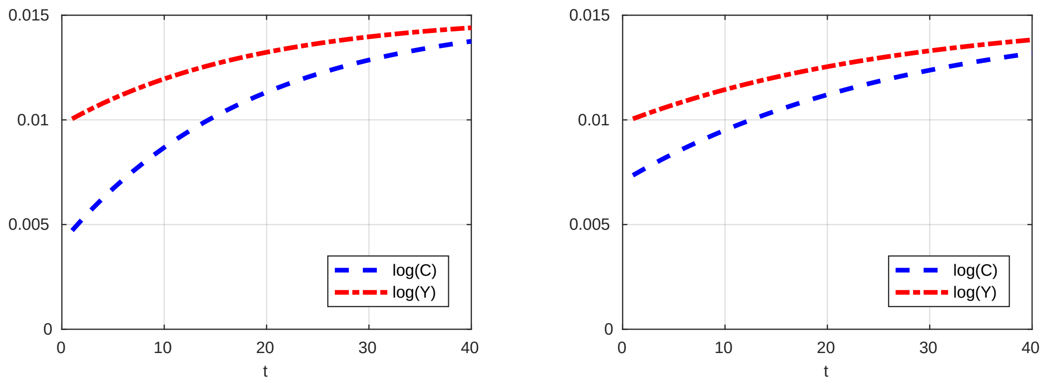

Figure 2.2: Impulse responses of log consumption and output to a 1 s. d. (permanent) productivity shock. Right panel: models RA (representative agent) and HA2 (het. agents, with β and ρ adjusted to match RA model dynamics). Left panel: model HA1 (het. agents, with β adjusted to match RA model steady state).

course, it has been calibrated to match), as well as autocorrelation of consumption growth. The implied Sharpe ratio of about 12% is lower than observed, yet still quite substantial compared to its value in a model with separable utility (approximately 0.6%). The second variant (HA1 column) is a model with idiosyncratic risk parameters calibrated as described above and otherwise the same as a representative-agent model, with the exception of the discount factor β which has been adjusted to obtain the same steady state. Looking at our main object of interest, we see that the presence of countercyclical idiosyncratic risk has increased the market price of risk (proxied here by the Sharpe ratio of excess returns) by approximately a third, but the dynamics of macroeconomic quantities has also changed significantly: with idiosyncratic risk, aggregate consumption growth has volatility closer to that of the output growth and autocorrelation closer to zero, which worsens the empirical fit of the model. In the third version (HA2 column), both the discount factor and the intertemporal elasticity of substitution are modified to maintain the same dynamics of output and consumption as in the RA variant of the model. We can see that the market price of risk remains high, so that by using a suitable choice of preference parameters, the model can be relatively succesful along both dimensions.

Even though the model with idiosyncratic risk has a higher price of risk relative to the representative agent model, the overall level of the Sharpe ratio still does not achieve the observed values. In principle, one could achieve a higher Sharpe ratio by cranking up the risk aversion. However, high values of γ are often considered unrealistic, as they imply implausibly conservative behavior by agents faced with a risky choice. In addition, higher risk aversion would make it harder to match the behavior of consumption in a model with idiosyncratic risk by requiring excessive adjustment of the intertemporal elasticity parameter (see also the discussion in the following subsection). The results presented here should then be interpreted as offering a partial resolution of the equity premium puzzle in a model with a moderate amount of risk aversion, but to explain the observed Sharpe ratio fully would likely require a richer model.

The bottom part of the table presents decomposition of the risk premium based on loglinear approximation, similar to the discussion in section 2.2 (see also the next subsection and the appendix for more details about loglinear solution). Dispersion of idiosyncratic shocks constitutes a bit less than a third of the overall long run risk contribution and around a third of the overall short run risk contribution. The overall contribution of long run risk is 61% in representative agent model and 56% in the HA2 model, but it is only 32% in the HA1 model, due to the overall amount of predictability in the economy being lower (the aggregate consumption is closer to a random walk).

To better understand how the introduction of idiosyncratic risk affects the behavior of output and consumption, figure 2.2 plots impulse responses to a productivity shock of output and consumption (log) levels for both RA and HA1 variants of the model (impulse responses in HA2 calibration are by construction close to the RA variant). The representative agent version shows both consumption and output growing over time toward their new, permanently higher, values implied by the permanent increase in productivity, but the response of consumption on impact is about half of output response (in line with calibration targeting volatility of consumption growth being half of output growth volatility). Thus households are willing to spread consumption increases over a longer horizon and to accept variation in future consumption growth rates in order to accumulate capital stock more quickly and thus to obtain more benefits from the increased productivity. However, in the model with idiosyncratic risk, the response of consumption on impact is much stronger and essentially consumes the whole productivity gain straight away at the cost of slower accumulation of capital, as if households were much more averse to intertemporal substitution of consumption.

This effect on consumption smoothing also complicates the analysis of asset prices, since the price of risk can be affected by the presence of idiosyncratic risk, in addition to its direct impact on the stochastic discount factor described in section 2, also through the changes in the endogenous process for aggregate consumption caused by a lower steady state interest rate and lower "aggregate" intertemporal elasticity of substitution.

Specifically, with less predictable consumption growth, the long run consumption risk emphasized by Kaltenbrunner and Lochstoer (2010) becomes less important, although the overall market price of risk has gone up in our case. On the other hand, as can be seen from the final column of table 2.2, it is possible to counteract such impacts by increasing IES (i.e. decreasing ρ) of individual households, although in general the size of the adjustment will depend on both the level and cyclicality of idiosyncratic risk, as well as households risk aversion, as discussed in more detail in the next subsection.

2.4.3 Qualitative analysis

To gain better intuition about the implications of idiosyncratic risk, we shall inspect a loglinear approximation to the model solution along the lines of Campbell (1994). Since the productivity process is a random walk, the detrended model has just one relevant state variable, (log) ratio of capital and productivity $k_t^* = \log(K_t/Z_t)$ (in terms of notation, lowercase symbols shall denote logs and starred variables are detrended by productivity). The dynamics of capital, output and consumption are determined by the deterministic steady state and by the sensitivity of detrended consumption to detrended capital: $\tilde{c}_t^* = \eta_{ck}\tilde{k}_t^*$, with a tilde denoting deviation from the steady state value.

A complete derivation can be found in the appendix, but it is possible to show that the steady state depends on preference and idiosyncratic risk parameters only through their effect on steady state return to capital $\bar{r}^k = -\log(\beta) + \rho\mu_z - \frac{1}{2}\gamma(1+\rho)\mu_x$. The coefficient η_{ck} depends on the steady state, as well as on the "effective" inverse of IES $\hat{\rho} = \rho - \frac{1}{2}\gamma(1+\rho)\phi_x$. In other words, any combinations of parameters $\beta, \rho, \gamma, \mu_x, \phi_x$ which imply the same $\hat{r}^k$ and $\hat{\rho}$ will lead to identical dynamics of output and consumption growth.

More specifically, if we start with a representative-agent model with parameters $\beta^{RA}, \rho^{RA}, \gamma^{RA}$ (i.e. $\mu_x^{RA} = \phi_x^{RA} = 0$), and then introduce idiosyncratic risk by setting $\mu_x > 0, \phi_x \neq 0$, we can maintain the same quantity dynamics in the heterogeneous-agent model by choosing parameters $\beta^{HA}, \rho^{HA}, \gamma^{HA}$ such that

$$-\log(\beta^{RA}) + \rho^{RA}\mu_z = -\log(\beta^{HA}) + \rho^{HA}\mu_z - \frac{1}{2}\gamma^{HA}(1+\rho^{HA})\mu_x$$
$$\rho^{RA} = \rho^{HA} - \frac{1}{2}\gamma^{HA}(1+\rho^{HA})\phi_x$$

If we, for example, decide to keep risk aversion the same: $\gamma^{HA} = \gamma^{RA}$, the above two equations pin down the new values of the discount rate and intertemporal elasticity of substitution. If the individual risk was acyclical ($\phi_x = 0$), the only necessary adjustment is in the discount rate, which should be set lower to counteract the precautionary saving effect pushing interest rates down. In the presence of countercyclical individual risk ($\phi_x < 0$), we would additionally need to make ρ^{HA} lower[9], to counteract the greater aversion of agents to intertemporal substitution.

Why do agents exhibit this aversion? We can gain some intuition by looking at the power utility case ($\gamma = \rho$). The individual Euler's equation can be then written approximately as

$$\log(\beta) + \rho E_t[\Delta c_{i,t+1}] - \frac{1}{2}\rho^2 Var_t[\Delta c_{i,t+1}] = r^b_{t+1}$$

Since $\Delta c_{i,t+1} = \Delta c_{t+1} + \eta_{i,t+1}$, if we ignore the small normalization shift in $\eta_{i,t+1}$, expected individual consumption growth moves one to one with aggregate expected consumption growth. However, with countercyclical risk, the conditional variance of individual consumption growth will vary inversely to Δc_{t+1}, and thus the whole left hand side will be more sensitive to $E_t[\Delta c_{t+1}]$. As a result, if we considered only aggregate data, the agent behaves as if he had higher ρ (lower intertermporal substitution) than he really does, which is consistent with empirical estimates of IES finding higher values when estimated on micro data compared with findings from aggregate time series (Havranek 2015).

Moreover, if the agent has Epstein-Zin preferences with risk aversion differing from the inverse of IES, the above result suggests that the degree of required adjustment in ρ depends on risk aversion as well, or alternatively, that risk aversion affects the dynamics of macroeconomic aggregates even at a first order approximation. The separation property described by Tallarini (2000) (i.e. that risk aversion affects the risk premia but not the behavior of quantities) thus does not hold outside the representative-agent model. A related issue with the proposed adjustment might be that, if idiosyncratic risk is strongly cyclical (ϕ_x has large magnitude) or households are very risk averse (γ is high), the adjustment might imply parameter values for ρ that are too low or even negative. It is possible that introducing other extensions affecting intertemporal choice, such as habit formation, might counteract this tendency, although I do not follow this direction in the current paper.

[9] A similar expression for "effective" intertemporal substitution in CRRA case was derived in Constantinides and Duffie (1996).

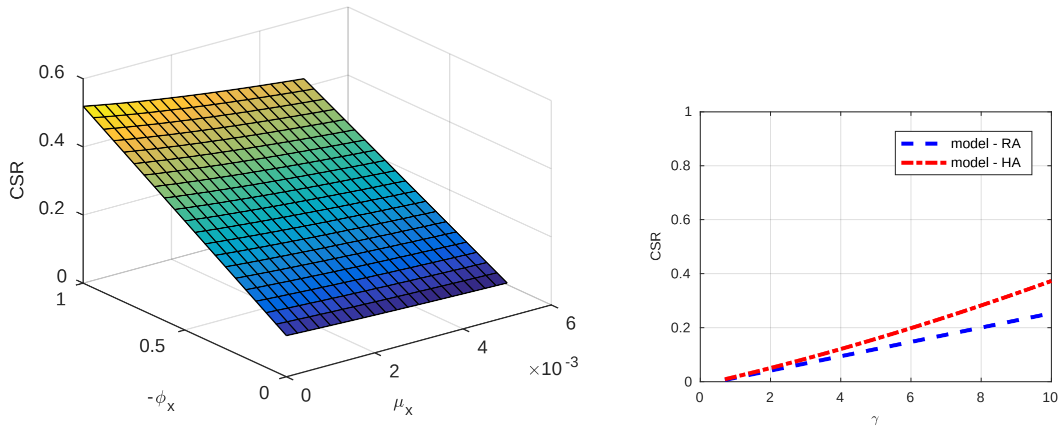

Figure 2.3: Comparative static for the conditional Sharpe ratio. Left: dependence on idiosyncratic risk parameters. Right: dependence on risk aversion. At each point, ρ and β are recalibrated to imply the same dynamics of aggregate quantities.

Even though the above discussion would suggest that the effect of idiosyncratic risk (at least as modelled here) does not affect qualitative properties of the representative-agent model conditional on suitable recalibration of preference parameters, the equivalence does not carry over to asset prices. Up to a linear approximation, log of scaled value function $v_t = \log(V_{i,t}/C_{i,t})$ can also be solved for as a function of capital stock, so that in terms of deviations from steady state, $\tilde{v}_t = \eta_{vk}\tilde{k}^*_t$. The coefficient η_{vk} is a function of the steady state and η_{ck}, but depends also on both μ_x and ϕ_x. With countercyclical risk ($\phi_x < 0$), the value function will be more sensitive to detrended capital stock and thus also to a productivity shock. The innovation to log SDF can be written as

$$m_{t+1} - E_t[m_{t+1}] = \underbrace{-\left[\left(\gamma - \frac{1}{2}\gamma(1+\gamma)\phi_x\right)\eta_{cz} + (\gamma-\rho)(-\eta_{vk})\right]}_{\eta_{m\epsilon}}\epsilon_{t+1} = \eta_{m\epsilon}\epsilon_{t+1}$$

implying a conditional Sharpe ratio

$$\frac{\log\left(E_t[R^k_{t+1}]\right) - r^b_{t+1}}{sd_t[r^k_{t+1}]} = -\eta_{m\epsilon}\sigma_z$$

Therefore, even if we recalibrate the parameters to maintain the same dynamics of aggregate consumption, market price of risk will still differ from the one implied by the representative-agent model with the same dynamics.

The left panel of figure 2.3 plots the (annualized) conditional Sharpe ratio as a function of μ_x, ϕ_x when preference parameters are recalibrated to match the quantity dynamics of the representative-agent model solved previously. Each point on the graph thus implies the same consumption process so that we can distinguish the pure effects of idiosyncratic risk on the risk premium. If the risk was acyclical ($\phi_x = 0$), the price of risk would actually go slightly down due to lower required discount rate, which in turn weakens the impact of long-run consumption risk (this effect is present only when consumption growth is not iid, otherwise acyclical idiosyncratic risk would have no impact, as in Krueger and Lustig (2010)). However, making the risk countercyclical increases the price of risk substantially. Note that Epstein-Zin preferences are crucial for this result, since if we imposed $\gamma = \rho$, we would obtain $\eta_{m\epsilon} = -\hat{\rho}\eta_{cz}$ and thus the recalibration procedure would imply the same price of risk for any combination of parameters.

The right panel of figure 2.3 plots the dependence of the risk premium on the risk aversion parameter, for a representative-agent model and for a model with idiosyncratic risk calibrated as in the previous section, again while keeping the quantity dynamics the same. We can observe that the presence of idiosyncratic risk not only makes the risk premium rise faster with higher risk aversion, but it causes it to do so at an increasing rate, leading to a convex relationship (whereas the dependence is linear in RA model). This confirms that the combination of Epstein-Zin preferences with idiosyncratic risk leads to an interaction that makes it easier to match observed risk premia with lower levels of risk aversion.

2.4.4 Cyclical skewness

Recent research (Guvenen, Ozkan, and Song 2014) suggests that it is cyclical variation in skewness, rather than variance of idiosyncratic shocks that is more consistent with data. Although cyclical variance, as analyzed in the previous sections, is especially tractable given the loglinear form of moment generating function for Gaussian distribution, the model allows the use of other distributions as well, as long as their moment generating function can be expressed in closed form. To see how much the results described above depend on specific form of idiosyncratic risk, I solve the model with $\eta_{i,t}$ following a mixture of three normal distributions with time varying means, as proposed by McKay (2017)[10].

[10] To be precise, I use the distribution of the permanent component of income shock faced by employed agents in the model described in that paper.

Specifically I assume that

$$\eta_{i,t} \sim \text{constant} + \begin{cases} N(\mu_{1,t}, \sigma_1^2) & \text{with prob. } p_1 \\ N(\mu_{2,t}, \sigma_2^2) & \text{with prob. } p_2 \\ N(\mu_{3,t}, \sigma_3^2) & \text{with prob. } p_3 \end{cases}$$

where the constant captures normalization, so that $E[\exp(\eta_{i,t})] = 1$, the means are given by

$$\begin{aligned} \mu_{1,t} &= 0 \\ \mu_{2,t} &= \mu_2 - x_t, \ \ \mu_2 < 0 \\ \mu_{3,t} &= \mu_3 - x_t, \ \ \mu_3 > 0 \end{aligned}$$

and, as before, x_t is a function of aggregate consumption growth:

$$x_t = \phi_x(\Delta c_t - \mu_z).$$

Individual consumption growth can belong either to the first mixture component, which stands for the "normal" experience faced by a majority of households, or to one of the other two components which represent negative or positive jumps. Movements in x_t then shift the position of the second and third components relative to first one, making the size of negative jumps larger during recessions (provided $\phi_x < 0$) and thus making the cross-sectional distribution of consumption growth more negatively skewed.

The calibration of means, variances and probabilities of the mixture elements follows McKay (2017), although I scale the overall size of the shock (i.e. means and standard deviations of mixture components) by one half to achieve a variance comparable to lognormal calibration used in previous sections. Sensitivity of x_t is estimated by regressing the time series for x_t provided by Alisdair McKay on his website[11] on US consumption growth, and the resulting coefficient is also scaled by one half. The chosen parameters are thus: $\mu_2 = -0.835$, $\mu_3 = 0.1970$, $\sigma_1 = 0.0319$, $\sigma_2 = \sigma_3 = 0.1668$, $p_1 = 94.87\%$, $p_2 = 3.24\%$, $p_3 = 1.89\%$ and $\phi_x = -7.285$. At the steady state, standard deviation of η with given parameters is 6.1%, or around 12.2% annualized, while the coefficient of skewness is 1.05 and of kurtosis 27.6, so the distribution is slightly positively skewed

[11] http://people.bu.edu/amckay/files/risk_time_series.csv

	data	model: RA	model: HA3	model: HA4
moments:				
$\sigma[\Delta y_t]$	1.90%	2.02%	2.01%	2.02%
$\sigma[\Delta c_t]/\sigma[\Delta y_t]$	0.56	0.50	0.76	0.53
$\sigma[\Delta i_t]/\sigma[\Delta y_t]$	2.58	2.65	1.69	2.47
$\text{corr}(\Delta y_t, \Delta y_{t-1})$	0.37	0.03	0.02	0.03
$\text{corr}(\Delta c_t, \Delta c_{t-1})$	0.27	0.21	0.05	0.17
SR	0.39	0.121	0.189	0.184
risk price decomposition:				
short run, Δc	-	39.1%	41.1%	26.4%
short run, x	-	0.0%	26.1%	16.8%
long run, Δc	-	60.9%	20.8%	36.1%
long run, x	-	0.0%	12.0%	20.8%

Table 2.3: Comparison of model-implied annualized moments under cyclical skewness. Data: US quarterly series 1947-2016; see the appendix for definitions. Model RA: calibrated as in table 2.1 without idiosyncratic risk. Model HA3: as in table 2.1 and section 2.4.4, but setting $\beta = 0.974$ to match RA model steady state. Model HA4: as in table 2.1 and section 2.4.4, but setting $\beta = 0.974, \rho = 0.255$ to match RA model steady state and quantity dynamics. Standard deviations and Sharpe ratio are annualized by doubling from quarterly values. The bottom section shows relative contributions to the price of risk computed using the loglinear approximation.

and fat-tailed. Measured in terms of plus/minus two standard deviations of aggregate consumption growth, skewness ranges from -1.5 to 3.1 over the business cycle.

Table 2.3, organized similarly as table 2.2, contains unconditional moments from two versions of a model with cyclical skewness. Again, I compare a version of the model with β recalibrated to match steady state return to capital (HA3 column), and another (HA4 column) with β and ρ recalibrated to match the dynamics of output and consumption[12]. The results are largely comparable to those in table 2.2, although the Sharpe ratio of 18% under skewed idiosyncratic shocks is somewhat higher compared to 16% under lognormal shocks. Without adjusting individual intertemporal elasticity of substitution, we again observe a change in the behavior of aggregate consumption, although the change is not as strong as in the lognormal case. Decomposition of risk premium is qualitatively also similar to the lognormal case, but quantitatively the role of idiosyncratic risk is slightly higher in relative terms.

[12]It is possible to derive approximate formulas for adjusting the parameters as in the previous section, although they are somewhat more involved due to the necessity of loglinearizing MGF terms. However, qualitatively the direction of adjustment is same as before.

2.5 Conclusion

In this paper, I have studied how preferences for early resolution of uncertainty and idiosyncratic, uninsurable risk affect risk premia in a tractable macroeconomic model with production. On one hand, the combination of the two elements implies that households care about direct shocks as well as news about both aggregate consumption and the amount or shape of individual risk, and if the latter varies cyclicaly over time, both can increase the price of risk more than each element would in isolation. On the other hand, when households can shift consumption intertemporally by investing in productive capital, countercyclical risk affects their incentive to do so, and on the aggregate level, the economy behaves as if households had lower intertemporal elasticity of substitution, potentially leading to different behavior of macroeconomic quantities. Nevertheless, at least in the setting analyzed here, one can maintain the same quantity dynamics by suitably recalibrating preference parameters. Specifically, if we are willing to assume that individual agents have higher intertemporal elasticity of substitution, it is possible to compensate for the effect of cyclical risk on aggregate consumption while keeping the price of risk higher.

There are several directions that could be pursued in further research. Introducing elastic labor supply or habit formation would allow for greater flexibility in matching macroeconomic dynamics. It might be also interesting to investigate independent shocks to the process describing distribution of idiosyncratic risk, either as a source of macroeconomic fluctuations or as an asset pricing factor, although identifying such shocks might present a challenge. An additional direction to consider would be to include stochastic volatility of aggregate shocks, which is another channel of time-varying uncertainty often analyzed in the literature, in order to compare and contrast the effects of "macro" and "micro" uncertainty on the economy. Finally, closer comparison to models with more realistic structure of household heterogeneity and trade between households would be useful in establishing the validity of the modelling approach used in the present paper.

2.A Appendix

2.A.1 Detrended model equations

Notation:

Lowercase variable names usually denote logarithms, e.g. $k_t = \log(K_t)$. Starred variables denote variables detrended by productivity, i.e. $y_t^* = \log(Y_t/Z_t) = y_t - z_t$. Delta denotes 1st difference, e.g. $\Delta c_t = c_t - c_{t-1}$.

List of variables:

Variable	Description
Δz_t	productivity growth rate
y_t^*	log detrendend output
k_t^*	log detrended capital
c_t^*	log detrended agg. consumption
Δc_t	growth rates of output, consumption
r_t^k	log return to capital
p_t^b	log bond price
r_t^b	log return to risk-free bond
m_t	log of aggregate SDF
v_t	log of scaled value function
ψ_t	log of scaled certainty equivalent
x_t	variance of individual consumption growth rates
ϵ_t	productivity shock

Equations:

- The production block contains equations describing productivity growth, the production function, capital accumulation, marginal product of capital, the Euler equation for investment and definition of consumption growth:

$$\begin{aligned}
\Delta z_t &= \mu_z + \epsilon_t \\
y_t^* &= \alpha k_t^* \\
\exp\left(k_{t+1}^* + \Delta z_{t+1}\right) &= (1-\delta)\exp\left(k_t^*\right) + \exp\left(y_t^*\right) - \exp\left(c_t^*\right) \\
\exp\left(r_t^K\right) &= \alpha \exp\left((\alpha-1)k_t^*\right) + 1 - \delta \\
1 &= E_t\left[\exp\left(m_{t+1} + r_{t+1}^K\right)\right] \\
\Delta c_{t+1} &= c_{t+1}^* - c_t^* + \Delta z_{t+1}
\end{aligned}$$

- The household block contains equations describing scaled value function, certainty equivalent, process of variance of individual consumption growth rates and the stochastic discount factor:

$$
\begin{aligned}
v_t &= \frac{1}{1-\rho}\log\left(1-\beta+\beta\exp((1-\rho)\psi_t)\right) \\
\exp((1-\gamma)\psi_t) &= E_t\left[\exp((1-\gamma)(v_{t+1}+\Delta c_{t+1}-(\gamma/2)x_{t+1}))\right] \\
x_{t+1} &= \mu_x+\phi_x(\Delta c_{t+1}-\mu_z) \\
m_{t+1} &= \log(\beta)-\rho\Delta c_{t+1}+(\rho-\gamma)(v_{t+1}-\psi_t+\Delta c_{t+1})+(1/2)\gamma(1+\gamma)x_{t+1}
\end{aligned}
$$

- The remaining equations describe price and return of the risk-free bond:

$$
\begin{aligned}
\exp(p_t^b) &= E_t\left[\exp\left(m_{t+1}\right)\right] \\
r_t^b &= -p_{t-1}^b
\end{aligned}
$$

Steady state:

Setting productivity shocks to zero allows us to find a stationary steady state, which corresponds to the balanced growth path in terms of original, undetrended variables. We shall denote steady state values by dropping the time index and bars over the variables.

- Along the balanced growth path, productivity and consumption grow at the same rate, so $\overline{\Delta z}=\overline{\Delta c}=\mu_z$. Idiosyncratic risk is at its average level: $\bar{x}=\mu_x$.

- Given the constant consumption growth, we can solve for the value function and steady state SDF:

$$
\begin{aligned}
\bar{v} &= \frac{1}{1-\rho}\log\left(\frac{1-\beta}{1-\beta\mathrm{e}^{(1-\rho)(\mu_z-(\gamma/2)\mu_x)}}\right) \\
\bar{\psi} &= \bar{v}+\mu_z-(\gamma/2)\mu_x \\
\bar{m} &= \log(\beta)-\rho\mu_z+\frac{1}{2}\gamma(1+\rho)\mu_x
\end{aligned}
$$

- Steady state SDF determines the return to capital, which in turn allows us to solve

	model: RA	model: HA1	model: HA2
3rd order perturbation			
Sharpe ratio	0.121	0.163	0.161
projection			
Sharpe ratio	0.119	0.160	0.160

Table 2.4: Comparison of solutions from perturbation and projection methods.

for steady state capital, output and consumption:

$$\begin{aligned}
\bar{r}^k &= -\log(\beta) + \rho\mu_z - \frac{1}{2}\gamma(1+\rho)\mu_x \\
\bar{k}^* &= \frac{1}{\alpha-1}\log\left(\frac{\exp\left(\bar{r}^k\right) - 1 + \delta}{\alpha}\right) \\
\bar{y}^* &= \alpha\bar{k}^* \\
\bar{c}^* &= \log\left(\exp(\bar{y}^*) - \left(\exp(\mu_z) - 1 + \delta\right)\exp(\bar{k}^*)\right)
\end{aligned}$$

- Finally, the SDF determines the bond price and return, which equals the return to capital:

$$\begin{aligned}
\bar{p}^b &= \log(\beta) - \rho\mu_z + \frac{1}{2}\gamma(1+\rho)\mu_x \\
\bar{r}^b &= -\log(\beta) + \rho\mu_z - \frac{1}{2}\gamma(1+\rho)\mu_x
\end{aligned}$$

2.A.2 Local vs. global solution

To find whether solving the model numerically with perturbation omits any substantial nonlinearities, I also solve a version of the model with counter cyclical variance also by using a projection method. I approximate consumption and value functions as combinations of Chebyshev polynomials up to the 10-th degree and solve for polynomial coefficients such that forward-looking conditions (i.e. the definition of the value function and the Euler equation, with expectations evaluated by 5-point Gauss-Hermite quadrature) hold exactly at a set of corresponding collocation nodes. Table 2.4 shows the resulting Sharpe ratios (obtained as averages from a simulation with each solution), which are very similar. Other moments are omitted as they were virtually identical up to 3 decimal places. Thus it seems that for the model and calibration studied here, nonlinearities do not matter very much.

2.A.3 Linearized solution

The model summarized above has a single state variable, detrended capital k_t^* and thus its inearized solution can be found explicitly. We shall denote deviations from a steady state value by tilde, e.g. $\tilde{k}_t^* = k_t^* - \bar{k}^*$. First, linearize key equations around the steady state:

$$\begin{aligned}
\tilde{k}_{t+1}^* &= \lambda_1 \tilde{k}_t^* - \lambda_2 \tilde{c}_t^* - \epsilon_{t+1} \\
\tilde{r}_t^K &= \lambda_3 \tilde{k}_t^* \\
E_t[\tilde{r}_{t+1}^K] &= -E_t[\tilde{m}_{t+1}] \\
\tilde{m}_{t+1} &= -\gamma \widetilde{\Delta c}_{t+1} + (\rho - \gamma)(\tilde{v}_{t+1} - \tilde{\psi}_t) + (1/2)\gamma(1+\gamma)\tilde{x}_{t+1} \\
\tilde{v}_t &= \kappa \tilde{\psi}_t \\
\tilde{\psi}_t &= E_t\left[\tilde{v}_{t+1} + \widetilde{\Delta c}_{t+1} - (\gamma/2)\tilde{x}_{t+1}\right] \\
\widetilde{\Delta c}_{t+1} &= \tilde{c}_{t+1}^* - \tilde{c}_t^* + \widetilde{\Delta z}_{t+1} \\
\widetilde{\Delta z}_{t+1} &= \epsilon_{t+1} \\
\tilde{x}_{t+1} &= \phi_x \widetilde{\Delta c}_{t+1}
\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ and κ are defined as

$$\begin{aligned}
\lambda_1 &= \exp\left(\bar{r}^k - \mu_z\right) \\
\lambda_2 &= \exp\left(\bar{c}^* - \bar{k}^* - \mu_z\right) \\
\lambda_3 &= \alpha(\alpha - 1)\exp\left((\alpha - 1)\bar{k}^* - \bar{r}^K\right) \\
\kappa &= \beta \exp((1-\rho)(\mu_z - (\gamma/2)\mu_x))
\end{aligned}$$

We are looking for consumption policy in the form of $\tilde{c}_t^* = \eta_{ck}\tilde{k}_t^*$.

Claim: if we can write the expected log SDF as $E_t[\tilde{m}_{t+1}] = -\hat{\rho}E_t\left[\widetilde{\Delta c}_{t+1}\right]$ for some $\hat{\rho}$, then η_{ck} can be found by using the method of undetermined coefficients as a (positive) solution to the quadratics

$$\hat{\rho}\lambda_2\eta_{ck}^2 + (\hat{\rho} - \lambda_2\lambda_3 - \hat{\rho}\lambda_1)\,\eta_{ck} + \lambda_1\lambda_3 = 0.$$

Proof: substitute law of motion for capital and consumption policy into the linearized Euler equation, take expectation (simply cancels shock), rearrange. There will be two real roots, one positive, one negative (since $\hat{\rho}\lambda_2 > 0$ and $\lambda_1\lambda_3 < 0$), and the positive one corresponds to the stable solution. □

Claim: our model satisfies the above with

$$\hat{\rho} = \rho - \frac{1}{2}\gamma(1+\rho)\phi_x.$$

Proof: since

$$\tilde{v}_{t+1} - \tilde{\psi}_t = \tilde{v}_{t+1} - E_t[\tilde{v}_{t+1}] - E_t[\widetilde{\Delta c}_{t+1}] + (\gamma/2)E_t[\tilde{x}_{t+1}]$$

and

$$E_t[\tilde{v}_{t+1} - \tilde{\psi}_t] = -E_t[\widetilde{\Delta c}_{t+1}] + (\gamma/2)E_t[\tilde{x}_{t+1}]$$

after bit of algebra, we get

$$E_t[\tilde{m}_{t+1}] = -\left(\rho - \frac{1}{2}\gamma(1+\rho)\phi_x\right) E_t\left[\widetilde{\Delta c}_{t+1}\right]$$

□

Finally, we can also solve for the value function in the form of $\tilde{v}_t = \eta_{vk}\tilde{k}_t^*$, also by using the method of undetermined coefficients. The result:

$$\eta_{vk} = \frac{\kappa\left(1 - \frac{\gamma}{2}\phi_x\right)\eta_{ck}\left(\lambda_1 - \lambda_2\eta_{ck} - 1\right)}{1 - \kappa\left(\lambda_1 - \lambda_2\eta_{ck}\right)}.$$

Having solved for the consumption and value functions, innovation to the log SDF can be expressed as

$$m_{t+1} - E_t[m_{t+1}] = \left(\gamma(1-\eta_{ck}) + (\gamma-\rho)(-\eta_{vk}) + \frac{1}{2}\gamma(1+\gamma)(-\phi_x)(1-\eta_{ck})\right)(-\epsilon_t)$$

Since typically $\gamma > \rho$, $\eta_{vk} < 0$ and $\phi_x < 0$, each of the three added terms inside the large parentheses is positive and can be understood as standing for short-run aggregate consumption risk, long run risk and short-run idiosyncratic risk, respectively. To further decompose long run risk, iterate forward on the definition of $\tilde{v}_t$ to obtain

$$\tilde{v}_t = \sum_{i=1}^{\infty}\kappa^i\left(E_t[\widetilde{\Delta c}_{t+i}] - \frac{1}{2}\gamma E_t[\tilde{x}_{t+i}]\right) = \left(1 + \frac{1}{2}\gamma(-\phi_x)\right)\sum_{i=1}^{\infty}\kappa^i E_t[\widetilde{\Delta c}_{t+i}]$$

so that the share of long run risk attributable to news about x can be taken as $\frac{\frac{1}{2}\gamma(-\phi_x)}{\left(1+\frac{1}{2}\gamma(-\phi_x)\right)}$.

2.A.4 Linearized solution with general MGF

The previous derivation of loglinear approximation can be relatively easily extended to the case of a general moment-generating function describing the distribution of idiosyncratic shocks. Specifically, let $G(t,x)$ be the MGF as described in the main text (normalized so that $G(1,x)=1$), and denote the cumulant generating function $g(t,x)=\log(G(t,x))$. We will continue to assume that x is a scalar following $x_t=\mu_x+\phi_x\widetilde{\Delta c}_t$. The relevant equations for the value function and log-SDF are modified as follows:

$$\begin{aligned}\exp((1-\gamma)\psi_t) &= E_t\left[\exp\left((1-\gamma)\left(v_{t+1}+\Delta c_{t+1}+\frac{1}{1-\gamma}g(1-\gamma,x_{t+1})\right)\right)\right]\\ m_{t+1} &= \log(\beta)-\rho\Delta c_{t+1}+(\rho-\gamma)(v_{t+1}-\psi_t+\Delta c_{t+1})+g(-\gamma,x_{t+1})\end{aligned}$$

and their steady state values, given that $\bar{x}=\mu_x$, are

$$\begin{aligned}\bar{v} &= \frac{1}{1-\rho}\log\left(\frac{1-\beta}{1-\beta \mathrm{e}^{(1-\rho)\left(\overline{\Delta c}+\frac{1}{1-\gamma}g(1-\gamma,\bar{x})\right)}}\right)\\ \bar{\psi} &= \bar{v}+\overline{\Delta c}+\frac{1}{1-\gamma}g(1-\gamma,\bar{x})\\ \bar{m} &= \log(\beta)-\rho\overline{\Delta c}+\frac{\gamma-\rho}{1-\gamma}g(1-\gamma,\bar{x})+g(-\gamma,\bar{x})\end{aligned}$$

To solve for dynamics, linearize g wrt. x at $t=-\gamma$ and $t=1-\gamma$:

$$\begin{aligned}g(-\gamma,x) &\approx g(-\gamma,\bar{x})+\theta_{(-\gamma)}\tilde{x}\\ g(1-\gamma,x) &\approx g(1-\gamma,\bar{x})+\theta_{(1-\gamma)}\tilde{x}\end{aligned}$$

where $\theta_{(t)}=\frac{\partial g(t,\bar{x})}{\partial x}$. Linearized equations then become

$$\begin{aligned}\tilde{v}_t &= \kappa\tilde{\psi}_t\\ \tilde{\psi}_t &= E_t\left[\tilde{v}_{t+1}+\widetilde{\Delta c}_{t+1}+(1/(1-\gamma))\theta_{(1-\gamma)}\tilde{x}_{t+1}\right]\\ \tilde{m}_{t+1} &= -\gamma\widetilde{\Delta c}_{t+1}+(\rho-\gamma)(\tilde{v}_{t+1}-\tilde{\psi}_t)+\theta_{(-\gamma)}\tilde{x}_{t+1}\end{aligned}$$

where $\kappa=\beta\exp\left((1-\rho)\left(\overline{\Delta c}+\frac{1}{1-\gamma}g(1-\gamma,\mu_x)\right)\right)$. Everything else is the same as in the previous case, and following the same argument we can derive effective inverse IES:

$$\hat{\rho}=\rho+\frac{\gamma-\rho}{\gamma-1}\theta_{(1-\gamma)}\phi_x-\theta_{(-\gamma)}\phi_x$$

and then η_{ck} is (the positive) solution to

$$\hat{\rho}\lambda_2\eta_{ck}^2 + \left(\hat{\rho} - \lambda_2\lambda_3 - \hat{\rho}\lambda_1\right)\eta_{ck} + \lambda_1\lambda_3 = 0$$

Using the method of undetermined coefficients, η_{vk} can be derived to be

$$\eta_{vk} = \frac{\kappa\left(1 + \frac{1}{1-\gamma}\theta_{(1-\gamma)}\phi_x\right)\eta_{ck}\left(\lambda_1 - \lambda_2\eta_{ck} - 1\right)}{1 - \kappa\left(\lambda_1 - \lambda_2\eta_{ck}\right)}$$

Then one can show that the innovation to log-SDF is

$$m_{t+1} - E_t[m_{t+1}] = \Big(\gamma(1-\eta_{ck}) + (\gamma - \rho)(-\eta_{vk}) + \theta_{(-\gamma)}(-\phi_x)(1-\eta_{ck})\Big)\left(-\epsilon_{t+1}\right)$$

which can again be used to decompose the risk premium, with the share of long run risk attributable to news about x being $\frac{\left(\frac{1}{1-\gamma}\theta_{(1-\gamma)}\phi_x\right)}{\left(1+\frac{1}{1-\gamma}\theta_{(1-\gamma)}\phi_x\right)}$.

2.A.5 Data sources

Data moments in table 2.2 for macroeconomic variables are obtained from quarterly national accounts data constructed by the U.S. Bureau of Economic Analysis and published in the St. Louis Fed FRED database. The sample period is 1947Q1 - 2016Q2. Output and investment growth (Δy, Δi) are computed as logarithmic growth rates of GDP and gross private domestic fixed investment quantity indices (NIPA table 1.1.3) divided by population (NIPA table 7.1). Consumption growth (Δc) is computed as a weighted average of logarithmic growth rates in quantity indices for nondurables and services consumption (NIPA table 1.1.3) divided by population, with weights determined by nominal shares of both consumption components in combined nominal nondurable+services consumption (NIPA table 1.1.5), i.e. using the Tornqvist index method (however, simply summing both series in real chained dollars yields almost identical results).

Data for financial returns are constructed from monthly dataset on Fama-French 3 factors published on Kenneth French's website[13]. In place of the return on capital/firm stock (R^s) I use the market return (i.e. the return on value-weighted portfolio of all firms listed at NYSE, AMEX or NASDAQ), while the risk-free rate (R^b) is represented by the return on 1-month Treasury bill. Returns are expressed in real terms by subtracting CPI inflation (series CPIAUCSL from FRED) and aggregated to quarterly frequency by

[13] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

summing monthly returns over the given quarter. The resulting sample period is 1947Q1 - 2016Q3.

Chapter 3

Uncertainty shocks with heterogeneous firms and firm owners

3.1 Introduction

The role of time-varying uncertainty[1] in the economy has been subject of much attention in recent years (summarized, e.g., by Bloom (2014)). Various measures of uncertainty are usually countercyclical. Recessions are associated with greater variance of shocks, larger dispersion of outcomes across firms and households and uncertain beliefs about future events. A natural question is whether and to what extent uncertainty might be a cause of economic fluctuations and through which mechanisms would an exogenous increase in uncertainty lead to economic downturns.

There are multiple dimensions of uncertainty which can affect the economy in different ways. One proposed channel involves an exogenous increase in the volatility of idiosyncratic productivity shocks faced by individual firms. In the presence of irreversible investment, higher uncertainty may depress new investment by raising the value of waiting and postponing investment or hiring decisions. Such a freeze in firm activity is then reflected in lower output and less efficient reallocation of production factors. This "real option" channel has been studied in partial equilibrium by Bloom (2009b) and in general equilibrium by Bloom et al. (2012) and by Bachmann and Bayer (2013), who find that an uncertainty shock can cause a noticeable drop in economic activity, even though the effect is somewhat muted in the latter case.

Even when embedded in a general equilibrium, this mechanism has so far been studied under the assumption of either risk-neutrality or with a representive household owning all firms in the economy. In such a case, however, households are insulated from the uncertainty itself, because although firm-specific shocks are more dispersed, this dispersion will wash out in the aggregate. And while the uncertainty shock may affect the level of future aggregate capital, output or prices, households will not face a larger amount of uncertainty about their future income or consumption than before. In reality, ownership of firms is not fully diversified, especially in case of private, unlisted companies (Moskowitz and Vissing-Jørgensen 2002), and more volatile firm profits can spill over to more volatile entrepreneur incomes. This paper aims to investigate how micro uncertainty in firm-specific productivity affects investment, saving and consumption decisions when households are exposed to (a part of) firm-specific volatility.

[1] The term "uncertainty" sometimes refers to ambiguity, i.e. a situation where agents lack well-defined probabilistic beliefs about random events they face. In the paper, in accordance with previous literature in this area, I will understand uncertainty in terms of risk, i.e. situations where random events are described by (objective or subjective) probabilities.

First, I illustrate the proposed mechanism in a simple two-period model with idiosyncratic but no aggregate uncertainty. When risk-averse entrepreneurs fully own their firm and are unable to insure against the risk they face, they will prefer to lower investment into new capital when faced with more volatile profits, both due to the real option effect (arising here from irreversible investment) and to the portfolio choice effect (a more risky asset is less attractive). With convex marginal utility, they will also want to lower their current consumption due to the precautionary saving motive. In contrast, when a representative household owns all the firms, its consumption is effectively deterministic and thus not affected by the precautionary motive, and the investment decision will be equivalent to that of a risk-neutral firm. i.e. higher compared to the non-diversified case. With higher investment, the drop in subsequent output will be relatively less severe. Theory suggests that abstracting away from the heterogeneity of firms owners and their exposure to profit volatility can potentially understate the effect of an uncertainty shock.

The sequence of events described above seems consistent with existing VAR estimates presented in Bachmann and Bayer (2013) and Gilchrist, Sim, and Zakrajšek (2014), who find that an increase in idiosyncratic firm volatility is associated with a subsequent drop in both consumption and investment. There is also evidence that higher volatility is associated with increases in the net foreign assets of a country (Fogli and Perri 2015). In this paper, I investigate a cross-country panel with different uncertainty measures collected by Baker and Bloom (2013) and their impact on several macroeconomic aggregates. I find that both "macro" (time-varying volatility of broad index stock returns) and "micro" (cross-sectional dispersion of individual stock returns) proxies of uncertainty are linked with subsequent falls in output, consumption and investment and increases in current account balance. Moreover, when including interaction with a measure of financial development (standing in for the degree of diversification in firm ownership), I find that the impact is stronger in less financially developed countries.

Next, I construct a dynamic model of an open economy with heterogeneous firms and entrepreneurs, who face idiosyncratic productivity shocks and irreversibility constraint on investment. Each entrepreneur controls the investment decisions of their own company, as well as choices about consumption and savings in foreign financial assets. Incomplete diversification is introduced by assuming that entrepreneurs own a share $1-\theta$ of their own company, and the remaining θ share is owned jointly by a collective of all entrepreneurs. The profits received in each period are then a combination of the entrepreneur's own firm profits and aggregate (average) profits. This approach offers a tractable way to vary the degree of diversification, from autarky ($\theta = 0$) to full risk-sharing ($\theta = 1$).

Using the model, I study how such an economy reacts to an exogenous uncertainty shock that raises the volatility of idiosyncratic shocks to a firm's productivities, and especially how such a reaction depends on the diversification parameter θ. The model predicts that the degree of diversification can substantially affect the economy's reaction to the uncertainty shock, where lower diversification makes the drop in investment larger and the recovery longer, while at the same time depressing consumption and increasing savings. On the other hand, diversification is less relevant for analyzing the behavior of output and investment in response to other kinds of shocks (such as fluctuations in productivity or interest rate), although even then it matters to some extent for the dynamics of consumption and savings. Interestingly, while a lack of diversification lowers the average level of capital and output in the economy, it does not seem to impact the efficiency of capital allocation across firms.

Related literature. The theory of investment under uncertainty with irrevesibilities has been studied extensively (Dixit and Pindyck 1994; Abel and Eberly 1994). Consequently, macroeconomists have paid increasing attention to the role of firm heterogeneity in determining aggregate investment (Caballero 1999), as well as macroeconomic fluctuations more generally (Veracierto 2002; Khan and Thomas 2008). The current paper is most closely related to the literature studying macroeconomic effects of changing cross-sectional uncertainty in firm productivity. In a well known contribution, Bloom (2009b) shows that, in the presence of irreversibility and fixed costs, exogenous increases in the volatility of firm-specific productivity shocks leads to a drop in investment and hiring in the short-run. Subsequent research, such as Sim (2007), Bloom et al. (2012) and Bachmann and Bayer (2013), has studied this mechanism in general equilibrium.

More recent papers have looked at differences in response to uncertainty shocks between durables and nondurables industries (Kehrig 2015), interaction of cyclical uncertainty with a financial accelerator mechanism (Chugh 2016), effects of fiscal stimulus (Winberry 2016) and the role of cyclical skewness in dispersion of firm productivity (Kuehn, Schreindorfer, and Ehouarne 2016). Alternatively, in the presence of financial frictions, an increase in volatility can also depress investment by raising credit spreads and interfering with firm's external financing, as described by Arellano, Bai, and Kehoe (2012), Gilchrist, Sim, and Zakrajšek (2014) and Christiano, Motto, and Rostagno (2014). All of these contributions model the consumer side of the economy via a representative household and thus abstract away from the precautionary channel studied here. One exception can be found in Dou

(2016), who also presents a model with underdiversified risk-averse entrepreneurs and uncertainty shocks, with a primary focus on its asset-pricing implications.

More broadly, the issue of comovement in consumption and investment in response to an increase in uncertainty has been discussed in Basu and Bundick (2017). The paper is also related to literature on how entrepreneurs with an inside stake affect the decisions of the firm (Chen, Miao, and Wang 2010; Panousi and Papanikolaou 2012) and on the implications of idiosyncratic entrepreneurial risk (Angeletos and Calvet 2006). The effect of precautionary and portfolio channels on investment has been also studied by Bayer et al. (2015) in a setting with labor income risk. On the empirical side, interaction between the response of aggregate output to macroeconomic uncertainty and financial development has been previously studied in Karaman (2015). This paper complements previous results by looking at the responses of several macroeconomic outcomes to a cross-sectional uncertanty shock.

Organization. Section 3.2 illustrates the main mechanism in a two-period model and presents empirical evidence. Section 3.3 desscribes the dynamic model and section 3.4 presents model results. More technical details can be found in the appendix.

3.2 Motivation

3.2.1 Theory

To illustrate the relationship between uncertainty, investment and consumption, consider a simple two-period model. There is a population of entrepreneurs, each with initial resources W. In the first period, i-th agent decides how to spend their wealth on consumption, savings in risk-free asset yielding fixed return (here normalized to 1) and physical capital in their own firm:

$$W = C_{i,1} + A + K_i.$$

In the second period, each firm faces a random realization of idiosyncratic productivity Z_i. A portion p of firms also gets a chance, independent of productivity, to adjust their firm's capital stock upward to a new value $K_i' \geq K_i$. In such a case, the investment is financed from within-period profits. The remaining $1-p$ share of firms will have capital stay fixed at their original choice: $K_i' = K_i$. Next, firms will produce output according to decreasing-returns production function $Y_i = Z_i \left(K_i'\right)^{\alpha}$, $0 < \alpha < 1$, and finally, second-period consumption is the sum of savings and output less any new investment:

$$C_{i,2} = A + Z_i \left(K_i'\right)^{\alpha} - \left(K_i' - K_i\right).$$

Each agent will make his initial choices and adjust capital conditional on the second-period state (if allowed) to maximize expected utility:

$$\max_{C_{i,1}, K_i, A_i, K'(K,Z)} u\left(C_{i,1}\right) + \mathbb{E}\left[u\left(C_{i,2}\right)\right]$$

Let us denote η_i a random variable capturing whether the firm can invest in the second period ($\eta = 1$) or not ($\eta = 0$). Iterating backwards, the second period investment is a simple static problem where the firm either chooses unconstrained optimum $K^*(Z) = (\alpha Z)^{\frac{1}{1-\alpha}}$ if it is higher than its original investment (i.e. when productivity is high), or is limited by the irreversiblity constraint so that $K' = K$, the same as firms without the opportunity to adjust. Then we can write total profits from production as

$$D(K, Z, \eta) = \begin{cases} ZK^{\alpha} & \text{if } \eta = 0 \text{ or } \eta = 1 \text{ and } K^*(Z) < K \\ \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)Z^{\frac{1}{1-\alpha}} + K & \text{if } \eta = 1 \text{ and } K^*(Z) \geq K \end{cases}$$

and the utility maximization problem, symmetric in the first period, can be written as

$$\max_{C_1, A, K} u(C_1) + \mathbb{E}\left[u\left(A + D(K, Z, \eta)\right)\right] \quad \text{s. t.} \quad C_1 + A + K = W.$$

The above model treats entrepreneurs as acting independently, essentially in autarky. With the financial market limited to a risk-free asset, each agent is fully exposed to second-period idiosyncratic risk. An opposite extreme would be a situation in which all agents collectively own all the firms and split profits equally, so that the idiosyncratic risk is fully diversified away. Individual consumption would be equal to average consumption, and thus the decision problem changes to

$$\max_{C_1, A, K} u(C_1) + u\left(\mathbb{E}\left[A + D(K, Z, \eta)\right]\right) \quad \text{s. t.} \quad C_1 + A + K = W.$$

The main point of interest is how the optimal choice varies with the degree of uncertainty in Z both in cases of autarky and full diversification. The following result (derived more precisely in the appendix) summarizes the qualitative properties of both solutions.

Claim: Assume $0 < p < 1$ and $u'''() > 0$. Let $A^{\mathrm{a}}, K^{\mathrm{a}}, C_1^{\mathrm{a}}$ denote the solution for the autarky model, $A^{\mathrm{d}}, K^{\mathrm{d}}, C_1^{\mathrm{d}}$ for the diversified model. Denote $\bar{Z} = \mathbb{E}[Z]$.

1. If there is no uncertainty in productivity, i.e. $Z = \bar{Z}$ always, both solutions coincide: $A^{\mathrm{a}} = A^{\mathrm{d}} = \bar{A}$, and similarly for capital and consumption.

2. If there is uncertainty, the initial capital choice of diversified entrepreneurs is lower than in the deterministic case, $K^{\mathrm{d}} < \bar{K}$.

3. Capital choice in the case of autarky under uncertainty is lower than diversified choice under uncertainty: $K^{\mathrm{a}} < K^{\mathrm{d}}$.

4. Similarly, consumption choice in the case of autarky is lower than diversified choice: $C^{\mathrm{a}} < C^{\mathrm{d}}$, and thus necessarily savings are larger: $A^{\mathrm{a}} > A^{\mathrm{d}}$.

Regardless of the degree of diversificaiton, initial investment is lower under uncertainty. This is the well known real option channel, since, due to the irreversibility constraint, having too much capital is more costly than having too little. In addition, physical capital is even less attractive for nondiversified entrepreneurs who would require a risk premium to hold an asset with uncertain return. Moreover, if marginal utility is convex (i.e. preferences exhibit "prudence"), nondiversified agents will also further decrease their consumption because of the precautionary saving motive. As a result, we can expect that an introduction of uncertainty in the case of autarky will lead to lower investment, lower initial consumption and higher savings compared to the case of full diversification.

Figure 3.1 provides a quantitative illustration by plotting optimal first period choices and average second period output against a mean-preserving dispersion of individual productivity[2]. The pattern described above is clearly visible and quantitatively not trivial. Capital is decreasing with dispersion, more so for the nondiversified case. For this specific parametrization, the consumption of diversified agents actually increases in response to uncertainty, as they choose to consume part of the funds freed by lower capital investment. Consumption of nondiversified entrepreneurs falls, in contrast, as they channel their resources into the safe asset. Average production also behaves differently, mostly falling with uncertainty in the autarky model, but increasing in the diversified model, in spite of

[2]Parameters used: $W = 1$, $\alpha = \frac{1}{2}$, $p = \frac{1}{2}$, $u(c) = \frac{1-\mathrm{e}^{-\gamma c}}{\gamma}$ with $\gamma = 5$, $\log(Z) \sim N\left(-\frac{\sigma^2}{2}, \sigma^2\right)$. Expectation is approximated with 7-point Gauss-Hermite quadrature, and the objective function is then maximized numerically.

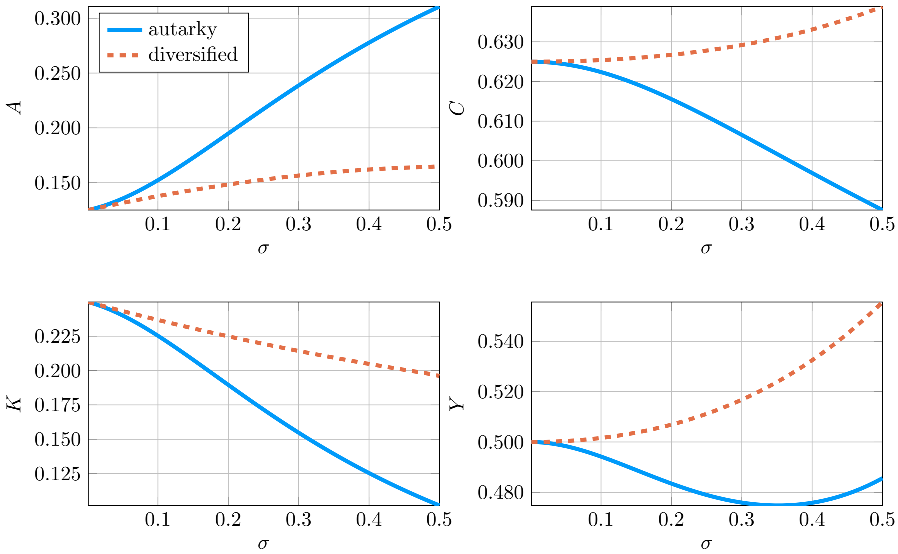

Figure 3.1: Two-period model: comparative static with respect to dispersion in Z.

lower initial investment. This is due to the Oi-Abel-Hartmann effect, as output is convex in Z when capital can be freely adjusted, which is true for some of the firms in the model.

3.2.2 Empirics

The preceding discussion suggests that an increase in uncertainty faced by firms at the micro level will possibly lower investment, but its direct effect on consumption will depend on the amount of risk that passes through to firm owners. In general, uncertainty is considered countercyclical and therefore negatively correlated with both investment and consumption. Previous research specifically estimating VAR responses to a rise in firm-specific shock volatility (Bachmann and Bayer 2013; Gilchrist, Sim, and Zakrajšek 2014) indicates negative impacts on both components of economic activity. Such comovement is, however, not easily obtained in general equilibrium models in which firms are owned by a representative household, as in Bloom et al. (2012). Similar issue arises even with other forms of uncertainty shocks in a class of real business cycle models, as discussed by Basu and Bundick (2017) who instead achieve a negative response of consumption through aggregate demand effects under sticky prices. The mechanism proposed in this paper would, on the other hand, affect incentives for consumption directly through the precautionary savings channel, and thus can potentially work alongside general equilirbrium forces. In addition, the precautionary motive would also lead to higher net savings in an open economy setting, where higher volatility has previously been found to be associated with increases in net foreign assets (Fogli and Perri 2015).

For further empirical evidence, in this section I investigate effects of uncertainty shocks on macroeconomic aggregates using a cross-country panel dataset collected by Baker and Bloom (2013), which covers 60 countries over 1970-2013. The dataset includes several measures of uncertainty, of which I will use volatility of daily broad index returns as a proxy for "macro" uncertainty, and cross-sectional dispersion of individual firm stock returns as a proxy for "micro" uncertainty. The main outcome variables of interest, obtained from the World Bank's WDI database, are output, consumption and investment growth, and change in the ratio of current account relative to GDP, all at annual frequency. The response of economic activity to uncertainty is estimated by a local projection approach (Jordà 2005), i.e. by regressing future outcome y on current uncertainty x and other controls at different horizons:

$$y_{i,t+h} = \alpha S_{i,t} + \beta^{(h)} x_{i,t} + \epsilon_{i,t},$$

where $y_{i,t}$ for country i at year t is growth rate or change in ratio to GDP of respective macroeconomic outcome expressed in percentage points, $x_{i,t}$ is a logarithm of respective volatility measure and controls $S_{i,t}$ include $x_{i,t-1}$, $y_{i,t-1}$, country and year fixed effects. Including contemporaneous value of uncertainty but not of the outcome variable is equivalent to standard VAR recursive identification with uncertainty ordered first. Of course, an increase in uncertainty could be either an exogenous impulse or an endogenous response to other shocks (or both), and disentangling the two with coarse annual data is a difficult challenge. The coefficients thus do not necessarily represent causal effects[3]. The main objective of these regressions is to provide additional evidence consistent with the hypothesis that an uncertainty shock, in both the "macro" and the "micro" sense, leads to a comovement in major macroeconomic aggregates, including consumption and savings. Such a comovement is a fact for which the hypothesis in current paper could provide one possible explanation.

Results are shown in figure 3.2, which plots estimated coefficients $\beta^{(h)}$ against horizon h for each combination of four outcome and two input variables. Given variable definitions, a coefficient $\beta = -2$ would, for example, correspond to an increase in volatility by one fifth leading to drop in the growth rate or ratio by roughly 0.4 percentage points. We

[3] In their original study, Baker and Bloom (2013) used only output as outcome and their main focus was on estimating causal effects of uncertainty using disasters as instruments. In this paper I use their dataset together with multiple outcome variables, but do not seek to explicitly identify causal effects.

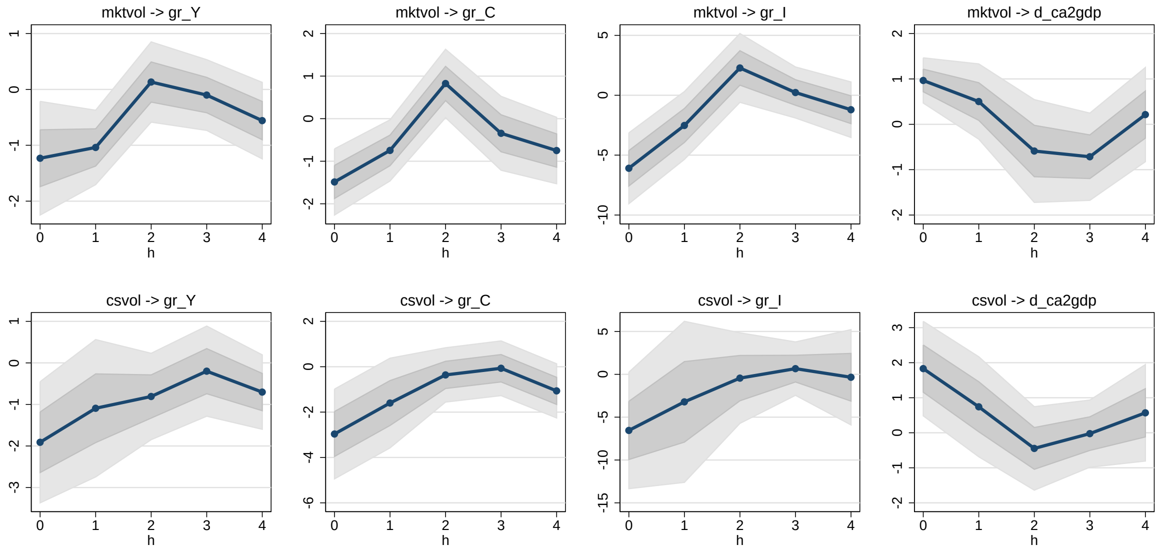

Figure 3.2: Cross-country panel estimates of the response of macroeconomic variables to uncertainty shock at different horizons. First row: response to log "macro" volatility; second row: response to log "micro" volatility. Outcome variables in columns from left to right: GDP growth, consumption growth, investment growth, change in CA/GDP ratio. Shaded bands represent ±1 and ±2 (clustered, robust) standard errors.

can see that response to uncertainty is negative on impact for both consumption and investment, regardless of the uncertainty measure. The current account also increases on impact, so that countries export more or import less and accumulate foreign assets. After 2-3 years, the economy recovers and the situation reverses, although some estimates are more imprecise at that point. The evidence thus supports the observation that cross-sectional uncertainty affects both investment and consumption decisions, consistent with the explanation proposed in the previous section.

An additional question of potential interest is whether the response is heterogeneous across countries. Specifically, if the precautionary response of entrepreneurs plays an important role, we would expect those economies where they are more exposed to idiosyncratic productivity risk to react more strongly to a cross-sectional uncertainty shock. One way to investigate this hypothesis would be to regress macroeconomic outcomes on uncertainty interacted with a measure of financial development as an indicator of the degree of diversification in the economy. A similar approach has been used by Karaman (2015), who finds evidence in favor of a heterogeneous response of output to macro uncertainty shock (ie.e overall stock market volatility). More relevant for this paper, I also find similar

outcome:	(1) growth Y	(2) growth C	(3) growth I	(4) change $\frac{CA}{Y}$
CSVOL	-6.80***	-7.30***	-14.6**	2.13
	(-4.6)	(-3.9)	(-2.4)	(1.8)
CSVOL × FINDEV	8.41***	8.47***	16.1*	-1.33
	(3.4)	(2.8)	(1.9)	(-0.8)
$(\beta + \delta z) \mid p25(z)$	-3.54	-4.01	-8.36	1.64
$(\beta + \delta z) \mid p75(z)$	-0.77	-1.21	-3.01	1.16
N	993	984	966	890

Table 3.1: Cross-country panel estimates of the response of macroeconomic variables to uncertainty shock and its interaction with financial development. t-statistics based on clustered robust s. e. in parentheses. Stars: * p<0.10, ** p<0.05, *** p<0.01.

results also for responses of consumption and investment to a cross-sectional uncertainty shock by estimating a regression

$$y_{it} = \alpha S_{i,t} + \beta x_{it} + \gamma z_{it} + \delta x_{it} z_{it} + \epsilon_{it}$$

where the aditional variable z captures the degree of financial development and S_{it} includes country and year fixed effects. The actual sensitivity of the economy to uncertainty is then given by $\beta + \delta z_{i,t}$, and from the previous discussion we may expect $\delta > 0$ for growth rates and $\delta < 0$ for current account.

As for the choice of z, there is a large literature studying the effects of financial development on economic growth (see, e.g., recent review in Popov 2017) which uses various proxies of development such as size of credit to private sector or stock market capitalization. Since it is not clear which specific indicator would correspond most closely to the mechanism analyzed here, I use the financial development index constructed by Svirydzenka (2016) that combines many commonly used variables collected in the World Bank's GFDD dataset. Results are presented in table 3.1 and show that the coefficients have the expected signs and most of the time are statistically significant. The heterogeneity implied by the interaction term is quantitatively quite substantial. The bottom two bottom rows present implied values $\beta + \delta z$ at values of z corresponding to 25-th or 75-th percentile of its distribution. In this comparison, a less financially developed country is more than twice as sensitive to an uncertainty shock as a more financially developed country.

3.3 Model

In order to study more closely the relation between the degree of diversification and economy's response to an uncertainty shock, this section describes a dynamic model of a small open economy with heterogeneous firms and entrepreneurs. Time is discrete and indexed by t. There is a population of entrepreneurs indexed by i, each of which operates a distinct firm. Each firm produces output Y from capital K and labor H using Cobb-Douglas technology with decreasing returns to scale:

$$Y_{i,t} = Z_{i,t} K_{i,t}^{\alpha_k} H_{i,t}^{\alpha_h}.$$

Each firm faces exogenous idiosyncratic productivity process $Z_{i,t}$. The firm hires labor on a competitive market at wage W_t in each period, so that its profits Π are

$$\Pi_{i,t} = \Pi(K_{i,t}, Z_{i,t}, W_t) = \max_H Z_{i,t} K_{i,t}^{\alpha_k} H^{\alpha_h} - W_t H.$$

Profits are used to finance investment I into capital stock, which is owned by the firm and evolves according to

$$K_{i,t+1} = (1-\delta) K_{i,t} + I_{i,t}.$$

The investment is irreversible, meaning that the firm is bound by constraint $I_{i,t} \geq 0$. Remaining funds are paid out to owners as a dividend $D_{i,t} = \Pi_{i,t} - I_t$. It is possible for a dividend to be negative, in which case the funds flow from owners to the firm (effectively, investment is financed as if by issuing new equity).

A share θ of each firm is owned collectively by all entrepreneurs, but the remaining $1-\theta$ share is owned by the individual entrepreneur who manages the firm. Dividends are distributed accordingly, so that i-th entrepreneur receives a share θ of average dividend $\bar{D}_t = \int D_{i,t} \, \mathrm{d}i$ and $1-\theta$ share of dividends from i-th firm. Each entrepreneur has a full control over the split of profits between investment and dividends in their own firm (we shall assume this is the case even if $\theta > \frac{1}{2}$). In addition, they can (individually) invest into risk-free bonds with return R. The entrepreneur's objective is to maximize discounted expected CARA utility from consumption

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta_t U(C_{i,t})\right], \quad U(C) = \frac{1}{\gamma}\left(1 - \mathrm{e}^{-\gamma C}\right)$$

subject to budget constraint

$$C_{i,t} + \frac{1}{R} B_{i,t+1} = B_{i,t} + (1-\theta) D_{i,t} + \theta \bar{D}_t.$$

One reason to use exponential utility is due to modelling convenience[4], since it leads to a closed-form steady-state policy function when $\theta = 1$ that can be used as starting point for numerical solution. It also allows, to some extent, for negative consumption, a situation which may occasionally happen if a firm with low capital receives a very good productivity shock and wishes to expand capital rapidly.

To keep the savings distribution stationary (c.f. Schmitt-Grohe and Uribe 2003), I will also assume that time preference is lower for richer individuals, or more formally, that the discount rate β_t (from period 0 to period t) satisfies

$$\beta_{t+1} = \beta_0 \exp(-\phi C_{i,t}) \beta_t.$$

so that the discount factor between periods t and $t+1$ is $\tilde{\beta}(C_{i,t}) = \beta_0 \exp(-\phi C_{i,t})$.

In addition to entrepreneurs, there is a population of homogeneous workers who supply labor. They do not have access to financial markets and thus consume their earnings in each period. The labor supply is determined by maximizing GHH preferences over consumption and leisure:

$$U^w(C^w, H^w) = C^w - \theta \frac{(H^w)^{1+\chi}}{1+\chi}$$

subject to budget constraint $C_t^w = H_t^w W_t$, where C^w, H^w describe worker's consumption and labor. The corresponding labor supply curve is then

$$H_t^w = \left(\frac{W_t}{\theta} \right)^{\frac{1}{\chi}}.$$

In equilibrium, we require that the market for labor clears, so that $\int H_{i,t} \mathrm{d}i = H_t^w$.

The process for firm-specific productivity evolves as AR(1) in logs, subject to time-varying volatility v_t:

$$z_{i,t+1} = \log(Z_{i,t+1}) = \rho_z z_{i,t} + \sigma_z \exp(v_t) \epsilon_{i,t+1} + \xi_t, \quad \epsilon_{i,t+1} \sim \mathcal{N}(0,1)$$

[4] For its tractability, CARA utility has been previously used to study consumption/savings problems with idiosyncratic risks, see e.g. Caballero (1990), Wang (2003) and Toda (2017).

where idiosyncratic standard normal shocks $\epsilon_{i,t}$ are iid over time and over households and ξ_t presents a normalizing factor ensuring $\int Z_{i,t} \mathrm{d}i = 1$ (see appendix for more details). Volatility, which serves as the source of exogenous variation for the economy, fluctuates over time according to

$$v_{t+1} = \rho_v + \sigma_v \eta_{t+1}, \quad \eta_{t+1} \sim \mathcal{N}(0, 1).$$

Let $S_{i,t} = (K_{i,t}, Z_{i,t}, B_{i,t})$ denote individual state variables. The aggregate state of the economy can be summarized by exogenous variable v_t and cross-sectional distribution over individual state, Γ_t. A *recursive equilibrium* can be characterized in terms of: policy functions for consumption $C(S, \Gamma, v)$, new capital choice $K'(S, \Gamma, v)$, and bond choice $B'(S, \Gamma, v)$, wage function $W(\Gamma, v)$, aggregate dividend function $\bar{D}(\Gamma, v)$ and law of motion for the distribution $\Gamma' = H(\Gamma, v)$ such that: 1) policy functions solve the entrepreneur's investment and consumption-saving problem, taking as given process for aggregate state, wages and aggregate dividends; 2) aggregate dividends is consistent with aggregation over individual dividends; 3) wage clears the labor market; 4) distribution evolves consistently with individual policies.

Looking closer at the entrepreneur's problem, it is possible to derive that their optimal choice must satisfy the following first order conditions:

$$\begin{aligned} 1 &= \tilde{\beta}(C_{i,t}) R \mathbb{E}_t \left[\frac{U'(C_{i,t+1})}{U'(C_{i,t})} \right] \\ 1 - \mu_{i,t} &= \tilde{\beta}(C_{i,t}) \mathbb{E}_t \left[\frac{U'(C_{i,t+1})}{U'(C_{i,t})} \left(\frac{\partial}{\partial K} \Pi(K_{i,t+1}, Z_{i,t+1}, W_{t+1}) + (1 - \delta)(1 - \mu_{i,t+1}) \right) \right] \\ & K_{i,t+1} \geq (1 - \delta) K_{i,t}, \ \mu_{i,t} \geq 0, \ \mu_{i,t} \left(K_{i,t+1} - (1 - \delta) K_{i,t} \right) = 0 \end{aligned}$$

The first condition, which describes the usual consumption smoothing, holds with equality, since we did not impose a binding borrowing constraint. The second condition, which captures optimal capital choice, involves a multiplier on the irreversibility constraint (scaled by marginal utility). The presence of multiplier effectively means that if the constraint is binding in current period, the expected marginal product of capital is lower than the current cost of investment, and if it is binding in the next period, remaining future capital has a lower value. Note that the investment FOC does not depend directly on the degree of risk-sharing θ, since higher risk-sharing lowers the entrepreneur's cost and benefit from investing in their own firm equally. It does, however, depend on entrepreneur's marginal utility when $\theta < 1$, since the condition puts higher weight on future states in which the entrepreneur individually experiences low consumption. On the other hand, if there is full risk-diversification, i.e. $\theta = 1$, each entrepreneur receives an aggregate dividend and is not affected by the profits of their company at all, so there is no heterogeneity across entrepreneurs. If we abstract from fluctuations in aggregate consumption, investment decisions in such a case will correspond to that of a risk-neutral manager discounting future profits at rate R.

In terms of solving the model, obtaining the solution without aggregate uncertainty ($v_t = 0$) is relatively straightforward. One can solve the entrepreneur's problem numerically for given wage and aggregate didvidend, then simulate a panel of agents to obtain (discrete approximation of) the cross-sectional distribution. Equilibrium values of wage and dividend are found in the outer loop imposing consistency and market clearing. To solve for dynamics with aggregate uncertainty, I follow Reiter (2009b) and use perturbation to solve for dynamics of coefficients describing the policy functions and cross-sectional distribution. Since the distribution is over a 3-dimensional state that would be hard to store directly (e.g. by histogram), I keep track only of its first and second moments. For any current moments, I reconstruct the discrete approximation to a cross-sectional distribution[5] so that it matches the moments while being close to the steady-state distribution using the maximum entropy approach described in Tanaka and Toda (2013). Given the distribution and current policy, next-period moments required to evaluate optimality conditions can be calculated easily. More details are provided in the appendix.

3.4 Results

The parameters of the model have been mostly calibrated to match selected macroeconomic moments or based on standard values used in the literature. The time period is one quarter. The persistence of the firm productivity process is set to 0.9, consistent with the range of values used in other work: for example, Gilchrist, Sim, and Zakrajšek (2014) use 0.8, Cooper and Haltiwanger (2006) estimates 0.885 at annual frequency (i.e. approx. 0.95 at quarterly freq.), Bachmann and Bayer (2013) choose a value close to unit root. The standard deviation is chosen at $\sigma_z = 0.11$ to match the average range of TFP values across firms reported in Syverson (2004). Following Veracierto (2002), labor share α_h

[5] Other approaches based on mapping from moments to space of "reference" distributions can be found in Algan, Allais, and Den Haan (2008) and Winberry (2018).

is set to 0.64 and capital share α_k to 0.21. $\delta = 0.025$ corresponds to 10% depreciation over the year. Interest rate $R = 1.015$ matches annual return to capital of 6%. Utility curvature $\gamma = 10$ implies relative risk aversion on average in the range of 2-6 (depending on average consumption of entrepreneurs). Given $\chi = 1$, I set η to match average hours worked equal to one third. Given $\phi = 0.1$, I choose β_0 to obtain zero average savings in a fully deterministic version of the economy (without aggregate or individual shocks). Regressing the (annual) measure of cross-sectional volatility from Baker and Bloom (2013) on its lag and country fixed effects yields coefficient of around 0.65 (similar results are obtained with an Arellano-Bond estimator), which corresponds to a quarterly persistence coefficient ρ_v of around 0.85. The shock to the volatility process has a standard deviation $\sigma_v = 0.13$ to match the unconditional standard deviation of the same volatility measure (after demeaning by country fixed effects).

A key parameter in the model is θ, which controls the degree of diversification in firm ownership. A model with a representative household would correspond to $\theta = 1$, while $\theta = 0$ would mean that each entrepreneur is the sole owner of their firm. The value that would best describe real economies lies somewhere in between, although likely significantly below 1. Holderness (2009) shows that, even for publicly traded firms in the US, large blockholders own, on average, 40% of a firm, and the largest shareholder owns 26%, and finds relatively similar results for other countries. The degree of diversification is even lower in case of private firms. Moskowitz and Vissing-Jørgensen (2002) report that the value of private equity in the US is comparable to the value of public equity, and for households which hold private equity, on average 70% of it is in their own business; similarly, Gentry and Hubbard (2004) find that US entrepreneurs hold 41.5% of all assets (i.e. not only equity) in their active businesses. In a large sample of European firms, Faccio, Marchica, and Mura (2011) estimate that across firms, the largest shareholder owns on average around a 63% share; across investors, the average Herfindahl index of their portfolio shares is 83% (which, if taken literally, would correspond to $\theta = 0.09$ in the model). Given the range of plausible values, I will compare two parametrizations, one with $\theta = 0.75$ and one with $\theta = 0.25$, to stand in for high and low diversification. All the parameters are summarized again in table 3.2.

Table 3.3 summarizes aggregate quantities in the steady state, i.e. with aggregate shocks turned off. In the less diversified economy, aggregate capital stock is somewhat lower compared to more diversified economy. As a result, the output and worker consumption is lower as well. Intuitively, capital is much riskier to hold for entrepreneurs and thus

parameter	value	description
β_0	0.99954	baseline discount factor
ϕ	0.1	discount sensitivity to consumption
γ	10	risk aversion
η	3.7	labor supply scale
χ	1	labor supply wage sensitivity
α_k	0.21	capital share
α_h	0.64	labor share
δ	0.025	depreciation rate
R	1.015	return on bonds
ρ_z	0.9	persistence of firm productivity
σ_z	0.11	baseline vol. of firm shocks
θ	$\{0.75, 0.25\}$	degree of diversification
ρ_v	0.85	persistence of volatility shock
σ_v	0.13	size of volatility shock

Table 3.2: Calibration of model parameters.

	K	B	Y	C_{entr}	C_{work}	$sd[\log(K_i)]$	$sd[B_i]$	effic.
$\theta = 0.75$	2.82	4.00	0.67	0.23	0.43	0.50	0.69	0.84
$\theta = 0.25$	2.17	27.89	0.62	0.58	0.39	0.53	1.11	0.84

Table 3.3: Moments of steady state without aggregate uncertainty.

they prefer to save through risk-free asset instead. Therefore the average savings in a less diversified economy is substantially larger and also more dispersed due to more volatile income of entrepreneurs. Interestingly, larger savings imply larger capital gains from financial assets, so that consumption of entrepreneurs is actually higher in a less diversified economy. We can also observe that distribution of capital across firms has roughly the same dispersion in both economies, suggesting that lack of diversification affects the overall level of capital but not its relative allocation across firms. The last column shows one possible measure of allocative efficiency, the ratio of actual output to a hypothetical output obtained if capital were allocated optimally across firms (see appendix for the precise definition), and in both cases the efficiency is around 84%.

Consumption and investment functions of entrepreneurs, as well as cross-sectional distributions over individual state variables (capital, productivity and bonds) are plotted in figure 3.3. Investment is decreasing in current capital stock and increasing in productivity, with a substantial portion of firms being constrained and investing zero. Investment is, in general, a bit lower in the low risk sharing economy, reflecting higher volatility of profits, which makes it a less attractive way of saving, and as a result the distribution of capital is also shifted to lower values. Consumption (plotted in deviations from the mean, as the two economies have different average consumption levels) is more sensitive to the state of the firm when θ is lower (which is natural, given that individual profits consistute a larger part of income in such a case), but interestingly is also more sensitive to the holding of bonds. As a result, marginal propensity to consume out of a wealth shock is higher under lower risk sharing, perhaps due to the presence of endogenous discounting and higher average consumption. Finally, the amount of savings in bonds does not seem to affect the investment choice (in the plotted case, the firm with average state variables is constrained, but the policy function for investment seems to be horizontal in the bond dimension at other points also). This is not entirely unexpected, as the model does not include more realistic forms of financial frictions such as borrowing constraints, collateral requirements or varying interest rate schedules. Further, the result also shows that the lack of diversification is not simply equivalent to other forms of financial frictions, so that any effects of uncertainty shocks must work through different mechanisms.

Turning to model dynamics, impulse responses of the main aggregates to a volatility shock are plotted in figure 3.4. The effects of shock on output are about three times stronger in the less diversified economy, causing a fall in output by about third of a percentage point (i.e. 1.2 points in terms of annualized growth rate). As predicted by the real otion channel, both economies respond to heightened volatility with a rapid drop in investment, but the drop is larger and more persistent in the less diversified economy. This leads to a larger and more persistent drop in capital stock, which is the main cause of lower output. The behavior of consumption is also different, with larger drop in the less diversified economy in agreement with the precautionary channel. In both cases, the initial drop in investment and/or consumption results in an increase in financial savings, since the volatility shock does not affect the productive capabilities of the economy upon impact. The excess savings are quickly spent again in the more diversified economy, as the investment rebounds, whereas with less diversification the rise in savings is much more persistent.

Next, figure 3.5 displays the responses of several additional variables. The reaction of aggregate capital mirrors the behavior of investment, so that the drop is more severe and persistent with less diversification. Lower capital stock generates lower demand for labor, resulting in lower wage and worker consumption. The fifth panel confirms that the

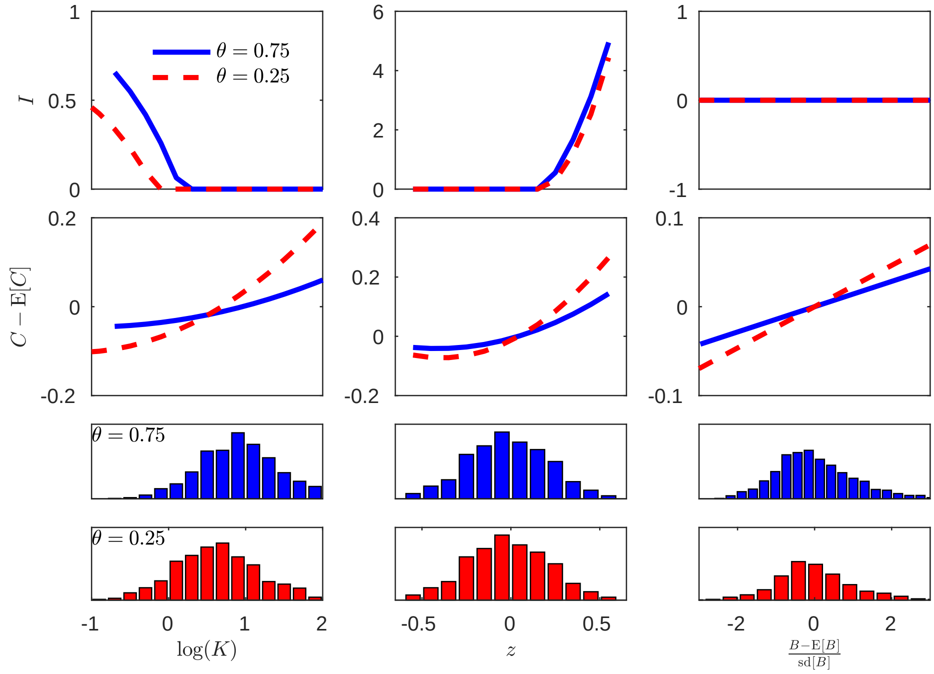

Figure 3.3: Policy functions (slices in single dimension, holding other individual state variables at their mean) and cross-sectional distributions (marginal) for entrepreneurs in the steady state of a high risk-sharing ($\theta = 0.75$) and a low risk-sharing ($\theta = 0.25$) calibration. Consumption is plotted in deviations from (calibration-specific) average value. Bond holdings are normalized by (calibration-specific) mean and standard deviation.

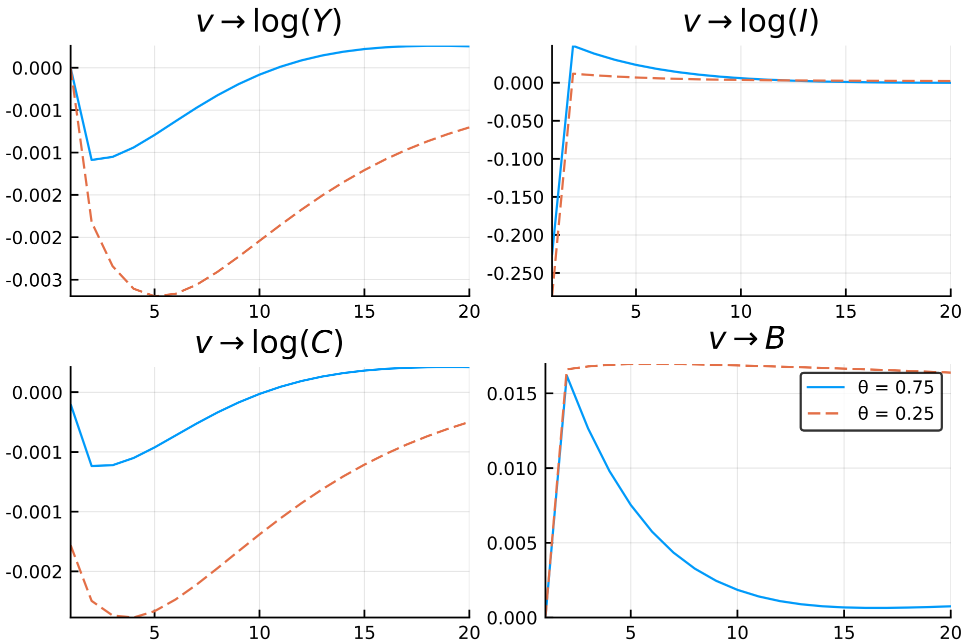

Figure 3.4: Impulse responses to a one st. dev. volatility shock.

response of entrepreneur consumption is qualitatively very different in the two economies and is responsible for most of the difference in the contemporaneous reaction of aggregate consumption. Finally, the last panel shows the response of the efficiency measure defined earlier and finds that the degree of diversification again does not seem to matter, since the drop in efficiency caused by a freeze in investment looks very similar in both economies.

Together, results shown in figure 3.4 seem to support the claims that the degree of diversification in firm ownership can play an important role in determining how the economy responds to a cross-sectional uncertainty shock affecting firm productivity. A natural question is whether the degree of diversification also matters for responses to other kinds of shocks. Figure 3.6 plots impulse responses to a shock in aggregate productivity and in the return to financial savings[6]. We can observe that the reactions of output and investment are virtually identical in the two economies, reacting positively to higher productivity and negatively to higher interest rate (which represents higher cost of capital). The behavior of consumption and savings is, however, somewhat different, with the response of consumption being more muted in the less diversified economy.

[6]More formally, the firm's production function is extended to $Y_{i,t} = \mathrm{e}^{a_t} Z_{i,t} K_{i,t}^{\alpha_k} H_{i,t}^{\alpha_h}$ and constant interest rate is replaced with $R_t = R\mathrm{e}^{r_t}$, in which a_t and r_t follow independent zero-mean AR(1) processes with parameters of ρ_a, σ_a and ρ_r, σ_r. The parameters used to generate the figure are $\rho_a = \rho_r = 0.9$ and $\sigma_a = 0.01, \sigma_r = 0.0025$.

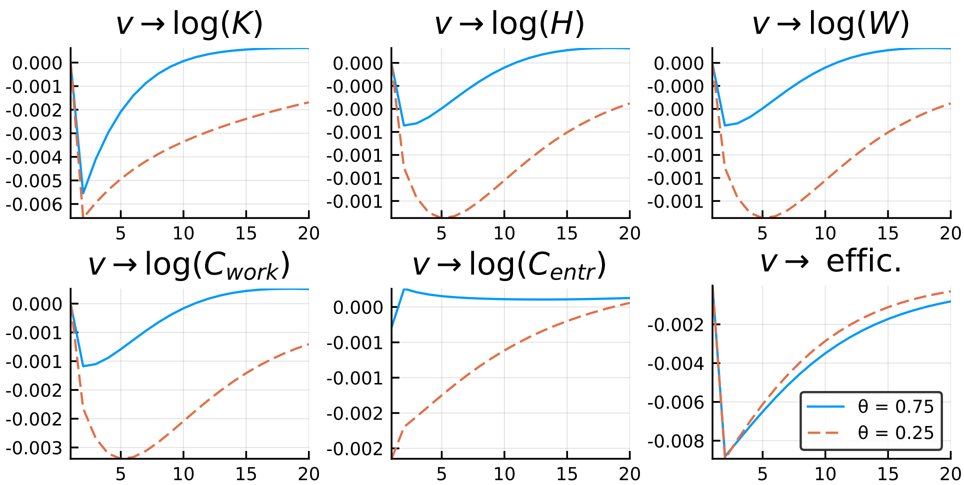

Figure 3.5: Additional impulse responses to a one st. dev. volatility shock.

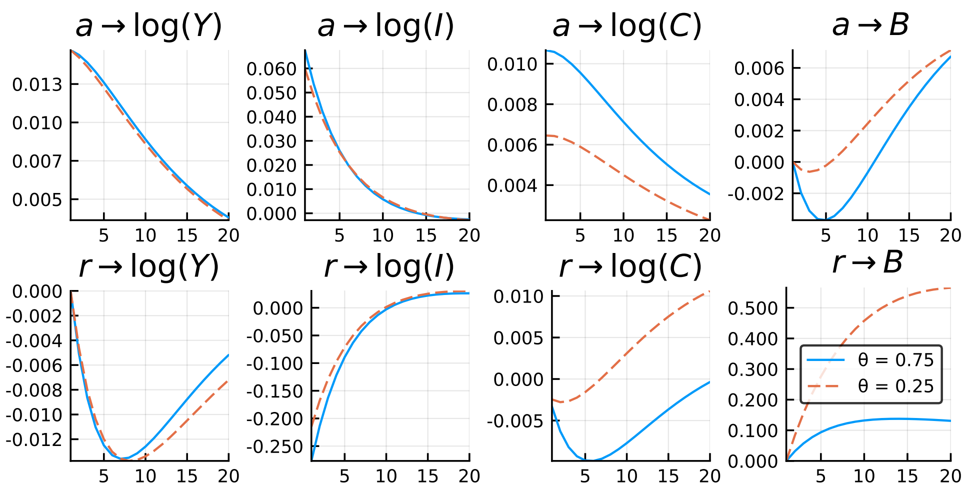

Figure 3.6: Impulse responses to productivity and interest rate shocks.

3.5 Conclusion

It is well known that an increase in the volatility of idiosyncratic productivity shocks can lead to a drop in economic activity due to the real option effect. This paper uses a dynamic model with heterogeneous firms and firm owners as well as cross-country empirical evidence to investigate an additional channel through which an uncertainty shock can be amplified and propagated. When risk-averse entrepreneurs are exposed to an increase in volatility because lack of diversification in their portfolio makes their income process more risky, they will respond to higher volatility by further cutting back on their consumption. Moreover, the fact that returns to capital become more risky will provide an additional incentive to reallocate their savings from risky capital to risk-free asset. As a result, the response of the economy to the uncertainty shock is stronger and more persistent if the degree of diversification is lower.

The model presented is relatively simple and includes only one, rather extreme form of investment irreversibility. It also abstracts from specific sources behind the lack of diversification faced by firm owners, which is instead introduced in a stylized way. It could thus be potentially useful to study a model with a richer structure of adjustment costs and more explicit microfoundations in future. In addition, it might be interesting to introduce nominal rigidities that would allow for simultaneous drops in investment and consumption to further depress output through the aggregate demand channel. It is likely that in such setting a low degree of diversification would lead to even greater amplification of an uncertainty shock.

3.A Appendix

3.A.1 Two-period model

Consider first order conditions:

- autarky:

$$u'(W-A-K) = \mathbb{E}\left[u'\left(A+D(K,Z,\eta)\right)\right]$$
$$u'(W-A-K) = \mathbb{E}\left[u'\left(A+D(K,Z,\eta)\right)\times\frac{\partial D}{\partial K}(K,Z,\eta)\right]$$

- diversification:

$$u'(W-A-K) = u'\left(\mathbb{E}\left[A + D(K,Z,\eta)\right]\right)$$
$$u'(W-A-K) = u'\left(\mathbb{E}\left[A + D(K,Z,\eta)\right]\right) \times \mathbb{E}\left[\frac{\partial D}{\partial K}(K,Z,\eta)\right]$$

1. With Z known in advance, every firm chooses optimal capital right away. η thus plays no role in determining payoff and with no uncertainty, first order conditions of both problems will coincide.

2. Combining diversified FOCs and writing out expectation wrt. η explicitly, we have conditions for $\bar{K}$ and K^{d}

$$1 = (1-p)\frac{\partial D}{\partial K}\left(\bar{K},\bar{Z},0\right) + p\frac{\partial D}{\partial K}\left(\bar{K},\bar{Z},1\right)$$
$$1 = \mathbb{E}\left[(1-p)\frac{\partial D}{\partial K}\left(K^{\mathrm{d}},Z,0\right) + p\frac{\partial D}{\partial K}\left(K^{\mathrm{d}},Z,1\right)\right]$$

   Note that marginal profitability of initial capital $\frac{\partial D}{\partial K}$ is a mix of functions which are linear and convex in Z. Moreover for $\eta = 1$ and values $Z > \bar{Z}$ (and since $\bar{Z}$ is mean of Z, such values exist with positive probability) irreversibility will not be binding and $\frac{\partial D}{\partial K}$ will be strictly convex. From Jensen inequality we then have

$$1 > \mathbb{E}\left[(1-p)\frac{\partial D}{\partial K}\left(\bar{K},Z,0\right) + p\frac{\partial D}{\partial K}\left(\bar{K},Z,1\right)\right]$$

   and thus the expected marginal profit is too low at the deterministic capital choice. Since $\frac{\partial D}{\partial K}$ is decreasing in capital, the optimal choice must necessarily be lower than $\bar{K}$.

3. By combining autarky FOC we get

$$\mathbb{E}\left[u'\left(A + D(K^{\mathrm{a}},Z,\eta)\right)\right] = \mathbb{E}\left[u'\left(A + D(K^{\mathrm{a}},Z,\eta)\right) \times \frac{\partial D}{\partial K}(K^{\mathrm{a}},Z,\eta)\right]$$

   Decompose the term on the right side according to $\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y] + \mathrm{Cov}(X,Y)$ and rearrange to get

$$1 - \mathbb{E}\left[\frac{\partial D}{\partial K}(K^{\mathrm{a}},Z,\eta)\right] = \mathrm{Cov}\left(\frac{u'\left(A + D(K^{\mathrm{a}},Z,\eta)\right)}{\mathbb{E}\left[u'\left(A + D(K^{\mathrm{a}},Z,\eta)\right)\right]}, \frac{\partial D}{\partial K}(K^{\mathrm{a}},Z,\eta)\right)$$

Marginal profit is increasing in Z while the marginal utility of consumption in the second period is decreasing in Z, so the covariance term will be negative and expected marginal profit is thus greater than 1. Recalling FOC for diversified choice, we have

$$\mathbb{E}\left[\frac{\partial D}{\partial K}(K^{\mathrm{a}}, Z, \eta)\right] > 1 = \mathbb{E}\left[\frac{\partial D}{\partial K}(K^{\mathrm{d}}, Z, \eta)\right]$$

and therefore $K^{\mathrm{a}} < K^{\mathrm{d}}$.

4. Consider autarkic FOC for savings written in terms of consumption: $u'(C_1^{\mathrm{a}}) = \mathbb{E}[u'(C_2^{\mathrm{a}})]$. Since we assumed marginal utility is convex, this implies $u'(C_1^{\mathrm{a}}) > u'(\mathbb{E}[C_2^{\mathrm{a}}])$, meaning that first-period consumption is expected to be lower than second-period consumption, or

$$C_1^{\mathrm{a}} < \frac{1}{2}\left(C_1^{\mathrm{a}} + \mathbb{E}\left[C_2^{\mathrm{a}}\right]\right)$$

The expected sum of consumption over the two periods can be written as

$$\left(C_1^{\mathrm{a}} + \mathbb{E}\left[C_2^{\mathrm{a}}\right]\right) = W + \mathbb{E}\left[D(K^{\mathrm{a}}, Z, \eta)\right]$$

and similarly for the diversified case

$$\left(C_1^{\mathrm{a}} + C_2^{\mathrm{a}}\right) = W + \mathbb{E}\left[D(K^{\mathrm{d}}, Z, \eta)\right]$$

(note that diversified consumption is deterministic)

Recalling earlier FOC for diversified capital choice K^{d}, we see that it maximizes the right hand expression $W + \mathbb{E}\left[D(K, Z, \eta)\right]$. Since we have also established that $K^{\mathrm{a}} < K^{\mathrm{d}}$, we must have total average autarky consumption lower than total diversified consumption:

$$\frac{1}{2}\left(C_1^{\mathrm{a}} + \mathbb{E}\left[C_2^{\mathrm{a}}\right]\right) < \frac{1}{2}\left(C_1^{\mathrm{d}} + C_2^{\mathrm{d}}\right)$$

The last expression is, however, equal to C_1^{d} since diversified entrepreneurs will choose same consumption in both periods, so then $C_1^{\mathrm{a}} < C_1^{\mathrm{d}}$.

3.A.2 Data

Uncertainty. Measures of uncertainty are taken from a cross-country panel dataset collected by Baker and Bloom (2013). These include the log of stock market volatility (lavgvol) computed from daily returns in a quarter (then averaged over last four quarters), and the log of cross-sectional volatility (lavgcs_vol) computed from the standard deviation of quarterly individual stock returns of different firms (then also averaged over 4 quarters). The original dataset is quarterly, but I use fourth quarter values to obtain the annual uncertainty measure.

Financial development. I use the financial development index constructed by Svirydzenka (2016) based on indicators collected in the World Bank's Global Financial Development Database, more specifically the main overall index (FD).

The two datasets were downloaded from N. Bloom's website[7] and the IMF's website[8] respectively. I am grateful to the authors for making their data available.

Macroeconomic variables. From the World Bank's WDI dataset, I use observations on real GDP growth (NY.GDP.MKTP.KD.ZG), real household final consumption expenditure growth (NE.CON.PRVT.KD.ZG), real gross capital formation growth (NE.GDI.TOTL.KD.ZG) and current account balance relative to GDP (BN.CAB.XOKA.GD.ZS), with the last entering estimation in differences.

3.A.3 Dynamic model

Normalizing productivity: Since productivities follows linear AR(1) process in logs, fluctuations in volatility would change the first moment of productivity distribution through the Jensen inequality term. To ensure that $\int \exp(z_{i,t})\mathrm{d}i = 1$, the law of motion for z is modified to

$$z_{i,t+1} = \rho_z z_{i,t} + \sigma_z \exp(v_t)\epsilon_{i,t+1} + \xi_{t+1}$$

where the shift ξ_{t+1} depends on exogenous volatility and auxiliary state variable tracking cross-sectional dispersion x_t:

$$\xi_{t+1} = \frac{1}{2}\left(\rho_z(1-\rho_z)x_t - \sigma_z^2 \exp(2v_t)\right)$$

[7] https://people.stanford.edu/nbloom/sites/default/files/bakerbloom2.zip
[8] http://www.imf.org/external/pubs/cat/longres.aspx?sk=43621

and x_t follows

$$x_{t+1} = \rho_z^2 x_t + \sigma_z^2 \exp(2v_t).$$

Efficiency: Define the firm's output after optimizing over labor

$$Y^*(K, Z; W) = ZK^{\alpha_k} (H^*)^{\alpha_h}, \quad H^* = \arg\max_H ZK^{\alpha_k} H^{\alpha_h} - WH$$

I define efficiency as the ratio of actual output to "potential output", where the latter would be obtained if capital were optimally reallocated across firms, taking distribution of productivities Z_i and wage W as given. For that to be the case, the marginal products of capital must be equalized across firms, so that $\frac{\partial Y^*}{\partial K}$ is constant for each firm. This implies that capital must be proportional to $Z_i^{\frac{1}{1-\alpha_k-\alpha_h}}$, meaning that

$$K_i = \left(\frac{Z_i^{\frac{1}{1-\alpha_k-\alpha_h}}}{\int Z_j^{\frac{1}{1-\alpha_k-\alpha_h}} \, \mathrm{d}j} \right) \bar{K}$$

if $\bar{K}$ is average capital. After some algebra, the corresponding optimal output is given by

$$\bar{Y}^{\text{opt}} = \left(A \left(\frac{\alpha_L}{W} \right)^{\alpha_L} \left(\int Z_i^{\frac{1}{1-\alpha_K-\alpha_L}} \, \mathrm{d}i \right)^{1-\alpha_K-\alpha_L} \bar{K}^{\alpha_K} \right)^{\frac{1}{1-\alpha_L}}$$

which can be easily computed given wage and cross-sectional distribution of productivity.

Individual control variables: The solution method requires first order conditions expressed in terms of equalities. The approximated decision variables will consist of: (for simplicity I supress endogenous discounting and aggregate uncertainty)

- "pseudomultiplier" $\tilde{\mu}(K, Z, B)$ (based on Sim 2007 satisfying

  $$\tilde{\mu}(K, Z, B) =$$
  $$1 - \beta \mathbb{E}_{Z'|Z} \left[\frac{U'(C(K', Z', B'))}{U'(C(K, Z, B))} \left(\frac{\partial}{\partial K} \Pi(K', Z') + (1-\delta)(1 - \max\{\tilde{\mu}(K', Z', B'), 0\}) \right) \right]$$

  evaluated at $K' = (1-\delta)K$ and B' implied from capital and consumption choice. When the irreversibility constraint is binding, the condition is the same as for the true multiplier, and when it is not binding, the pseudomultiplier is nonpositive. The actual multiplier is thus $\mu = \max\{\tilde{\mu}, 0\}$.

- unconstrained target capital $K^*(K, Z, B)$ that the firm would choose if the irreversibility constraint were dropped in current period:

$$1 = \beta \mathbb{E}_{Z'|Z} \left[\frac{U'(C(K', Z', B'))}{U'(C(K, Z, B))} \left(\frac{\partial}{\partial K} \Pi(K', Z') + (1 - \delta)(1 - \max\{\tilde{\mu}(K', Z', B'), 0\}) \right) \right]$$

evaluated at $K' = K^*(K, Z, B)$ and B' implied from capital and consumption choice. Actual future capital will then be the higher of target capital or bound from irreversibility.

- consumption, which satisfies the original Euler equation evaluated at actual future capital:

$$1 = \beta R \mathbb{E}_{Z'|Z} \left[\frac{U'(C(K', Z', B'))}{U'(C(K, Z, B))} \right]$$

evaluated at $K' = \max\{K^*(K, Z, B), (1 - \delta)K\}$ and B' implied from capital and consumption choice.

Approximating policy functions: the policy functions defined above are approximated with the tensor product of Chebyshev polynomials. Let vector b collect all the polynomial coefficients. With aggregate uncertainty, policy functions expressed as functions of an individual state will change over time according to the overall state of the economy, so b_t varies over time. To pin down its dynamics, require that the above conditions evaluated with given coefficients hold with equality at a set of collocation nodes. This implies a system of forward-looking equations that links b_t and b_{t+1} (conditional on the current and future state of the economy, but averaging over individual shock realizations).

Steady state solution: For given aggregate wages and dividends, coefficients of individual policy functions are found by using a projection method, i.e. by solving the nonlinear system in b implied by the steady-state version of the residual for b_t, b_{t+1} described above. Given the policy functions, a panel of agents is simulated and final period distribution is used to compute the aggregate dividend and market-clearing wage implied by the aggregate labor demand. Wages and dividends themselves are found in an outer fixed-point loop. To achieve convergence, I start by solving the model for $\theta = 1$, which has an analytic solution for the consumption function, use that solution as the starting point for the model with lower θ, and so on in multiple steps until the actual desired value of θ is reached.

Approximating cross-sectional distribution: The steady state solution yields a simulated panel of agents from which I construct a discrete approximation to the cross-sectional distribution by using an empirical distribution of simulated households in the last period (i.e. the points of the distribution are observed individual states, the weights of the distribution are uniform). Evaluating first order conditions for the dynamic solution requires current wage and aggregate dividend, which in turn depend on time-varying cross-sectional distribution over (K, Z, B), but storing the full distribution as a state variable is intractable. I approximate the distribution by its first and second moments (with the exception of purely exogenous moments of productivity), which become state variables in the model. Given moments, I "reconstruct" the full distribution by tweaking the weights on discrete points of the reference, steady state distribution so that the reconstructed distribution satisfies the moments and is close to the reference distribution in the Kullback-Leibler information sense. This approach, described in Tanaka and Toda (2013), first solves a minimization problem for a vector of multipliers (one for each moment), then expresses new weights as an explicit function of multipliers. I include multipliers as additional control variables and append first order conditions from the minimization problem as another set of equations to the model. Once the current distribution is obtained, applying the policy functions yields the next-period distribution and thus implies the law of motion for the moments.

Dynamic solution: Let X_t be a vector collecting all aggregate state variables in the model (exogenous variables, distribution moments and possibly others), and Y_t collect all aggregate control variables (coefficients of policy function approximation, multipliers on distribution moments and any other variables of interest). Collect all the model equations into residual function F that satisfies $\mathbb{E}_t \left[F(X_{t+1}, Y_{t+1}, X_t, Y_t) \right] = 0$ (expectation is wrt. aggregate shocks). The system is linearized around the steady state (with derivatives evaluated by automatic differentiation)[9] and the linear solution is then obtained by standard methods (Klein 2000).

Accuracy: There are two main approximations involved when solving the model. First, individual policy functions are approximated by low order polynomials. Table 3.4 shows the mean and median absolute value of residuals from an individual entrepreneur's Euler equations (as defined previously in this appendix), where the moments are computed in the steady state using the reference cross-sectional distribution. The residuals are hard to interpret straightforwardly, but since the first order conditions are defined in the form of a ratio or product of terms being equal to one, the Euler equation terms implied

[9] The model is implemented in Julia 0.6 using `ForwardDiff.jl` package (Revels, Lubin, and Papamarkou 2016) for differentiation and `BasisMatrices.jl` package (`https://github.com/QuantEcon/BasisMatrices.jl`) for function approximation.

	$\theta = 0.75$		$\theta = 0.25$	
	mean	median	mean	median
$\lvert e_\mu \rvert$	0.0361	0.0245	0.0689	0.0470
$\lvert e_k \rvert$	0.0405	0.0340	0.0818	0.0658
$\lvert e_c \rvert$	0.0352	0.0267	0.0745	0.0574

Table 3.4: Mean and median absolute residual from model optimality conditions in the steady state.

by the approximate policy functions seem to be within a couple of percentage points of their correct values. Second, when solving for the aggregate dynamics, the cross-sectional distribution is summarized with a finite number of moments. Figure 3.7 plots a comparison of impulse responses when a different number of moments is used for this purpose. Either only the first moments are used (line denoted "less"), second moments as in the rest of the results section (line "base"), or second moments with added third moments in capital and bonds (line "more"). The results seem to be qualitatively comparable, although the responses with only first moments seem to be sufficiently different to justify the inclusion of higher moments for tracking the distribution.

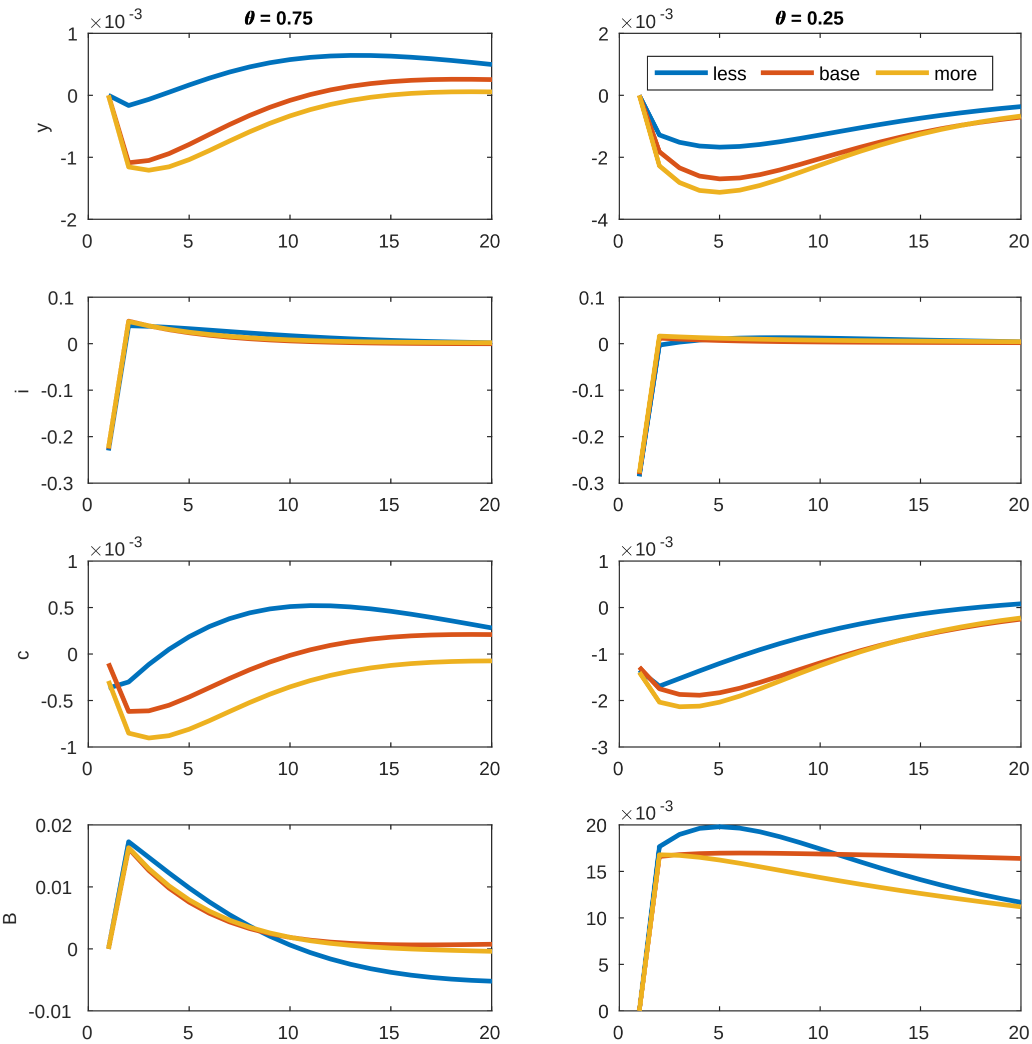

Figure 3.7: Impulse response with different number of moments for tracking the cross-sectional distribution.

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