EQUITY PRICES UNDER BAYESIAN DOUBT ABOUT MACROECONOMIC FUNDAMENTALS

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ABSTRACT. I present a consumption-based explanation of a number of phenomena in the aggregate equity market. The model invokes the recursive utility function of Epstein and Zin (1989), configured with the plausible parameters of the average coefficient of the aversion to late resolution of uncertainty of about 22, and the elasticity of intertemporal substitution of 1.5. Statistically hard to discriminate in less than 80 years of data from the ubiquitous *i.i.d.* model of real consumption growth, the endowment process is specified as being subject to sporadic large shocks and incessant small shocks. The large infrequent shocks, modelled by means of a four-state hidden Markov chain, display interesting macroeconomic regularities, occuring at both the business-cycle, and a lower, frequencies. Despite the fact that the levels of endowments are observable, the source of their variation cannot be detected perfectly, facing investors with a complex signal-extraction problem. The associated posterior probabilities provide a natural link between the observed asset value fluctuations and the economic uncertainty within the rational Bayesian learning framework. Although computationally arduous, having to be solved on a high-performance computing machine in a low-level language, the model is able to account for (i) the observed magnitude of the equity premium, (ii) the low and stable risk-free rate, (iii) the magnitude and the countercyclicality of risk prices, (iv) the average levels and the procyclicality of price-dividend and wealth-consumption ratios, (v) the long-horizon predictability of risk premia, and (vi) the overreaction of price-dividend ratio to bad news in good times, all within the conceptually simple representative-agent framework.

Key words and phrases. Asset Pricing, Equity Premium Puzzle, Risk-Free Rate Puzzle, Time-Series Predictability, Bayesian Uncertainty, Stochastic Differential Utility, Low-Frequency Risk, High-Performance Computing.
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I. Introduction

This paper explores an ingeniously simple channel that links the real and financial sectors of the macroeconomy. Its arguably ambitious aim is to retort to such key questions as why asset values swing over time, soaring in booms just to tumble in recessions, or why the risk premiums are so sizable and persistently time-varying whereas real consumption growth appears so smooth and unpredictable. Previous macroeconomic literature until fairly recently completely missed this seemingly immaterial link due to its undue reliance on the expected utility framework, wherein the axiom of the reduction of compound lotteries has rendered a Bayesian doubt about macroeconomic growth prospects irrelevant.

The kernel of the paper consists in the macroeconomic fundamentals being modelled jointly at two leading spectral frequencies by the statistically independent composition of a pair of two-state continuous-time hidden Markov chain models (HMM), an intriguing idea first raised in Hansen 2007 Ely lecture. The critical implication is that the modeller's leeway in her choice of the degree of the persistence of the corresponding expected consumption growth is dramatically restricted; the persistence is in fact fully dictated by the respective durations of the Markov chain states. This check is unfortunately absent in many current models of the stock market wherein the modeller simply feeds in a highly persistent AR(1) sequences of latent variables.

The incompleteness of information leads to an intricate signal-extraction problem (Wonham 1964, Liptser and Shiryaev 2001), the outcome of which is a vector of posterior probabilities of the macroeconomy being in a particular growth state, and it is these likelihoods, rather than, for instance, the conditional volatilities of consumption growth, that naturally fulfill the role of yardsticks of Bayesian doubt about macroeconomic fundamentals.

The dynamics of these beliefs is again dictated, this time by the highly restrictive Bayes' theorem, engendering a surprisingly strong nonlinear dynamics, in particular in the assets' risk premiums, which the current linear (log-linearized) models at the forefront of research completely miss. Although latent, these likelihoods are easily estimable from the NIPA real nondurable consumption and services data by means of the well-known Hamilton (1989) filter.

The first two-state HMM model approximates the income growth dynamics at the business-cycle frequency; its companion transition probability matrix exactly matches the point estimates obtained by Cagetti, Hansen, Sargent and Williams (2004). Their data correspond to the time-series of the Solow residual, and the posterior probabilities from the Hamilton (1989) filter closely track the NBER recessions. Expected income growth variation at this particular frequency is one of the keys to reproduce the observed countercyclicality of risk prices, and the concomitant long-horizon predictability of assets' risk premiums, but fails utterly in engendering a successful match of the respective magnitudes. It has been introduced in recent studies of endowment economies, featuring expected utility households, most notably in David (1999) and Veronesi (1999, 2000), and as lately as 2008 in Chen and Pakoš, who consider the case of the Kreps and Porteus (1978) households. In a parallel strand of literature, Hansen (2007) explores the asset-pricing ramifications of investors' worry about potential misspecification of such a model of fundamentals in an exchange economy, whereas Cagetti et. al analyze the dynamics of asset prices from the vantage point of a stochastic growth economy.

The second two-state HMM model captures the income growth dynamics at a lower frequency, with the average duration of high-growth state around 12 years and the low-growth state around 5 years. Chen and Pakoš (2008) explore this channel with a two-state Markov chain, and find that this time the model matches the magnitudes of observed risk premia, but unfortunately not the time-series properties of asset prices. My calibration of the low-frequency dynamics of the income growth, however, differs from Chen et. al. I closely follow the ideas of Hansen and Sargent by matching the income process in the model to the NIPA real consumption, and consequently using the detection-error probabilities methodology to ensure that (i) the mistake probability between the integrated four-state HMM model and the ubiquitous *i.i.d.* model is more than 5% in less than 80 years of data, making precise inference statistically hard, and (ii) the relative entropy of these two competing models is maximized so that I achieve a meaningful low-frequency risk.

The households' utility function that naturally prescribes an aversion to Bayesian uncertainty is introduced by means of the recursive Epstein and Zin (1989) preferences. The behavioral axioms of choice behind such preferences specifically do not list the postulate of the reduction of compound lotteries. Compared to the case of expected utility, investors cease to be indifferent to the temporal distribution of risk. They may either love or abhor, depending on the parameter configuration, the way uncertainty about future economic prospects unfolds. And it is the aversion to such an uncertainty, hereafter to be gauged by the likelihoods of the macroeconomy being in a particular growth state, that is the key behavioral premise of the model. For illustration, a suddenly heightened doubt about future macroeconomic growth prospects will tend to lengthen the average time up to the future resolution of the respective uncertainty, a situation particularly feared by Bayesian investors with recursive preferences, who immediately respond by bidding down stocks and up real bonds. And as these likelihoods about fundamentals are updated by Bayes' theorem upon the arrival of new information, prices and hence expected returns change. The model therefore naturally generates variation in the conditional distribution of returns and prices, with homoscedastic consumption growth.

The state-independent preferences case is configured with the aversion to late resolution of uncertainty[1] of about 35.5, the elasticity of intertemporal substitution EIS of 1.5, and the subjective rate of time preference of 1.5% per annum. I subsequently refine the model's implications by considering the case of variable aversion to late resolution of economic uncertainty, holding its average at 22, below the Bansal and Yaron's (2004) value of 28, and variable consumption impatience, holding its average close to 1% per annum. All parameter constellations, across all states of the world, do imply a feature called "preference for early resolution of uncertainty", which makes households shy away from assets with random payoffs the uncertainty of which resolves in the distant future. This essential characteristic of the model is precluded within the expected utility framework due to the axiom of the reduction of compound lotteries. Interestingly, I find many asset-pricing results qualitatively robust to the introduction of the preference state-dependence, but quantitatively enhanced.

Impelled partially by tractability, I endogenize the structural preference parameters by rendering them mildly wealth contingent. I justify the wealth dependence of consumption impatience by appealing to the well-known Becker and Mulligan's (1997) hypothesis that wealth causes patience. The endogeneity of the aversion to late resolution of uncertainty is motivated by the economic intuition that households tend to fear a given economic uncertainty about future growth prospects significantly more in bad times (times of low consumption growth).

The asset pricing implications of this economic setting are rich and surprisingly wildly nonlinear. The uncertainty about the low-frequency income growth dramatically raises the risk prices, by more than an order of magnitude, in comparison to Hansen (2007); the model succeeds in matching the observed equity premium with a small term premium (Abel 1999, Pakoš 2008). The mean price-dividend ratio is close to its sample counterpart, and is significantly below the mean wealth-consumption ratio, broadly consistent with the recent empirical study of Lustig and Nieuwerburgh (2008). As conjectured, the time-series properties of risk prices are driven predominantly by the business-cycle component of the expected income growth that engenders their countercyclical variation. In consequence, aggregate wealth and equity risk premiums are forecastable by the wealth-consumption and price-dividend ratios. The respective linear-projection coefficients are of the correct signs and the magnitudes commensurate with real data. In case of state-dependent preferences, the time variation in the aversion to late resolution of uncertainty further enhances these effects. Hence, the model is broadly consistent with the empirical variance decompositions of the price-dividend ratio, tending to attribute the bulk of the large swings in prices to expected returns, especially in short samples.

[1] The yardstick of the aversion to late resolution of uncertainty is, in Epstein and Zin (1989) notation, defined to be $1 - \alpha/\rho$.

There are important related papers, mostly penned by Professor Bansal and his coauthors. In particular, the provocative Bansal and Yaron (2004) investigates the ability of a hard-to-detect persistent consumption growth to account for stylized features of the equity market. Their economic story, however, is a polar opposite to mine. Although configuring their recursive utility specification so that it exhibits a preference for early resolution of uncertainty, they do so implicitly, without emphasizing its gravity[2]. In fact, the information structure in their model is complete, and they are forced to couch their analysis in terms of households preferring a less persistent process to a more persistent one, that is to say, a constant expected income growth to a highly autocorrelated one. Therefore, their interpretation is different to the one in both this paper, and its precursor in Chen and Pakoš (2008), who explicitly feature incomplete information, attributing the swings in expected returns, Sharpe ratios, price-dividend ratios, and other variables of immense interest to financial economists, to a variation in macroeconomic uncertainty about growth prospects. That there undoubtedly is an uncertainty about growth prospects in the canonical Bansal and Yaron model, but still no variation, in for example, expected (excess) returns, or Sharpe ratios, only further illustrates the diametrically opposite economic mechanisms at work. To get these into motion, Bansal and Yaron are forced to lift the assumption of the homoscedasticity of consumption growth innovations. In such a case, the expected returns do rise and fall but precisely because the consumption volatility does. In contrast, the expected returns, Sharpe ratios, etc., in this paper are in a natural state of motion despite homoscedastic consumption growth innovations, due to the fluctuating confidence about macroeconomic fundamentals.

Finally, in a recent paper, Ai (2007) analyzes the role of the degree of information quality in determining the level of the equity premium. Although his setting is different, he makes a noteworthy discovery that risk premiums tend to rise with higher uncertainty about the long run risk.

II. Model

A. **Primitives.**

a. *Preferences.* Consider an exchange economy of Lucas (1978), with a single perishable consumption good c_t, populated by a continuum of identical investors of measure one. Assume further that their preferences are recursive as in Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1989), later extended to the continuous-time setting by Duffie and Epstein (1992ab). Formally,

$$J_0 \;=\; \mathbb{E}\left[\int_0^\infty f\left(c_s,\, J_s\right) ds\right],$$

[2] I was unable to find a single reference to the term 'preference for early resolution of uncertainty' in their paper.

with the indirect utility function as of time t denoted J_t. The aggregator function[3] $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is of the constant-elasticity-of-substitution (CES) form, which in continuous-time setting means that

$$f(c,\, J) \;=\; \frac{\delta}{\rho}\,\frac{c^\rho - (\alpha J)^{\rho/\alpha}}{(\alpha J)^{(\rho/\alpha)-1}}. \tag{II.1}$$

The parameter δ is the measure of the consumption impatience of households, $(1-\rho)^{-1}$ denotes the elasticity of intertemporal substitution EIS, and $1-\alpha$ is the coefficient of the relative risk aversion, commonly defined in terms of atemporal gambles over consumer's wealth as $W \times \partial_{WW} J / \partial_W J$. These structural parameters are all constant in the benchmark state-independent preference setting, and mildly wealth-dependent[4] in the refined state-dependent case, consistent with the Becker and Mulligan's (1997) hypothesis that "wealth causes patience", and the economic intution that households tend to be significantly more averse to a given economic uncertainty about future growth prospects in bad times.

I note here, and demonstrate hereafter, that it is the variability in the coefficient of the preference for early resolution of uncertainty $1-\alpha/\rho$, not the risk aversion $1-\alpha$ itself, that is behind the enhanced performance of the state-dependent calibration. It just turns out that the recursive preferences, configurable with only 3 parameters, tightly link the aversion to risk and aversion to late resolution of economic uncertainty (technically termed the preference for the early resolution of uncertainty). The risk aversion tells how much money households are willing to give up to avoid a fair gamble taken right now, the uncertainty is resolved immediately. The preference for early resolution of uncertainty tells, in contrast, how much households dislike the fact that the uncertainty about their endowment is about to be resolved far in the future, not immediately. In fact, shutting off such aversion to the economic uncertainty, one needs wildly implausible variation in the relative risk aversion coefficient to match, for example, the time-series predictability results from the empirical finance literature.

As I demonstrate hereafter, the equilibrium wealth-endowment ratio is a function of the vector of the posterior probabilities $\boldsymbol{\pi}$, defined later, that capture the likelihood of the economy being in various states, such as a recession or a boom. As a result, I may parameterize the parameters δ and α as linear functions of these likelihoods, namely,

$$\begin{aligned}\delta\,(\boldsymbol{\pi}) &= [\delta_1, \ldots, \delta_4] \times \boldsymbol{\pi}\\ \alpha(\boldsymbol{\pi}) &= [\alpha_1, \ldots, \alpha_4] \times \boldsymbol{\pi}\end{aligned}$$

[3]This is the ordinarily equivalent aggregator for which the variance multiplier is zero. See Duffie and Epstein (1992ab) for more details.

[4]The assumption that δ and α depend on the aggregate rather than individual wealth is purely for tractability reasons; in computing the first-order conditions I do not have to take partial derivatives with respect to δ and α, which immensely simplifies the mathematics. This point is also related to recent debates of internal vs. external habit specification. Of course, fully endogenizing these parameters may further enrich the model's implications.

where $\delta_i < \delta_j$ and $\alpha_i < \alpha_j$ for $i < j$.

b. *Endowments.* Each investor is endowed with an identical Lucas tree, yielding a perfectly observable endowment flow at the rate C_t per unit of time, the dynamics of which is given by the following differential equation

$$\frac{dC_t}{C_t} = \kappa_t \, dt + \sigma_{\mathrm{C}} \, dZ_{1t} \tag{II.2}$$

where the Brownian shock dZ_{1t} is normally distributed with mean zero and variance dt, the endowment growth volatility σ_C is constant, and the true expected endowment growth rate κ_t is latent, being subject to sporadic large changes at random times. It follows a hidden Markov chain with four states that are known, and are denoted κ_i, $i = 1, \ldots, 4$. The four states allow me to model the variation in the expected endowment growth rate at the business-cycle (BC) and a lower (LF) frequencies jointly. The states decompose into statistically-independent BC component, denoted θ_t, and the LF component, denoted ϑ_t. Formally, $\kappa_t = \theta_t + \vartheta_t$; its transition matrix is denoted $\boldsymbol{P}_\kappa \, (dt)$. Each component itself follows a two-state hidden Markov chain with $\theta_t = \overline{\theta}, \underline{\theta}$, and $\vartheta_t = \overline{\vartheta}, \underline{\vartheta}$. Without loss of generality, the states are ordered as $\kappa_1 = \overline{\theta} + \overline{\vartheta}$, $\kappa_2 = \underline{\theta} + \overline{\vartheta}$, $\kappa_3 = \overline{\theta} + \underline{\vartheta}$ and $\kappa_4 = \underline{\theta} + \underline{\vartheta}$, where $\underline{\theta} < \overline{\theta}$ and $\underline{\vartheta} < \overline{\vartheta}$. The respective transition probability matrices for the time interval $(t, \, t + dt)$ are denoted $\boldsymbol{P}_\theta(dt)$ and $\boldsymbol{P}_\vartheta(dt)$.

Lemma 1. *The 4×4 transition probability matrix $\boldsymbol{P}_\kappa \, (dt)$ may be written as the Kronecker product of the associated 2×2 transition matrices for the low-frequency dynamics, $\boldsymbol{P}_\vartheta(dt)$, and the business-cycle dynamics $\boldsymbol{P}_\theta \, (dt)$. Formally*

$$\boldsymbol{P}_\kappa \, (dt) \quad = \quad \boldsymbol{P}_\vartheta(dt) \otimes \boldsymbol{P}_\theta \, (dt)$$

Proof. This is just an application of simple probability laws, given the statistical independence of θ_t and ϑ_t, and the ordering of the states of κ_t. □

c. *Market Completeness.* The asset market is complete despite the signal extraction problem as investors are able to attain any claim that is contingent on observable events simply by trading in the Lucas tree.

d. *Equity Market.* Following Abel (1999), I model the aggregate stock market as a zero net-supply contingent asset, so-called leveraged consumption claim, that yields the following dividend flow

$$D_t \propto C_t^{\phi} \exp \{X_t\} \tag{II.3}$$

where X_t is a random process which drives a wedge between the otherwise deterministic relationship between endowment and dividends. The parameter $\phi > 0$ is the elasticity of the dividend flow with respect to the endowment flow,

$$\phi \quad = \quad \frac{\partial \log D_t}{\partial \log C_t},$$

and it is a yardstick of the dividend leverage; its magnitude tells how sensitive the aggregate dividend flow from the equity market is to changes in the underlying fundamentals - the endowment flow[5].

The choice of the time-series properties of X_t is essential. Economic intuition may suggest that endowment and dividends are cointegrated, and therefore X_t, as the respective cointegrating residual, ought to be statistically stationary. Intuitive though it may be, such a choice necessarily introduces another state variable to already extremely computationally intensive problem. Being fully aware of potential pitfalls, the curse of dimensionality forces me to assume that X_t is a random walk[6], exhibiting no temporal dependence; formally

$$X_t = \sigma_{D,2} Z_{2t}$$

where the Brownian shock dZ_{2t} is normally distributed with mean zero and variance dt, but uncorrelated with the consumption growth shock dZ_{1t}, and $\sigma_{D,2}$ is a constant volatility of the random process X_t.

Application of Itô lemma to the relationship (II.3) yields the following differential equation for the growth rate of the dividend flow

$$\begin{aligned} \frac{dD_t}{D_t} &= \left[(1-\phi)\,\mu_C^s + \phi\,\kappa_t\right] dt \; + \\ &+ \; \phi\sigma_C dZ_{1t} \; + \; \sigma_{D,2} dZ_{2t} \end{aligned}$$

where μ_C^s is the unconditional mean of the endowment growth rate. This specific normalization implies that endowment and dividend growth rates share, on average, a common drift, partially mitigating the lack of, potentially important, cointegration between the levels in the model.

B. **Time Evolution of Investors' Confidence.** The investors' information sets are denoted $\mathcal{I}_t$ and they contain all past realizations of the endowment and the dividend flows up to time t. I measure investors' confidence about the macroeconomic fundamentals by the following posterior probabilities

$$\pi_{jt} = \mathcal{P}\left\{ \kappa_t \; = \; \kappa_j \,|\, \mathcal{I}_t \right\}, \quad j \in \{1, \ldots, 4\} \tag{II.4}$$

[5]Hereafter, I carefully estimate the leverage parameter ϕ. As the reader will notice, I cannot reject the hypothesis of no cointegration between the per-capita nondurable consumption and services (NIPA data), and aggregate dividends (Prof Shiller's data); therefore, I run the regression in first differences, using a particular linear filter to emphasize business-cycle and lower frequencies. This is meant to avoid the criticism of dividends being sticky in the short-run and hence ϕ being close to zero. After all, the model is designed to explain asset prices at business-cycle and lower frequencies.

[6]Proper modelling of the cointegrating residual is definitely worthwhile but the reader should be aware that the linear system that would arise from augmenting the current model and subsequently discretizing the relevant partial differential equation may be huge. Depending on the discretization mesh (I use $\delta\pi = 0.02$ and am forced to approximate the relevant partial derivatives up to 4th order, even one-sided, in order to avoid an explosion of the solution on the boundary), it may be even $1,000,000 \times 1,000,000$!

As probabilities sum to one, one of them is redundandant; without loss of generality I choose π_{4t}. In consequence, the remaining posterior probabilities π_{1t}, π_{2t}, and π_{3t} live in a three-dimensional tetrahedron Δ^3, defined hereafter for future reference.

The open set $\Delta^3 \subset \mathbb{R}^3_+$ is a three-dimensional tetrahedron defined by

$$\Delta^3 \;=\; \left\{ (\pi_1, \pi_2, \pi_3) \in \mathbb{R}^3_+ \,|\, \sum_{i=1}^{3} \pi_i \;<\; 1 \right\}.$$

Note that the differential equations for the consumption and the dividends all contain unobservable drifts. In order to make analytical progress, rendering the analysis amenable to Markov dynamic programming, I use a diffusion representation of these equations. The following lemma compactly summarizes the result.

Lemma 2. *The* 2×1 *process*

$$\mathbf{Y}_t = \left[\frac{dC_t}{C_t}, \frac{dD_t}{D_t} \;-\; (1-\phi)\,\mu^s_C\,dt \right]' \tag{II.5}$$

may be written as a diffusion with respect to the Brownian process $\widetilde{\mathbf{Z}}_t = \left[\widetilde{Z}_{1t}, \widetilde{Z}_{2t}\right]$*, so-called 'innovation' process, as follows*

$$d\mathbf{Y}_t \;=\; \mathbb{E}\left[d\mathbf{Y}_t \,|\, \mathcal{I}_t\right] + \left(\mathbf{\Sigma}\mathbf{\Sigma}'\right)^{1/2} d\widetilde{\mathbf{Z}}_t \tag{II.6}$$

where the volatility matrix is

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_C & 0 \\ \phi\,\sigma_C & \sigma_{D,2} \end{pmatrix} \tag{II.7}$$

Proof. Liptser and Shiryaev (2001). See also Appendix that develops the necessary additional notation. □

The interpretation of the innovation process is as a vector of innovations in the given economic series *perceived* by investors. Note that due to the incompleteness of information, dZ_i and $d\widetilde{Z}_i$ in fact differ.

Investors' confidence as gauged by the vector of posterior probabilities, also evolves as a diffusion, according to the following lemma.

Lemma 3. *The vector of the posterior probabilities* $(\pi_1, \pi_2, \pi_3) \in \Delta^3$ *follows the diffusion process*

$$d\pi_{jt} \;=\; m_{jt}\,dt \;+\; \widetilde{\boldsymbol{h}}_{jt}\,d\widetilde{\mathbf{Z}}_t, \tag{II.8}$$

TABLE 1. The Calibration of the Endowment Process to Per-Capita Consumption Growth

Panel A. Gibbs Sampler Estimates of the BC Component with a Pre-Set Transition Matrix

States of BC component	Mean (% p.a.)	95% CI (% p.a.)
$\overline{\theta}$	2.40	(2.09,2.71)
$\underline{\theta}$	-0.68	(-1.01,0.88)

Panel B. Calibration of the LR Component with a Pre-Set Transition Matrix and Using Detection Error Probabilities

States of LR component	Mean (% p.a.)		Jump Intensity	Magnitude (p.a.)
$\overline{\vartheta}$	1.00		λ_{LR}	0.085
$\underline{\vartheta}$	-2.50		μ_{LR}	0.213

Notes: For the BC calibration, I borrow the annualized transition matrix P=[0.8457, 1.0-0.8457;1-0.2631, 0.2631] from Cagetti et al (2004). The sample period is 1948.Q1-2007.Q4, and I perform Gibbs Sampler with 2,000,000 Monte Carlo simulations, discarding the first 1 million for convergence reasons. For the LR calibration, I choose the annualized transition matrix to be P=[0.9265, 1-0.9265, 1-0.8162, 0.8162] (see main text for more details). I then choose the states $\overline{\vartheta}$ and $\underline{\vartheta}$ to maximize the Kullback-Leibler relative entropy distance between the *i.i.d.* and HMM models, subject to the constraint that the detection error probability (with non-informative prior) for 80 years is about 5%.

where π_{j0} *is given,* $j \in \{1, 2, 3\}$*, and*

$$m_{jt} = \sum_{i=1}^{4} \lambda_{ij}\, \pi_{it} \tag{II.9}$$

$$\Xi_t = (\kappa_t,\, \phi\, \kappa_t) \tag{II.10}$$

$$\widetilde{\boldsymbol{h}}_{jt} = \pi_{jt}\, [\Xi_j - \mathbb{E}\,(\Xi_t \,|\, \mathcal{I}_t)] \times \tag{II.11}$$

$$\times\, (\boldsymbol{\Sigma}\boldsymbol{\Sigma}')^{-1/2} \tag{II.12}$$

Proof. Wonham (1964) and Liptser and Shiryaev (2001), Theorem 9.1. See also Appendix on my ordering of states to construct Ξ_i, $i = 1, \ldots, 4$. □

The previous proposition suggests that investors in fact learn from both signals: consumption *and* dividends. This may appear at first counterintuitive; note that we may write the dividend signal as

$$\frac{dD_t}{D_t} \propto \phi \left(\frac{dC_t}{C_t} \right) + \sigma_{D,2} dZ_{2t} \tag{II.13}$$

Clearly, for the leverage yardstick $\phi \geq 1$, the dividend signal is noisier that the consumption signal. A naive solution to the signal extraction problem is to drop the nosier signal in Lemma 3. However, as the following proposition demonstrates, this argument is fallacious, holding only asymptotically as the volatility ratio $\sigma_{D,2}/\sigma_C$ goes to infinity.

Proposition 1. *The information content of the dividend signal as a function of the volatility ratio $\sigma_{D,2}/\sigma_C$ declines asymptotically ($\sigma_{D,2}/\sigma_C \rightarrow \infty$) to zero.*

Proof. See Appendix for details. □

C. **Time Evolution of Asset Prices.** Information in my economy is generated by innovations that are Brownian processes. As a result, all equilibrium asset prices follow diffusions (Chi-fu Huang 198x). Specifically, the price of the Lucas tree, which yields the endowment stream, solves the differential equation

$$dP_t = [m_P \, P_t - C_t] \, dt + P_t \, \widetilde{\mathbf{g}}_P d\widetilde{\mathbf{Z}}_t \tag{II.14}$$

The instantaneous conditional expected return m_P and the instantaneous conditional volatility $\widetilde{\boldsymbol{g}}_P$ are determined endogenously so that in the equilibrium each investor willingly holds his tree(s). Without loss of generality, I normalize the measure of the Lucas orchard to one; one tree for each investor.

The aggregate equity market is modeled as the leveraged consumption claim, with its value S_t dictated by the differential equation

$$dS_t = [m_S \, S_t - D_t] \, dt + S_t \, \widetilde{\mathbf{g}}_S \, d\widetilde{\mathbf{Z}}_t \tag{II.15}$$

Again, as with the Lucas tree, the instantaneous conditional expected return m_S and the instantaneous conditional volatility $\widetilde{\boldsymbol{g}}_S$ are determined endogenously in the competitive equilibrium by the market clearing condition that the demand for the leveraged consumption claim is zero.

Note that the *only* asset in net positive supply, normalized to one, is the Lucas tree that yields the endowment stream; all other asset prices are shadow prices.

III. Model Calibration

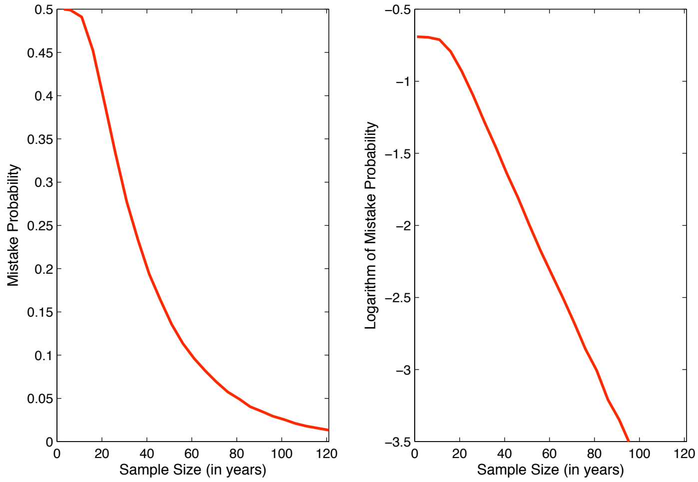
A. **Preference Parameters.**

a. *State-Independent Preferences.* This benchmark case is configured with the coefficient of the aversion to late resolution of uncertainty $1 - \alpha/\rho$ equal to 35.5, the elasticity of intertemporal substitution $(1 - \rho)^{-1} = 1.5$, and the subjective rate of time preference equal to 1.5% per annum.

b. *State-Dependent Preferences.* This refined case endogenizes the aversion to late resolution of uncertainty $1 - \alpha/\rho$ and household impatience δ by making them mildly wealth-dependent, consistent with the wealth-causes-patience hypothesis of Becker and Mulligan (1997), hence,

Assumption 1. *The household impatience δ is a decreasing function of wealth*

FIGURE III.1. Mistake Probabilities



Notes: I follow Hansen (2007), and use Monte Carlo with 50,000 simulations to construct these curves, by computing the LR ratio for models A and B given that data was generated with model A, and vice versa. I assume a non-informative prior over the two models.

and the intuition that households tend to particularly fear late unfolding of a given economic uncertainty in bad times (times of low consumption growth). Formally,

Assumption 2. *The coefficient of the aversion to late resolution of uncertainty* $1 - \alpha/\rho$ *is a decreasing function of wealth.*

As I show hereafter, the wealth-consumption ratio is a function of the vector of the posterior probabilities that measure the likelihoods of the macroeconomy being in various growth states. Therefore, I make the following parametrizations

$$\begin{aligned} \delta(\boldsymbol{\pi}) &= [0.005,\, 0.01,\, 0.02,\, 0.03] \times \boldsymbol{\pi} \\ (1 - \alpha/\rho)(\boldsymbol{\pi}) &= [1.03,\, 28.0,\, 58.0,\, 88.0] \times \boldsymbol{\pi} \end{aligned}$$

On average, with respect to the ergodic probability of the hidden Markov chain, the mean household impatience $\overline{\delta}$ is about 1% and the mean aversion to late resolution of uncertainty $\overline{(1 - \alpha/\rho)}$ is around 22, below the Bansal and Yaron's (2004) value of 28. The relatively large aversions to late resolution of economic uncertainty, 58 and 88, occur in states of the world when investors are pretty confident that the low-frequency component of the true expected endowment growth is in the low state $\underline{\vartheta}$. These, however, have relatively small probability, and moreover do not seem to have occurred throughout the past 60 years of the U.S. history.

B. **Data Description.** Quarterly consumption data are from the U.S. national accounts as available from the Federal Reserve Bank of St. Louis. I measure consumption C_t as the sum of real personal consumption expenditures (PCE) on nondurable goods and services.

TABLE 2. Calibration of the Dividend Process

1929.A - 2004.A		1948.Q1 - 2007.Q4	
ϕ	$\sigma_{D,2}$ (in %)	ϕ	$\sigma_{D,2}$ (in %)
1.72	8.95	1.75	3.47
(0.28,2.66)	(4.20,12.60)	(0.98,2.48)	(2.30,4.80)

Notes: Notation is as follows: c_t is the (demeaned) real per-capita log consumption growth, d_t is the (demeaned) real log dividend from S&P500 index, ϕ is the yardstick of dividend leverage defined by the regression $A(L)\,(1-L)\,d_t = \phi\,A(L)\,(1-L)\,c_t + A(L)\,(1-L)\,\sigma_{D,2}Z_{2t}$, and $\sigma_{D,2}$ is the annualized volatility of the dividend shock dZ_{2t} that is uncorrelated with the consumption growth shock. The linear filter $A(L)$ is a two-sided MA(3) low-pass filter that eliminates spectral components with periods less or equal to 3 quarters; the respective filter coefficients implied by the modified Daniell kernel are {0.083, 0.167, 0.167, 0.167, 0.167, 0.167, 0.083}. The values in parentheses are bootstrap percentile confidence intervals for the respective parameters. The replicates were generated using random block resampling with block lengths having a geometric distribution with mean 20. In total, 50 000 experiments were performed.

Nondurable goods and services are converted to per-capita values by dividing by the population. The dividend series are obtainable from Professor Shiller's web.

C. **Consumption Growth Dynamics.**

a. *Business-Cycle Component Calibration.* Initially[7], I estimate a Bayesian univariate 2-state HMM treating all parameters as unknown, and using diffuse priors. The confidence intervals turn out to be quite wide, in particular for the transition probabilities[8]; hence, the priors are important. I choose to fix the transition probability matrix at the magnitudes that Cagetti, Hansen, Sargent and Williams (2004) estimate, using the Solow residual[9]. I subsequently re-estimate the model. The estimated states with their confidence intervals (CI) are reported in Table 1, Panel A.

b. *Low-Frequency Component Calibration.* In calibrating the low-frequency (LF) component, I borrow from the work of Hansen and Sargent. Specifically, I consider two models of consumption, model **A** and model **B**.

The model **A** is the classic *i.i.d.* specification of consumption dynamics[10] (at quarterly frequency),

$$\widetilde{c}_{t+1} - \widetilde{c}_t = 0.0049 + 0.01\,\widetilde{\varepsilon}_{t+1} \tag{III.1}$$

with $\widetilde{\varepsilon}_t \sim i.i.d.\, N(0,1)$. The tilde means the variable is time-aggregated to quarterly value, for example, $\widetilde{c}_t = \log\left(\int_{t-1}^{t} C_\tau\, d\tau\right)$.

[7]I also estimate a full four-state Bayesian regime-switching model. However, sufficient conditions for the existence of the infinitesimal generator corresponding to the estimated transition probability matrix are as yet unknown. Although I do impose some known necessary conditions, I am unable to compute the generator for the transition matrix that I estimate.
[8]Results available upon request.
[9]They show that the implied posterior probabilities closely track NBER business cycles.
[10]I calibrate the volatility of the consumption growth at the conservative level of 2% per annum, below the Bansal and Yaron (2004) counterpart of 2.70 %.

TABLE 3. Summary Statistics of the Time-Averaged Fundamentals

Panel A: Simulated Data

	Consumption	Dividends
Mean	2.02 %	2.05 %
Std.Dev.	2.50 %	9.26 %
$AC(1)$	0.25	0.08
$corr\,(g_C,\, g_D)$	0.56	

Panel B: Real Data

	Consumption (s.e.)	Dividends (s.e.)
Std.Dev.	2.93 % (0.69 %)	11.49 % (1.98 %)
$AC(1)$	0.49 (0.14)	0.21 (0.13)
$corr\,(g_C,\, g_D)$	0.55 (0.34)	

Notes: The data are simulated for 400,000 quarters, and consequently time-aggregated over a year. Real data are taken from Bansal and Yaron (2004), Table 1.

The model **B** is a four-state HMM, with the BC dynamics specified as described above, and the LF dynamics carefully selected so that **A** and **B** are just indistinguishable in 80 years of data; that is, the detection error probability is around 5%. An attractive way to calibrate this LF dynamics is to choose $\overline{\vartheta}$, $\underline{\vartheta}$, and the transition probability matrix so that the relative entropy of model **B** with respect to model **A** is maximized, subject to the constraints that $\mathbb{E}\,(\vartheta_t) = 0$ and the detection error probability is greater or equal than 5% in 80 years of data. However, it turns out that the transition matrix is not identified by this method. I therefore fix the durations of the good state of LF component to be around 12 years, and the low state, around 5 years[11]. I subsequently numerically maximize the relative entropy of the model **B** with respect to the model **A**. The estimated states are reported in Table 1, Panel B.

D. **Dividend Growth Dynamics.** Consumption and dividend flows are related by

$$D_t \propto C_t^{\phi} \exp\{X_t\} \tag{III.2}$$

If the shock X_t were stationary, I may estimate the leverage parameter by running the cointegrating regression

$$\widetilde{d}_t \;\; \propto \;\; \phi\, \widetilde{c}_t + \widetilde{X}_t$$

where small letters with tildes are logs of time-aggregated values. However, recall that the curse of dimensionality forces me to assume that the shock X_t is a random walk, thereby

[11]Chen and Pakoš (2008) provide a Gaussian quadrature approximation of the Bansal-Yaron expected consumption growth rate in terms of a 2-state Markov chain. Their estimated intensities are around 0.125 each. They subsequently find that in order to match the temporal dynamics of asset prices, such as stock return predictability, they must tilt these intensities to around 0.085 for the good state and 0.19 for the bad one. These numbers imply durations of the good and bad states quite close, though not exactly, to 12 and 5 years, respectively, the magnitudes that I use in the main text. This observations provides a partial justification for having chosen those durations.

building into the model a lack of cointegration between the logs of the two series[12]. In fact, the model suggests to identify the leverage measure ϕ by running the equation in growth rates

$$(1-L)\,\widetilde{d}_t \;\; \propto \;\; \phi\,(1-L)\,\widetilde{c}_t + (1-L)\,\widetilde{X}_t$$

rather than in levels. This is quite tricky, though. Consider applying any linear filter $A(L)$ to both sides of the equation. The result is

$$\begin{aligned} & A(L)\,(1-L)\,\widetilde{d}_t \\ = & \;\; \phi\,A(L)\,(1-L)\,\widetilde{c}_t + A(L)\,(1-L)\,\widetilde{X}_t \end{aligned}$$

which also may be used to estimate the leverage. The choice of the linear filter appears to make a statistical difference. I select to utilize a *low-pass* filter to eliminate spectral components with periods less than 3 quarters[13] as the model is constructed to capture the dynamics of consumption and dividends at business-cycle and lower frequencies. In detail, the linear filter is a two-sided MA(3) filter with coefficients implied by the modified Daniell kernel[14]. I verify that before applying the filter, consumption has also high-frequency components, perhaps arising from seasonal adjustments, by estimating the smoothed power spectrum. After the transformation, all spectral components with period T less or equal 3 quarters are indeed eliminated; again, power spectrum visually inspected.

I run the regression with the filtered time series to focus on the consumption and dividend growths comovement at the business-cycle and lower, long-run, frequencies. In addition, I bootstrap the confidence intervals. Details are reported in Table 2. The leverage yardstick is estimated around 1.75 in both the annual sample (1929.A-2004.A) and the quarterly sample (1948.Q1-2007.Q4), although the 95% confidence bounds are wider in the annual data. In calibrating the model, I choose $\phi = 2$, well within the confidence bounds. In a related paper, Bansal and Yaron (2004) choose $\phi = 3$. Further, the point estimate of the volatility of the dividend shock $\sigma_{D,2}$, which is uncorrelated with the consumption shock, varies from around 3.5% in the quarterly sample up to nearly 9% in the annual sample. I calibrate $\sigma_{D,2}$ at a quite conservative value of 7.5%, out of the quarterly confidence bounds, but well within the annual confidence bounds.

E. **How Well Do Simulated Fundamentals Match Real Data ?** I calibrate the consumption process using detection-error probabilities; it is impossible in less than 80 years of data to statistically discriminate between the *i.i.d.* model and my four-state hidden Markov switching model as the mistake probability is above 5%. Still, some readers

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[12] I formally test this restriction in the annual Shiller data 1929.A-2004.A, and I cannot reject the null hypothesis that real dividends and real consumption are not cointegrated even at 10% significance level. The ADF t-statistics equals -2.94 and the Phillips-Perron t-statistics=-2.97. The 10%, 5% and 1% quantiles are -3.10, -3.42, -4.04, respectively.

[13] Whether we choose MA(2), MA(3) or MA(4) doesn't seem to make any statistical difference; MA(0) vs MA(3) does. If you do not filter the series, you get in quarterly data that ϕ is statistically less than one.

[14] MA(3) coefficients are $\{0.083, 0.167, 0.167, 0.167, 0.167, 0.167, 0.083\}$.

may find a comparison of the model-implied moments with those from the real data informative. In order to do that, I proceed as follows. First, I carefully time-aggregate the instantaneous consumption and dividend growth rates to annual magnitudes as follows[15]

$$\begin{aligned}\Delta c_{t+1} &= \log\left(\int_t^{t+1} C_\tau \, d\tau\right) - \log\left(\int_{t-1}^{t} C_\tau \, d\tau\right)\\ \Delta d_{t+1} &= \log\left(\int_t^{t+1} D_\tau \, d\tau\right) - \log\left(\int_{t-1}^{t} D_\tau \, d\tau\right)\end{aligned}$$

Consequently, I calculate the standard deviation and the first-order autocorrelation of the simulated series. The results are reported in Table 3. The magnitudes of the first-order autocorrelations of the time-aggregated simulated series are below the point estimates, but well-within the confidence bounds. This implies that the univariate predictability by means of lagged values is, if anything, weaker in the simulated data. Second, the respective volatilities are also a bit smaller as compared to the data, and raising these would only help improve the fit of the model. Finally, the contemporaneous cross-correlation between the simulated consumption and dividend growth rates is 0.56, practically exactly equal to the real data counterpart of 0.55. In view of this, it appears fair to conclude that the calibration of the consumption and dividends appears statistically consistent with the NIPA data.

IV. Household Problem

A. **Markov Dynamic Programming.** In order to solve the portfolio choice problem of the representative household, I use Markov dynamic programming. Define the indirect utility function

$$J_t = \mathbb{E}\left[\int_t^{\infty} f(c_s,\ J_s)\, ds \mid \mathcal{I}_t\right] \tag{IV.1}$$

The rate of return dR_t^W on the aggregate wealth W_t is equal to the weighted average of returns on the I-Owe-You-s, yielding the real interest rate $r_t dt$, and the return on the Lucas tree, equal to the capital gain dP_t/P_t plus the endowment yield $C_t\,dt/P_t$; formally,

$$dR_t^W = a_p\left(\frac{dP_t}{P_t} + \frac{C_t\,dt}{P_t}\right) + (1 - a_P)\ r_t\,dt, \tag{IV.2}$$

As a result, the household's dynamic budget constraint takes the form

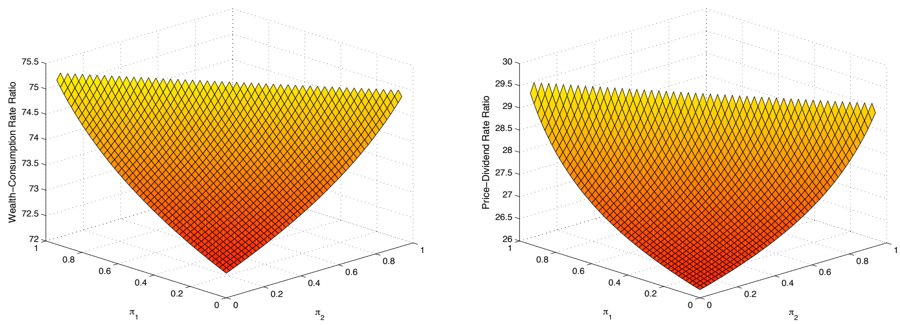
$$dW_t = W_t\,dR_t^W - c_t\,dt. \tag{IV.3}$$

where c_t is the consumption rate, and it is the portfolio outflow. Duffie and Epstein (1992ab) show that the associated Hamilton-Jacobi-Bellman (HJB) equation for this

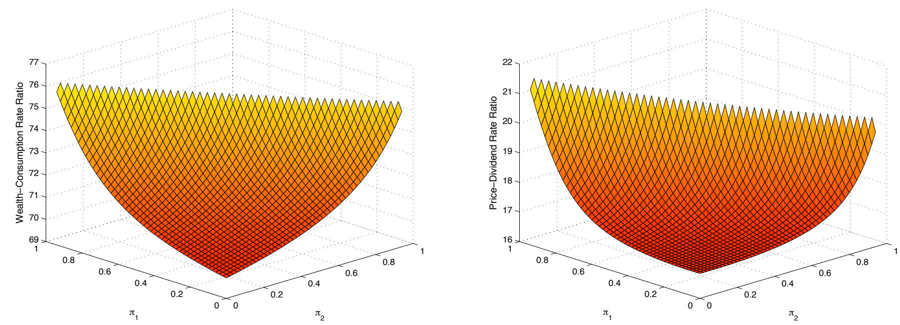
[15] Time is measured in years.

FIGURE III.2. Equilibrium Wealth-Consumption and Price-Dividend Surfaces

Panel A. Valuation Rate Ratios With State-Independent Preferences



Panel B. Valuation Rate Ratios With State-Dependent Preferences



***Notes*:** These pictures display the equilibrium wealth-consumption rate ratios (left), and the equity market value over dividend rate ratios (right), all annualized. Note that in plotting these surfaces, I set the posterior probability $\pi_3 = 0$.

decision problem is of the form

$$0 = \sup_{\{c,\, a_P\}} \left\{ f(c,\, J) + \mathcal{L}^J J \right\} \tag{IV.4}$$

where $\mathcal{L}^J$ is a second-order differential operator, so-called infinitesimal generator, and for brevity it is displayed in Appendix. Because preferences are homothetic, it is natural to conjecture that the value function is separable across the aggregate wealth W_t and the rest of the state variables, in my case, the vector of posterior probabilities $\boldsymbol{\pi}_t = (\pi_{1t}, \pi_{2t}, \pi_{3t})$. As a result, I guess and verify that the value function takes the shape

$$J_t \;\equiv\; J(W_t, \boldsymbol{\pi}_t) \;=\; \alpha^{-1}\, \delta^{\alpha/\rho}\, \left[\Psi(\boldsymbol{\pi}_t)\right]^{\frac{1-\rho}{(\rho/\alpha)}}\; W_t^{\alpha}, \tag{IV.5}$$

where I use my hindsight and parametrize the value function in terms of the equilibrium wealth-consumption rate ratio $\Psi(\boldsymbol{\pi})$, the detailed characterization of which is supplied in the following essential proposition.

Proposition 2. *The equilibrium wealth-consumption rate ratio* $\Psi : \Delta^3 \to \mathbb{R}^+$ *solves the nonlinear elliptic partial differential equation (PDE)*

$$0 = \mathcal{L}^{\Psi}\,\Psi + a_0\left(\boldsymbol{\pi}\right)\,\Psi\, + 1 \tag{IV.6}$$

subject to the boundary conditions

$$\lim_{\boldsymbol{\pi}\,\to\,\partial\Delta^3} \Psi(\boldsymbol{\pi}) \sim \text{finite} \tag{IV.7}$$

where for brevity the nonlinear second-order differential operator $\mathcal{L}^{\Psi}$*, and the function* $a_0(\boldsymbol{\pi})$*, are displayed in Appendix.*

Proof. See Appendix for details. □

Unfortunately, the consumption-wealth rate ratio $\Psi\left(\pi_1,\,\pi_2,\,\pi_3\right)$ is not separable across the posterior probabilities, and the partial differential equation is irreducible to a system of ordinary differential equations. I am forced to resort to numerical methods, decomposing the tetrahedron domain Δ^3 into a regular mesh, and subsequently employing a high-order finite difference scheme[16]. Note that I do not linearize but rather carefully deal with the nonlinearity by means of a particular iterative scheme. All computations have been done on high-performance machines[17].

The expected consumption growth is not constant, but rather a persistent continuous-time AR(1) process, and therefore the solution Ψ is *not* the wealth-consumption ratio but wealth-consumption *rate* ratio. I construct the discrete-time analogue of the wealth-consumption ratio as[18,19]

$$\frac{W_t}{\int_{t-1}^{t} C_\tau\, d\tau} \;=\; \frac{W_t}{C_t} \times \frac{C_t}{\int_{t-1}^{t} C_\tau\, d\tau}$$

Figure III.2 on page 17 (Panel A) portrays the equilibrium wealth-consumption rate ratio Ψ when the households' preferences exhibit state-indepedence. The portrayed surface lies in the plausible interval of about 72.5 up to 75. After accounting for the time-aggregation, the wealth-consumption ratio $W_t/\int_{t-1}^{t} C_\tau\, d\tau$ moves significantly more, from 70.7 up to 77.9.

To interpret the picture further, note that the partial derivatives $\partial_i\Psi$, $i = 1, 2, 3$, are all positive and hence the function is strictly increasing in the posterior probabilities π_1, π_2 and π_3. The derivatives are ordered as $\partial_1\Psi > \partial_2\Psi > \partial_3\Psi > 0$. This effect is a direct consequence of the ordering of my states κ_i, whereby the lower the index i, the higher is

[16]Technically, I use 5+ nodes to approximate partial derivatives, even the one-sided.

[17]I utilize several computational *C/C*$^{++}$*/FORTRAN* libraries, which are well-known in the field of numerical mathematics, but dramatically less so in economics. All codes are fully parallelized to run on multiple processing units by means of OpenMP and Message Passing Interface (MPI) protocols. The libraries I use include in particular GNU GMP, SuperLU_DIST_2.1, MPICH2, and Intel Math Kernel Library. In addition, I invoke the professional C/FORTRAN compiler set by Intel Inc.

[18]Time is measured in quarters.

[19]For presentation reasons, I annualize the ratio by multiplying it by four.

the expected growth rate of consumption, and hence the larger is the surge in the wealth-consumption ratio in response to a change in the posterior probability π_i. Furthermore, Ψ exhibits a mild convexity, which is related to the fact that expected returns turn out to be inverted U-shaped; the extra fall in Ψ coming from a dramatic increase in expected returns occurs precisely in the interior of the domain $int\,(\Delta^3)$.

Figure III.2 on page 17 (Panel B) portrays the wealth-consumption rate ratio when the households' preferences display mild state-dependence. The graph inherites many of the qualitative characteristics of its state-independent counterpart. The biggest difference is a surge in its variation. The discrete-time counterpart, $W_t / \int_{t-1}^{t} C_\tau d\tau$, lies in the hefty interval [72.7, 84.5], compared to the wealth-consumption rate ratio Ψ that moves from around 74.3 up to 82.1. As noted before, this effect is due to the persistence in the consumption growth.

Because both plots of the wealth-consumption rate ratios, Panels A and B, are qualitatively similar, I shall, for the sake of brevity, suppress a dual presentation where a loss in understanding is deemed negligible.

B. **Real Pricing Kernel and Risk Prices.** Duffie and Epstein (1992ab) derive the formula for the equilibrium pricing kernel $\{\, M_t,\, \mathcal{I}_t \,\}_{t\geq 0}$, for the case of the normalized aggregator (my case) as

$$M_t \;=\; \exp\left(\int_0^t \partial_J f\,(C_s,\, J_s)\; ds \right) \partial_C f\,(C_t,\, J_t)\,.$$

The following proposition identifies both its dynamics and the instantaneous risk price functions.

Proposition 3. *The equilibrium dynamics of the real pricing kernel is dictated by the differential equation*

$$\frac{dM_t}{M_t} \;=\; -r_t\, dt \;-\; \mathbf{\Lambda}_t d\widetilde{\boldsymbol{Z}}_t$$

where r_t is the instantaneous real interest rate, and $\mathbf{\Lambda}_t$ is the vector of the instantaneous risk price functions

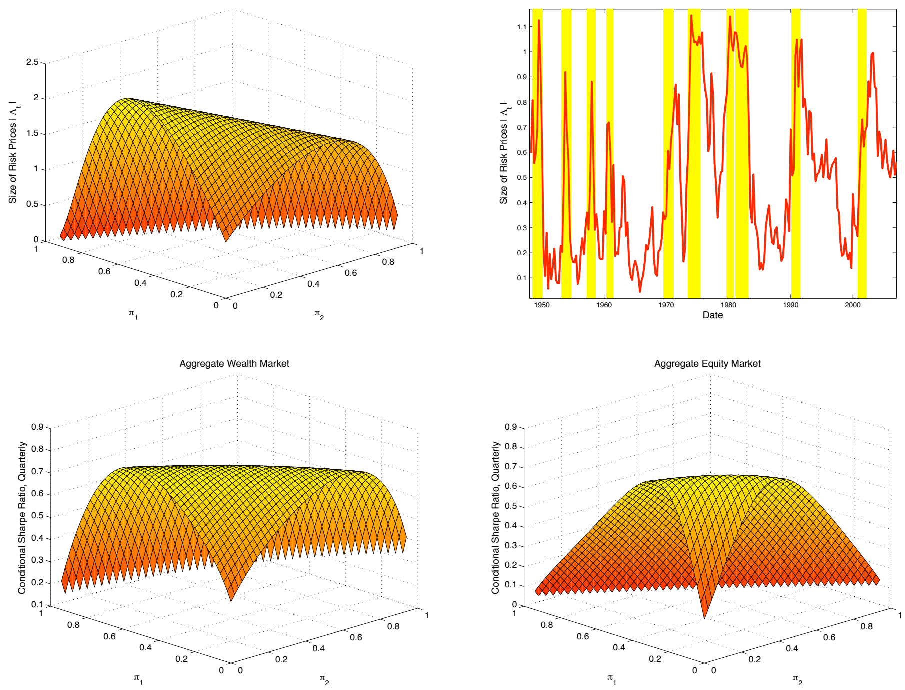
$$\mathbf{\Lambda}_t \;=\; (1-\alpha)\,\widetilde{\boldsymbol{g}}_C \;+\; \left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3} \widetilde{\boldsymbol{h}}_i \left(\frac{\partial_i \Psi}{\Psi}\right),$$

given as a sum of the Brownian motion consumption risk price (first term), and the uncertainty price (second term).

Proof. See Appendix for derivation. □

Note that the Brownian consumption risk is multiplied by the coefficient of the relative risk aversion, whereas the uncertainty premium depends on the coefficient of the aversion to late resolution of uncertainty.

Figure IV.1. Risk Price Function and Time-Series of Risk Prices: Case of State-Dependent Preferences



Notes: The top left picture displays the surface of the magnitude of the vector-valued instantaneous risk price function $|\mathbf{\Lambda}_t|$. The bottom two pictures display the conditional quarterly (i.e. finite-horizon as opposed to the instantaneous one) Sharpe ratios for the aggregate wealth market (left) and the aggregate equity market (right). Because these graphs are four-dimensional, I hold fixed the posterior probability at $\pi_{3t} = 0$. The top right picture displays the time-series of the instantaneous risk prices for the sample period 1948.Q1-2007.Q4, obtained by estimating the time-series of the vector of the posterior probabilities $\boldsymbol{\pi}_t = (\pi_{1t}, \pi_{2t}, \pi_{3t})$, corresponding to the real per-capita consumption growth, by means of the Hamilton (1989) filter.

Figure IV.1 on page 20 (top left) plots the size of the instantaneous market prices of risk $|\mathbf{\Lambda}_t|$ for[20] $\pi_3 = 0$ when preferences exhibit state-depedence. Several points are worthwhile discussing. First, there is a clear tendency for the market prices of risk to be high when ambiguity about both the business-cycle and long-frequency consumption growth rates are high. First, note that holding either π_1 or π_2 fixed, the graph attains its local maximum for $\pi_i \sim 1/2$. The reason that it is not exactly 1/2 is that the risk prices depend also on the partial derivatives of the value function[21]. Second, the graph reaches

[20]The graph is 4-dimensional and I choose to present it for $\pi_3 = 0$ for several reasons. First, the domain, which is a tetrahedron, is maximized for such a choice, allowing nicer presentation. Second, it allows better interpretation of the graph in terms of ambiguity about business-cycle and long-run consumption growth rates as discussed in the main text further.

[21]My presentation differs from Hansen (2007) in that I include the value function derivatives; if I do not, the maximum is always attained for $\pi_i = 1/2$, holding π_j fixed, $j \neq i$.

its global maximum for $\pi_2 = 0$. Recall the definition of the posterior probabilities

$$\begin{aligned}\pi_{1t} &= \mathcal{P}\left\{\theta_t = \overline{\theta},\, \vartheta_t = \overline{\vartheta} \,|\, \mathcal{I}_t\right\} \\ \pi_{2t} &= \mathcal{P}\left\{\theta_t = \underline{\theta},\, \vartheta_t = \overline{\vartheta} \,|\, \mathcal{I}_t\right\} \\ \pi_{3t} &= \mathcal{P}\left\{\theta_t = \overline{\theta},\, \vartheta_t = \underline{\vartheta} \,|\, \mathcal{I}_t\right\}\end{aligned}$$

Observe that the law of total probability dictates that

$$\begin{aligned}\pi_{1t} + \pi_{2t} &= \mathcal{P}\left\{\vartheta_t = \overline{\vartheta} \,|\, \mathcal{I}_t\right\} \\ \pi_{1t} + \pi_{3t} &= \mathcal{P}\left\{\theta_t = \overline{\theta} \,|\, \mathcal{I}_t\right\}\end{aligned}$$

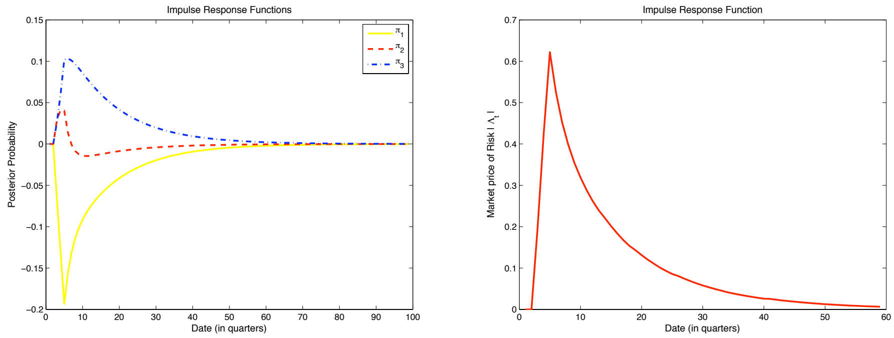
In words, the sum $\pi_{1t} + \pi_{2t}$ tells us the posterior probability of the long-frequency component of the consumption growth being in the high state, and the sum $\pi_{1t} + \pi_{3t}$ indicates the posterior probability of the business-cycle component of the consumption growth being in the high state. Consider two scenarios. In the first one, the posterior probability $\pi_{1t} = 0$ and $\pi_{2t} \sim 1/2$, hence, we analyze the local maximum of the graph along the $\boldsymbol{y}$ axis. Note that in such a scenario, the ambiguity about the long-frequency component is large ($\pi_{1t} + \pi_{2t} \sim 1/2$), but there is absolutely no ambiguity with respect to the business cycle component ($\pi_{1t} + \pi_{3t} = 0$). In the second case, $\pi_{2t} = 0$ but $\pi_{1t} \sim 1/2$, and hence, we are analyzing the graph along the $\boldsymbol{x}$ axis, in which case the ambiguity about the low-frequency component is large ($\pi_{1t} + \pi_{2t} \sim 1/2$), and the ambiguity about the business-cycle component is large ($\pi_{1t} + \pi_{3t} \sim 1/2$) . This analysis explains the pattern of the local maxima and why the global maximum (for $\pi_{3t} = 0$) is attained along the $\boldsymbol{x}$ axis.

Figure IV.1 on page 20 (top right) plots the time-series of the risk prices[22] for the sample period 1948.Q1-2007.Q4. In order to do that, I run the Hamilton (1989) filter, given the calibration, that is, the 4×4 transition probability matrix, the states of the hidden Markov chain and the consumption volatility, to estimate the posterior probabilities for the real per-capita consumption growth. We see that there is a clear business cycle pattern in the magnitude of the risk price funtion, rising dramatically during downturns. Also, the Brownian motion consumption risk component is very small, and hence most of the variation in the risk prices comes from the uncertainty premium as in Hansen (2007). Further, the magnitudes of the risk prices are noteworthy. Hansen (2007), Figures 4 and 5, plots the instantaneous risk prices for a two-state HMM model calibrated only to the business-cycle dynamics; his maximum risk price is around 0.09. It is the contribution of the low-frequency component of the consumption growth that significantly raises the risk prices; the maximum in Figure IV.1 on page 20 on the left is close to 2, more than an order of magnitude higher.

Another advantage of modelling the business-cycle and long-run components of the consumption growth jointly is related to the impact of the *repeated* observations of low consumption growth. In a two-state model, such a string of bad luck tends to reduce

[22]In fact, as the risk price function $\boldsymbol{\Lambda}_t$ is a $1{\times}2$ vector, I plot the magnitude $|\boldsymbol{\Lambda}_t|$.

Figure IV.2. Impulse Response Functions to a Four-Quarter Standard-Deviation Shock: Case of State-Dependent Preferences



risk prices because it leads to a partial resolution of consumption growth uncertainty; investors raise their posterior that they are in a bad state and risk prices drop. Hansen (2007), p. 24, correctly points out that the inclusion of more low-frequency components of consumption growth would tend to mitigate this problem. This may already be observed in this model which features only single low-frequency component. A string of bad luck raises the uncertainty associated with the long-run consumption growth states, and risk prices rise.

In order to illustrate this point, Figure 5 plots the impulse response functions corresponding to a four-quarter sequence of large[23], one-standard-deviation, shocks[24,25]. The left panel plots the reaction of the posterior probabilities. As may be observed, the posteriors move in the expected directions. The probability of both the BC and LF components being in the high state drops, whereas the posterior of at least one component being in the low state rises. Of course, the reaction for π_3 is stronger as investors are suddenly learning that the long-frequency risk component is unlikely to be in the high state. The right panel portrays the impulse response function for the magnitude of the instantaneous market prices of risk; risk prices soar in response to the persistently negative consumption growth shocks, just to decline steadily as these are slowly dying out.

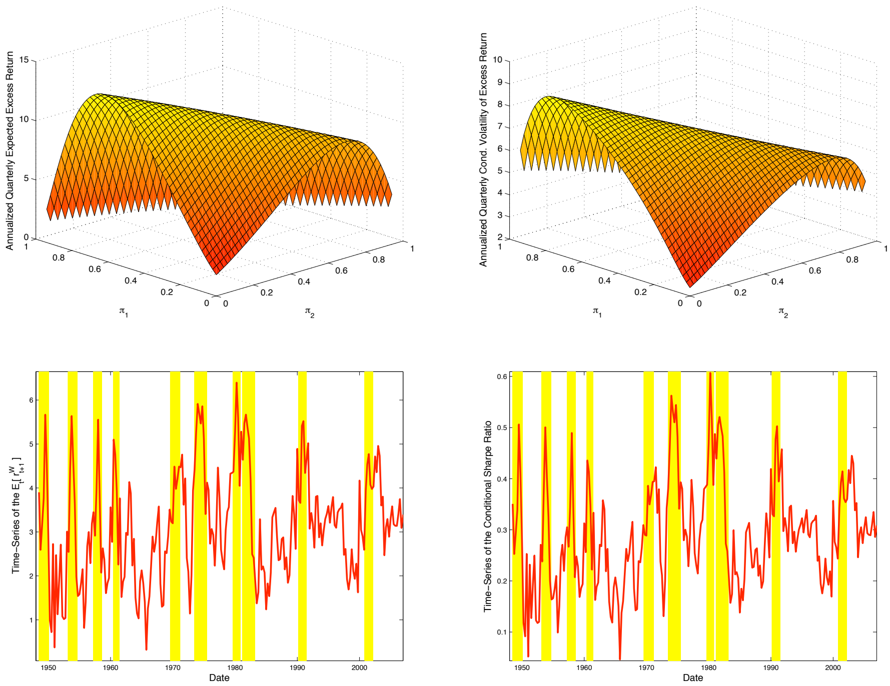
Although it is quite common in continuous time literature, with rare exceptions, to report the annualized instantaneous quantities, such a practice may be, at least partially, misleading. To counteract this valid criticism, I compute the annualized quarterly, as

[23]The largest annual fall in the real per-capita consumption growth in the sample 1948.A-2004.A (Shiller data) amounts to -1.68% p.a., which coincidentally occurred in the year of 1974. The shock I am considering amounts to a fall in the real consumption growth of 2% p.a., and hence by sample standards it is very large. By the way, the next largest fall occurred in the year of 1980; consumption growth fell by -1.43% p.a.

[24]I assume that the economy starts in the state where investors are quite confident that the long-run consumption growth component, and the business-cycle components, are both in the high state; that is, $P\left\{\vartheta_t = \overline{\vartheta} \,|\, I_t\right\} = P\left\{\theta_t = \overline{\theta} \,|\, I_t\right\} = 0.95$. This yields the following starting point $\boldsymbol{\pi}_t = \left(0.95^2,\, 0.95 \times 0.05,\, 0.05 \times 0.95\right)$. Technically, I continuously shock the whole economy for four quarters with $d\widetilde{\boldsymbol{Z}} = -(dt, 0, 0)$.

[25]I plot the impulse response function as a deviation from what would occur should there be no shocks, as the starting point is not the total ergodic probability for the four-state unobservable Markov chain, and the economy is in motion.

Figure IV.3. Annualized Quarterly Expected Excess Return and Conditional Volatility : Aggregate Wealth Market



Notes: The Sharpe ratio is quarterly, the conditional average return is annualized quarterly magnitude.

opposed to the annualized instantaneous, Sharpe ratios on the aggregate wealth and aggregate equity market. Mathematically, I do that by explicitly solving a certain class of linear parabolic partial differential equations for the conditional expected excess return, and the conditional volatility, of the respective returns. Consequently, I compute the Sharpe ratios from their definitions[26]

$$SR_t^i \quad = \quad \frac{\mathbb{E}\left(R_{t+1}^i - R_t^f \mid \boldsymbol{\pi}_t\right)}{std\left(R_{t+1}^i - R_t^f \mid \boldsymbol{\pi}_t\right)}$$

The results of this exercise are reported in Figure IV.1 on page 20, bottom left and right panels. As may be observed, the actual *annualized quarterly*, as opposed to the annualized instantaneous, Sharpe ratios, though large and time-varying, are quantitatively smaller, about a half, relative to their instantaneous counterparts. The culprit is, in view of the Feynman-Kač theorem, the time variation in the posterior probabilities, coupled with the quite strong concavity of the instantaneous risk prices[27].

C. **Real Interest Rate.** The instantaneous real interest rate is found by invoking the first-order condition from Hamilton-Jacobi-Bellman equation with respect to the portfolio

[26]Time t is measure in quarters.

[27]Mathematically, as it is well-known, parabolic PDEs have a tendency to smooth out their initial conditions.

weight a_p, and imposing the equilibrium condition $a_p = 1$. The following proposition summarizes the determinants of the level of the real interest rate.

Proposition 4. *The level of the real interest rate $r_t = r(\boldsymbol{\pi}_t)$ is given by the expression*

$$\begin{aligned} r &= \delta + (1-\rho)\, m_C \\ &\quad -\frac{1}{2}\left(1-\alpha\left(2-\rho\right)\right)\left(\widetilde{\boldsymbol{g}}_C \widetilde{\boldsymbol{g}}_C'\right) - \\ &\quad -\left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{g}}_C'\right)\left(\frac{\partial_i \Psi}{\Psi}\right) - \\ &\quad -\frac{1}{2}\left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_i \Psi\, \partial_j \Psi}{\Psi^2}\right) \end{aligned}$$

Proof. See Appendix. □

The interpretation of the first three terms is standard. The higher the subjective rate of time preference, the more impatient households are to consume and therefore in equilibrium the real interest rate has to rise to make them willingly consume their endowment rather than attempt at dissaving. The second term reflects the expected endowment growth rate. With a high expected endowment rate, most of the consumption occurs far in the future and household's desire to borrow against this future income drives up the real interest rate as there can be no borrowing in the aggregate. Finally, the third term is the standard precautionary savings motive as we know it from the canonical Lucas (1978) economies.

The last two terms are new and reflect time-varying precautionary savings motive. In order to convince the reader, I invoke the following lemma.

Lemma 4. *The following identities hold true*

$$\begin{gathered} \left(1-\tfrac{\alpha}{\rho}\right)\textstyle\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{g}}_C'\right)\left(\tfrac{\partial_i \Psi}{\Psi}\right) = \\ -\operatorname{cov}\left(\tfrac{dM_t}{M_t}, \tfrac{dC_t}{C_t} \mid \boldsymbol{\pi}_t\right) - (1-\alpha)\left(\widetilde{\boldsymbol{g}}_C \widetilde{\boldsymbol{g}}_C'\right) \end{gathered}$$

and

$$\begin{gathered} \left(1-\tfrac{\alpha}{\rho}\right)\textstyle\sum_{i,j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right)\left(\tfrac{\partial_i \Psi \partial_j \Psi}{\Psi^2}\right) = \\ \tfrac{1}{2}\left(1-\tfrac{\alpha}{\rho}\right)^{-1}\left|\Lambda_t\right|^2 + \\ +\tfrac{1-\alpha}{2}\left(1-\tfrac{\alpha}{\rho}\right)^{-1}\operatorname{cov}\left(\tfrac{dM_t}{M_t}, \tfrac{dC_t}{C_t} \mid \boldsymbol{\pi}_t\right) \end{gathered}$$

Proof. See Appendix. □

According to the lemma, the last term in the expression for the real interest rate (the term on the second line in the lemma) is an increasing function of the size of the risk prices in the economy, and the covariance between the real pricing kernel and the endowment growth rate[28]. Therefore, this term convincingly reflects a portion of the precautionary savings motive. In fact, the magnitude of the vector of risk prices $|\Lambda_t|$ attains its maximum when the uncertainty in the economy is largest[29]. This is the time when investors are particularly anxious, and their supply of extra savings drives down the equilibrium interest rate. This same story holds for the covariance term $\mathrm{cov}\left(\frac{dM_t}{M_t}, \frac{dC_t}{C_t} \mid \boldsymbol{\pi}_t\right)$. Investors exhibit particular anxiety not only in times when the volatility of consumption growth is large but also when it happens in bad times. One of the measures of such a concidence, in addition to the magnitude of the vector of the risk price discussed before, is the covariance between consumption growth and the real pricing kernel. Note that the term on the first line in the lemma is just a linear combination of all the terms already discussed. The reader may easily verify that the signs of such terms, when combined with the rest, do indeed deliver the aforementioned interpretation.

D. **Aggregate Wealth Market Dynamics.** The expected wealth return, and the corresponding volatility, fluctuate over time, depending on the state of the economy as summarized by the vector of the posterior probabilities $\boldsymbol{\pi}_t = (\pi_{1t}, \pi_{2t}, \pi_{3t})$. The following proposition characterizes the time evolution of these first two conditional moments.

Proposition 5. *(i) The instantaneous conditionally expected excess return on the aggregate wealth portfolio is*

$$\begin{aligned} \mathbb{E}\left(\frac{dW_t}{W_t} + \frac{C_t dt}{W_t} \mid \boldsymbol{\pi}_t\right) &- r_t\, dt = \\ &= -\mathrm{cov}\left(\frac{dM_t}{M_t}, \frac{dW_t}{W_t} \mid \boldsymbol{\pi}_t\right) \\ &= \left(\boldsymbol{\Lambda}_t \widetilde{\boldsymbol{g}}'_W\right) dt \end{aligned}$$

(ii) The instantaneous conditional volatility of the aggregate wealth return is

$$\widetilde{\mathbf{g}}_W = \widetilde{\boldsymbol{g}}_C + \sum_{i=1}^{3} \widetilde{\mathbf{h}}_i \left(\frac{\partial_i \Psi}{\Psi}\right) \tag{IV.8}$$

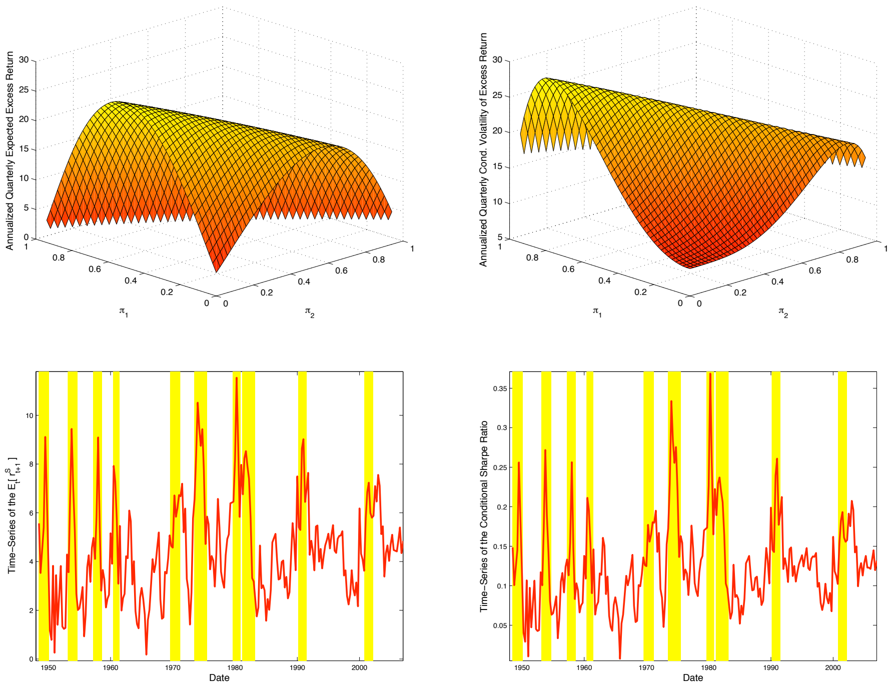
Proof. See Appendix for proofs. □

I present the first two annualized moments of wealth portfolio returns for the *finite-horizon*, quarterly, period, in contrast to the instantaneous ones. This choice matters as the moments are time-varying. The methodology to obtain these finite-horizon moments is technical, and is described at great length in Appendix.

[28]Strictly speaking, this holds true for $\alpha > 1$, which is the only case I consider in this paper. It is also a necessary condition for the preference specification to exhibit a preference for early resolution of uncertainty.

[29]Technically, $|\Lambda_t|$ does not attain its maximum when the uncertainty is largest because $|\Lambda_t|$ also hinges on the partial derivatives of the value function. Investors are not most worried when uncertainty is the largest but when the product of the partial derivatives of the value function and the 'uncertainty' is largest. These two points are, though, close to each other, and were it not for the partial derivatives of the value function, they would, of course, coincide.

Figure IV.4. Annualized Quarterly Expected Excess Return and Conditional Volatility : Aggregate Equity Market



Notes: The Sharpe ratio is quarterly, the conditional average return is annualized quarterly magnitude.

Figure IV.3 on page 23 (top left and right) plots the annualized quarterly first two conditional moments of the aggregate wealth return when preferences exhibit state-dependence. There is a notably large variation in the expected excess return and volatility in response to the investors' confidence. Figure IV.3 on page 23 (bottom left) displayes the implied time-series of the conditional excess mean return, obtained by estimating the posterior probabilities for the sample period 1948.Q1-2007.Q4, using Hamilton (1989) filter. The mean moves countercyclically, rising steeply in recessions. For illustrative purposes, Figure IV.3 on page 23 (bottom right) also portrays the related time-series of the conditional Sharpe ratio.

Note the highly nonlinear profile of the conditionally expected excess return and volatility. In fact, the expected return profile as a function of the posterior probabilities is inverted U-shaped, attaining its global maximum in the interior of the domain Δ^3, and its global minima on the boundary $\partial\Delta^3$. As investors become quite confident about the state of the economy, the uncertainty premium in expected returns falls to the level reflecting the Brownian consumption risk. This observation is in stark contrast to the Bansal and Yaron (1999) model, wherein the expected return is an affine function, rather than inverted U-shaped, of their state variables.

E. **Equilibrium Price-Dividend Ratio and Aggregate Equity Market Dynamics.** I model the aggregate equity market, with its value denoted S, as a leveraged consumption claim (Abel 19xx), with the leverage yarstick denoted ϕ. In our economy, the aggregate equity price is not statistically stationary whereas the price-dividend rate ratio is. Due to the preference homotheticity, the following functional relationship holds true

$$\frac{S_t}{D_t} = \Phi(\boldsymbol{\pi_t})$$

Proposition 6. *The equilibrium price-dividend ratio* $\Phi : \Delta^3 \to \mathbb{R}^+$ *solves the linear elliptic PDE*

$$0 = \mathcal{L}^{\Phi}\,\Phi + b_0\,(\boldsymbol{\pi})\,\Phi + 1 \tag{IV.9}$$

subject to the boundary conditions

$$\lim_{\boldsymbol{\pi} \to \partial\Delta^3} \Phi(\boldsymbol{\pi}) \sim \text{finite} \tag{IV.10}$$

where the elliptic differential operator $\mathcal{L}^{\Phi}$ *is displayed in Appendix.*

Proof. See Appendix for details. □

Figure III.2 on page 17 portrays the equilibrium price-dividend rate ratio $S/D \equiv \Psi(\boldsymbol{\pi})$ as obtained by applying a high-order finite-difference scheme to the linear elliptic PDE displayed in the above proposition. As already discussed in the case of the wealth-consumption rate ratio, $\Phi\,(\boldsymbol{\pi})$ is *not* the price-dividend ratio but rather price-dividend rate ratio. The difference again matters as the expected dividend growth is not constant but in fact a persistent continuous-time AR(1) process. I construct the discrete-time analogue of the quarterly wealth-consumption ratio as[30,31]

$$\frac{S_t}{\int_{t-1}^{t} D_\tau\, d\tau} = \frac{S_t}{D_t} \times \frac{D_t}{\int_{t-1}^{t} D_\tau\, d\tau}$$

and it turns out to vary sizably more, from 24 up to 35.4, compared to the price-dividend rate ratio Φ that moves only from around 26.4 up 32. All discussion related to the wealth-consumption rate ratio carries over to the price-dividend rate ratio. In particular, the graph is even more convex as expected returns in times of Bayesian uncertainty about growth prospects, which occurs in the interior of the domain, $int\,(\Delta^3)$, soar even higher, engendering a further discount in the in equity prices relative to confident times when $\boldsymbol{\pi}_t \to \partial\Delta^3$. In addition, the equity prices tend to overreact to bad news in good times, and underreact to good news in bad times as may be observed by evaluating the significantly different gradients close to the "opposite" boundaries. This effect was first unveiled in a rational-expectations expected-utility framework in Veronesi (1999).

As in the market for the aggregate wealth, the return moments in the aggregate equity market are time-varying.

[30] Time is measured in quarters.

[31] For presentation reasons, I annualize the ratio by multiplying it by four.

TABLE 4. Quantitative Asset Pricing Implications of the State-Dependent Calibration

	Model	**Real Data**
Average RRA	8.04	
EIS	1.5	
$E\left(r_{t+1}^S - r_t^f\right)$	5.60 %	6.33 % (2.15 %)
$E\left(r_t^f\right)$	1.50 %	0.86 % (0.42 %)
$\sigma\left(r_t^S - r_t^f\right)$	14.42 %	19.42 % (3.27 %)
$\sigma\left(r_t^f\right)$	1.64 %	0.97 % (0.28 %)
$E\left(\exp\left(p_t - d_t\right)\right)$	19.82	26.56 (2.53)
$\sigma\left(p_t - d_t\right)$	9.80 %	29 % (4 %)

Notes: I simulate 400,000 quarters of relevant time-series pertaining to the aggregate wealth at the frequency $dt = 0.001$. I subsequently compute the unconditional annual moments of the aggregate wealth and the leveraged consumption claim, and the concomitant return in excess of the yield-to-maturity r_t^f of the zero-coupon bond maturing in one quarter ahead, my yardstick of the risk-free rate. Small letters are in logs; returns are continously compounded. All variables are annualized quarterly series.

Theorem 1. *(i) The instantaneous conditional expected excess return on the aggregate equity is*

$$\begin{aligned} \mathbb{E}\left(\frac{dS_t}{S_t} + \frac{D_t dt}{S_t} \mid \boldsymbol{\pi}_t\right) &- r_t\, dt = \\ &= -\operatorname{cov}\left(\frac{dM_t}{M_t}, \frac{dS_t}{S_t} \mid \boldsymbol{\pi}_t\right) \\ &= \left(\boldsymbol{\Lambda}_t \widetilde{\boldsymbol{g}}_S'\right) dt \end{aligned}$$

(ii) The instantaneous conditional volatility of the aggregate equity return is

$$\widetilde{\mathbf{g}}_S = \widetilde{\boldsymbol{g}}_D + \sum_{i=1}^{3} \widetilde{\mathbf{h}}_i \left(\frac{\partial_i \Phi}{\Phi}\right) \tag{IV.11}$$

Proof. See Appendix for proofs. □

I present in Figure IV.4 on page 26 the first two moments of the aggregate equity portfolio returns for the *finite-horizon* period, in contrast to the instantaneous ones. Again, this choice matters as the moments are time-varying, and thus the expected return on the aggregate equity portfolio for a yearly holding period does not equal the annualized instantaneous return; the same holds for the conditional aggregate equity return volatility.

F. **Unconditional Quantitative Asset Pricing Implications.**

a. *Simulation of the Model.* As I work in continuous-time setting, I am forced to discretize time t in order to simulate the whole economy. The model is very nonlinear[32], and therefore I proceed carefully, selecting the discrete time small, $\Delta t = 0.001$, and approximating the stochastic differential equations for the vector of the posterior probabilities

[32] Although the wealth-consumption and price-dividend ratios do not look wildly nonlinear, the expected excess return profiles do.

using the Euler scheme. The sequence of *i.i.d.* Gaussian variables needed for the simulation is quite lengthy. In this respect, I avail myself of the state-of-the-art random number generator[33] (Saito and Matsumoto 2006). Having obtained the time-series of the posterior probabilities of length 400,000 quarters, I consequently assign to them the corresponding quantities, which I linearly interpolate[34,35]. All flow variables are time-aggregated as expounded in Appendix.

b. *Equity and Risk-Free Rate Puzzles.* Table 4 reports the relevant summary statistics for the equity markets (state-dependent case), and, for the reader's convenience, also the sample counterparts as estimated by Bansal and Yaron (2004). The average yield-to-maturity on an one-year zero-coupon real bond, my measure of the risk-free rate, comes out 1.50%, and has a volatility of 1.64% percent, all within the estimated asymptotic two-standard-error bounds in Panel C. The equity risk premium is 5.60 %, and though a bit lower compared to the point estimate of 6.33%, it is well within the asymptotic confidence bounds. The model is able to account for the observed risk premium in the aggregate stock market without engendering any risk-free rate puzzle. In addition, Abel (1999) cautions against accounting for a large equity premium with a large term premium. As demonstrated in the related paper by Pakoš (2008), the term premium is economically small. As regards the volatility of equity excess return, it comes out about 14.42%, well above the lower asymptotic two-standard error confidence bound 12.88%=19.42%-2.00×3.27% of the Bansal and Yaron point estimate.

c. *Matching Moments of Valuation Ratios.* The model nicely fits the mean price-dividend and wealth-consumption ratios, being exempt from the Lustig and Nieuwerburgh (2008) criticism. The average price dividend ratio comes out about 20 whereas the wealth-consumption ratio is around 75. If there is any problem with the model-implied moments, it probably lies in the relatively smaller variation in the price-dividend ratio, which comes out about half, $\sigma(p_t - d_t) = 9.80\% < 21\% = 29\% - 2.00 \times 4\%$. In this respect, I am considering several possible avenue for future research that may help fit this important moment better, such as introducing durable goods.

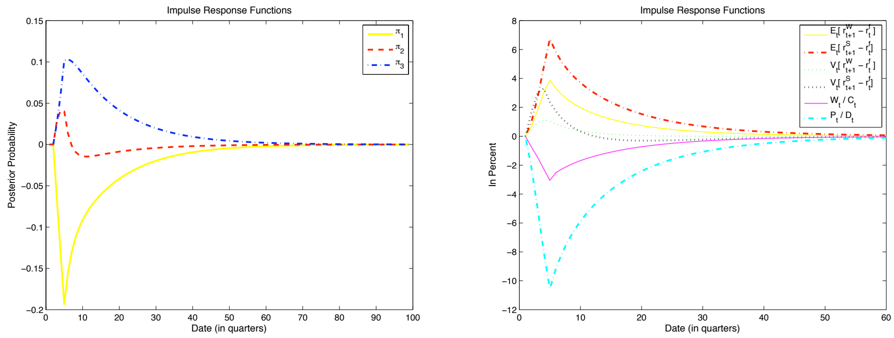
G. **Countercyclical Risk Premia and Stock Return Predictability.** Although the excess return profile is highly nonlinear (inverted U-shaped), the model is capable of generating countercyclical risk premia for careful, but still highly plausible, calibration. What is necessary is to live on the monotonic part of the profile most of the time. This condition is met in my parametrization as the transition probability kernels $\boldsymbol{P}_\theta(\Delta t)$ and $\boldsymbol{P}_\vartheta(\Delta t)$ are significantly asymmetric.

[33]In detail, I use the double-precision SIMD-oriented Fast Mersenne Twister (dSFMT) random number generator. I choose the optional period to be the largest $2^{216091} - 1$. The code itself is implemented in C directly by its authors as expounded in Saito and Matsumoto (2006), and is downloadable from their home Dept. of Mathematics at the University of Hiroshima.

[34]This is not restrictive as it appears. The domain Δ^3 is decomposed into a mesh with $\delta\pi = 0.02$.

[35]I have also experimented with higher-orders of approximation but the OpenMP parallel code running on 64 processors was unable to handle this exercise within the reasonable time of several hours.

FIGURE IV.5. Impulse Response Functions for the State-Dependent Case



Notes: I shock the economy with a yearly negative shock of size 1 std. I start the economy in the good state in terms of LR and BC. PD and WC as a percent of the stationary values.

In light of the discussion above, I construct the following thought experiment. Suppose the economy is doing well in the sense that investors attach posterior probabilities of $\mathcal{P}\left\{\vartheta_t = \overline{\vartheta}\,|\,\mathcal{I}_t\right\} = 0.95$ and $\mathcal{P}\left\{\theta_t = \overline{\theta}\,|\,\mathcal{I}_t\right\} = 0.95$ to the economy being in the high state of the low-frequency, and business-cycle, consumption growth components. All of a sudden, the macroeconomy gets buffeted by a year-long negative consumption shock[36] equal in magnitude to one standard deviation; quantitatively, we are talking about a drop in annual consumption growth[37] of 200 basis points. Figure IV.5 displays the resulting impulse response functions for the (annualized instantaneous) expected[38] excess returns, and conditional volatilities, on the aggregate wealth and equity portfolios, and for the wealth-consumption rate and price-dividend rate ratios.

As may be observed, the expected excess returns soar in response to the negative income shocks, just to slowly come down as those are slowly dying out. The peak for the aggregate stock market is a hefty six percent surge, whereas that for the aggregate wealth market is nearly four percent, magnitudes that are economically large. Furthermore, the conditional volatility and valuation ratios are also countercyclical and procyclical, respectively, moving in the expected directions.

This thought experiment illustrates the ability of the model to generate countercyclical movements in the conditional risk premiums. Therefore, excess returns ought to be predictable. I now proceed to analyze such predictability in the aggregate equity market.

H. **Volatility Tests.** Equity prices by definition equal the expected discounted value of future dividends

$$S_t \;=\; \mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^{\infty}\left(\frac{M_s}{M_t}\right)D_s\,ds\right\}$$

and, as a result, move up and down in response to changing expectations of future dividends and changing expectations of future discount rates (expected returns). Using

[36]Technically, I shock the economy continuously with $d\widetilde{\boldsymbol{Z}} = -\left(dt, 0, 0\right)$ for four quarters.
[37]Recall the calibration $\sigma_C = 2\%$.
[38]These are instantaneous expected excess returns. Working with finite-horizon moments is computationally intensive.

TABLE 5. Variance Decompositions
Panel A. Population Decompositions, 400 000 Quarters

	Assets	
	Consumption Claim	Dividend Claim
Return (in %)	9.47	31.27
Cash Flow Growth (in %)	80.37	43.11

Panel B. Small Sample Decompositions, 220 Quarters

	% of *var* $(w-c)$		% of *var* $(p-d)$	
	Mean	Median	Mean	Median
Return (in %)	77.01	101.34	53.46	58.28
Cash Flow Growth (in %)	16.46	-4.79	25.70	21.47

postwar data with the length of about 200 quarters, Campbell and Shiller (1988), and Cochrane (1992), among others, try to quantify the relative economic significance of these price determinants, and find that swings in expected dividend growth do not appear to account for the variation in equity prices. Formally, they consider log-linearizing the identity

$$1 + R^S_{t+1} = \frac{S_{t+1} + \int_t^{t+1} D_\tau \, d\tau}{S_t} \tag{IV.12}$$

with the transversality condition that there are no rational asset price bubbles, and arrive at the following approximate variance decomposition

$$\begin{aligned} 100\% \approx & \sum_{i=1}^{\infty} \rho^i \frac{\text{cov}\,(s_t - d_t,\, \Delta d_{t+i})}{\text{var}\,(s_t - d_t)} \\ & - \sum_{i=1}^{\infty} \rho^i \frac{\text{cov}\left(s_t - d_t,\, r^S_{t+i}\right)}{\text{var}\,(s_t - d_t)} \end{aligned} \tag{IV.13}$$

where I define $\rho = \exp\left(\overline{s-d}\right) / \left[1 + \exp\left(\overline{s-d}\right)\right]$, with the bar denoting the sample mean, and denote $s_t = \log(S_t)$, $r^S_t = \log\left(1 + R^S_t\right)$, $d_t = \log\left(\int_{t-1}^t D_\tau \, d\tau\right)$.

Table 5, Panel A, presents the population values of (IV.13) using 400,000 quarters of the simulated data. The items do not sum to one due to the highly nonlinear relationships between price-dividend ratio and expected returns; the gap should thus be attributed to expected returns.

As may be observed, wealth-consumption ratio varies mostly (around 80%) due to varying expectations of consumption growth. In contrast, only about 43% of the variance of the price-dividend ratio is attributable to the changing expectations of dividend growth, with 100%-43%=57% imputed to swings in expected returns.

Note that these decompositions are *population* values, and therefore may not be retrievable in short samples of 200 to 300 quarters of postwar data as the underlying variables are highly persistent. In response to this legitimate concern, table 5, Panel B, presents the means and medians obtained by performing 1000 simulations with the length of 220 quarters. The small sample bias is dramatic! If we re-ran the postwar history, around

50% of the time we would impute all the variation in the wealth-consumption ratio to the swings in expected returns, in stark contrast with the true population decomposition discussed above. And around 50% of time, we would also conclude that more than 100%-21.47%=78.53% of the variation in the price-dividend ratio is due to changing expectations of expected equity returns.

I. **Time-Series Predictability of Aggregate Wealth and Equity Returns.** In order to evaluate the ability of the model to match the empirical results in the predictability literature, I run the following regressions,

$$
\begin{aligned}
g^D_{t\to t+T} &= a + b\,(d_t - p_t) + \varepsilon^D_{t\to t+T} \\
r^S_{t\to t+T} - r^f_t &= a + b\,(d_t - p_t) + \varepsilon^S_{t\to t+T}
\end{aligned}
$$

The population slopes and R^2s from these regressions are conveniently summarized in Table 6, Panel A, with no need to report t-statistics as they are based on 400,000 quarters of simulated data. The risk premiums in the aggregate stock market are predictable with the correct signs, and plausible magnitudes of the linear projection coefficients. Furthermore, both slopes and R^2s rise with the horizon.

An astute reader may worry about the relatively high forecastability of the dividend growth rate. Table 6, Panel B, demonstrates, that we if re-ran the postwar history, about 50% of the time we would tend to conclude that the aggregate dividend growth rate is unpredictable with the respective price-dividend ratio.

V. Summary

I present a consumption-based asset pricing model that puts a macrocoenomic explanation on a number of phenomena observed in the aggregate equity market. This seemingly elusive link between the real and financial sectors of the economy is accomplished by means of an aversion to Bayesian uncertainty about macreconomic fundamentals, thereby evading the failures of much of the previous literature to connect the apparent turbulence in asset markets with a seemingly smooth and unpredictable consumption growth.

The great virtue of the model is its conceptual simplicity, the minor technical drawback the programming labor needed to solve it. In terms of the economics, I stay neoclassical; in terms of computations, I go a bit experimental, using high-performance machines, together with the freely-available C++/Fortran numerical libraries from the ingenious open-source community. The focus of the paper is economics wherefore I have throughout suppressed a description of the computational details, including the domain decomposition, the choice of the numerical scheme, or the code parallelization by means of OpenMP and Message Passing Interface (MPI) protocols, in order not to distract the reader as these issues are just too interesting.

I avail myself of the construct of the representative household, endowed with the recursive utility function of Epstein and Zin (1989), which I configure with plausible magnitudes of the aversion to late resolution of uncertainty and the elasticity of intertemporal

TABLE 6. Predictability Results for Aggregate Equity Market, State-Dependent Case : Regression Analysis on Simulated Data

Panel A. Population Predictive Regressions, 400 000 Quarters

T (in quarters)	$d_{t+1} - d_t$		$r^S_{t\to t+T} - r^f_t$	
	b	R^2	b	R^2
4	-0.26	8.27 %	0.46	11.45 %
8	-0.36	7.30 %	0.74	16.64 %
12	-0.44	6.69 %	0.92	19.20 %
16	-0.50	6.15 %	1.05	20.37 %
20	-0.54	5.57 %	1.15	20.83 %

Panel B. Predictive Regressions, Small Sample of 220 Quarters, Medians

T (in quarters)	$d_{t+1} - d_t$		$r^S_{t\to t+T} - r^f_t$	
	b	R^2	b	R^2
4	-0.23	6.63 %	0.53	13.66 %
8	-0.30	4.95 %	0.85	19.84 %
12	-0.32	3.78 %	1.07	23.62 %
16	-0.33	3.58 %	1.20	24.81 %
20	-0.31	3.28 %	1.30	26.05 %

Notes: Predictive regressions are run with overlapping observations. Risk-free rate is the YTM on a risk-free zero-coupon bond maturing in one quarter.

substitution. I further endow the household with a single asset, the Lucas tree, yielding a periodic stream of income. Although its level is perfectly observable, its growth rate is subject to sporadic large shocks and incessant small, Brownian, shocks. As it is statistically hard to discriminate between the source of the shocks, the information structure is incomplete, confronting investors with a complex signal-extraction problem.

The kernel of the paper lies in the specification of the large shocks, which are modelled with the help of a continuous-time four-state hidden Markov chain model (HMM). The two states of the HMM model are meant to encapsulate the business-cycle dynamics, with the remaining two tracking the variation in the expected income growth at a low frequency. I justify the calibration of the HMM model to the real per-capita consumption growth by means of detection-error probabilities; the probability of mistakenly believing in the income growth being identically and independently distributed is more than 5% with less than 80 years of data.

The model offers a novel laboratory to evaluate the effect of economic uncertainty about the macroeconomic fundamentals on asset prices, being able to account for (i) the observed magnitude of the equity premium, (ii) the low and stable risk-free rate, (iii) the magnitude and the countercyclicality of risk prices, (iv) the average levels and the procyclicality of price-dividend and wealth-consumption ratios, (v) the long-horizon predictability of risk premia, and (vi) the overreaction of price-dividend ratio to bad news in good times, all within the conceptually simple representative-agent framework.

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Appendix A. Wonham's (1964) Nonlinear Filter In Detail

Let the four-tuple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a filtered probability space, supporting a bivariate Brownian motion $\{\mathbf{Z}_t, \mathcal{F}_t\}_{t\geq 0}$, where $\mathbf{Z}_t = [Z_{1t}, Z_{2t}]'$. Let $\{\mathbf{Y}_t, \mathcal{F}_t\}_{t\geq 0}$ denote the vector of the endowment growth rate and dividend growth rate,

$$d\mathbf{Y}_t \quad = \quad \left[\frac{dC_t}{C_t}, \frac{dD_t}{D_t} - (1-\phi)\mu_C^s\, dt\right]'$$

Its dynamics is given by the vector stochastic differential equation

$$d\mathbf{Y}_t = \mathbf{\Xi}_t\, dt + \mathbf{\Sigma}\, d\mathbf{Z}_t \tag{A.1}$$

where the volatility matrix

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_C & 0 \\ \phi\sigma_C & \sigma_{D,2} \end{pmatrix}.$$

and the drift

$$\mathbf{\Xi}_t = [\kappa_t,\, \phi\kappa_t]'$$

switches between four states at random times. It follows a four-state continuous-time hidden Markov chain, with the transition kernel in the time period $(t,\, t + \Delta t)$ given by $\boldsymbol{P}(\Delta t) = \boldsymbol{I} + \boldsymbol{M}\,\Delta t + \bar{o}(\Delta t)$; the generator $\boldsymbol{M}$ is defined

$$\boldsymbol{M} = \lim_{\Delta t \downarrow 0} \frac{\boldsymbol{P}(\Delta t) - \boldsymbol{I}}{\Delta t} \tag{A.2}$$

and it is given by the 4×4 real matrix

$$\boldsymbol{M} = \begin{pmatrix} -\lambda_{12} - \lambda_{13} - \lambda_{14} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & -\lambda_{21} - \lambda_{23} - \lambda_{24} & \lambda_{23} & \lambda_{24} \\ \lambda_{31} & \lambda_{32} & -\lambda_{31} - \lambda_{32} - \lambda_{34} & \lambda_{34} \\ \lambda_{41} & \lambda_{42} & \lambda_{43} & -\lambda_{41} - \lambda_{42} - \lambda_{43} \end{pmatrix},$$

with $\lambda_{ij} \geq 0$, $i \neq j$. The four states correspond to four states for the true expected endowment growth rate κ_t. Formally,

$$\begin{aligned} \mathbf{\Xi}_1 &= [\kappa_1,\, \phi\kappa_1] \\ \mathbf{\Xi}_2 &= [\kappa_2,\, \phi\kappa_2] \\ \mathbf{\Xi}_3 &= [\kappa_3,\, \phi\kappa_3] \\ \mathbf{\Xi}_4 &= [\kappa_4,\, \phi\kappa_4] \end{aligned}$$

Define the orthogonalized innovation vector $\widetilde{\mathbf{Z}}_t$ as

$$d\widetilde{\mathbf{Z}}_t = \left(\mathbf{\Sigma}\mathbf{\Sigma}'\right)^{-1/2} [d\mathbf{Y}_t - \mathbb{E}\{d\mathbf{Y}_t \,|\, \mathcal{I}_t\}] \tag{A.3}$$

where the investor's information set, $\mathcal{I}_t = \sigma\left(\{\mathbf{Y}_\tau \,:\, 0 \leq \tau \leq t\}\right)$, is strictly smaller than $\mathcal{F}_t$ due to the unobservability of the true expected growth rates of consumption. To introduce appropriate notation, let the volatilities with respect to the innovation process $\widetilde{\mathbf{Z}}$ be denoted as

$$\begin{bmatrix} \widetilde{\mathbf{g}}_C \\ \widetilde{\mathbf{g}}_{\mathrm{D}} \end{bmatrix} = \left(\mathbf{\Sigma}\mathbf{\Sigma}'\right)^{1/2}$$

and let the 'new' drifts be denoted

$$\begin{aligned} m_{Ct} &= \sum_{i=1}^{4} \kappa_i\, \pi_{it} \\ m_{Dt} &= (1 - \phi)\, \mu_C^{\$} + \phi \left(\sum_{i=1}^{4} \kappa_i\, \pi_{it}\right) \end{aligned}$$

Hence, we may rewrite the Itô process for the vector Y_t as a homoscedastic diffusion

$$\begin{aligned} \frac{dC_t}{C_t} &= m_{Ct}\, dt + \widetilde{\mathbf{g}}_C\, d\widetilde{\mathbf{Z}}_t \\ \frac{dD_t}{D_t} &= m_{Dt}\, dt + \widetilde{\mathbf{g}}_D\, d\widetilde{\mathbf{Z}}_t \end{aligned}$$

where I have defined the posterior probabilities as

$$\pi_{it} = \mathcal{P}\{\,\mathbf{\Xi}_t = \mathbf{\Xi}_i \,|\, \mathcal{I}_t\,\}, \quad i \in \{1, 2, 3, 4\} \tag{A.4}$$

Clearly, $\sum_{i=1}^{4} \pi_{it} = 1$, and thus π_{4t} is redundant. Let us also denote

$$\mathbf{m}_t^{\Xi} = \sum_{i=1}^{4} \Xi_i \, \pi_{it} \tag{A.5}$$

The posterior probabilities follow the time-homogenous diffusion (Wonham 1964, Liptser and Shiryaev 2001)

$$d\pi_{it} = m_{it} \, dt + \widetilde{\mathbf{h}}_{it} \, d\widetilde{\mathbf{Z}}_t \tag{A.6}$$

where

$$m_{it} = \sum_{j=1}^{4} \lambda_{ji} \, \pi_{jt}, \quad i \in \{1,2,3\} \tag{A.7}$$

$$\widetilde{\mathbf{h}}_{it} = \pi_{it} \left[\Xi_i - \mathbf{m}_t^{\Xi} \right] \left(\Sigma \Sigma' \right)^{-1/2}, \; i \in \{1,2,3\} \tag{A.8}$$

Appendix B. Proof of Proposition 2

Recall the definition of the innovation process $d\widetilde{\boldsymbol{Z}}_t$, written explicitly as

$$\begin{aligned} d\widetilde{\mathbf{Z}}_t &= \left(\Sigma\Sigma'\right)^{-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left\{ \frac{dC_t}{C_t} - \mathbb{E}_t \left(\frac{dC_t}{C_t} \,|\, \mathcal{I}_t \right) \right\} + \\ &+ \left(\Sigma\Sigma'\right)^{-1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left\{ \frac{dD_t}{D_t} - \mathbb{E}_t \left(\frac{dD_t}{D_t} \,|\, \mathcal{I}_t \right) \right\} \end{aligned}$$

The term $\frac{dC_t}{C_t} - \mathbb{E}_t \left(\frac{dC_t}{C_t} \,|\, \mathcal{I}_t \right)$ corresponds to the consumption growth innovation, and the term $\frac{dD_t}{D_t} - \mathbb{E}_t \left(\frac{dD_t}{D_t} \,|\, \mathcal{I}_t \right)$ is the dividend growth innovation. Note that as long as $\left(\Sigma\Sigma'\right)^{-1/2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a nonzero vector, we do need the dividend signal even though it actually is nosier than the consumption signal. I now proceed to show that the second column of this matrix goes to zero as the orthogonal part the dividend volatility $\sigma_{D,2}$ grows without bounds. In such a case, investors learn exclusively from the consumption innovations $d\widetilde{Z}_1$; the dividend signal is irrelevant. What the proof also shows, though, is that as long as $\sigma_{D,2}$ is finite, the dividend signal cannot be disposed of in the inference problem, at least as long as $\phi \neq 0$.

Recall the volatility matrix

$$\Sigma = \begin{pmatrix} \sigma_C & 0 \\ \phi \sigma_C & \sigma_{D,2} \end{pmatrix}$$

Basic algebra shows that

$$\left(\Sigma\Sigma'\right) = \begin{pmatrix} \sigma_C^2 & \phi\sigma_C^2 \\ \phi\sigma_C^2 & \phi^2\sigma_C^2 + \sigma_{D,2}^2 \end{pmatrix} = \sigma_C^2 \times \begin{pmatrix} 1 & \mathcal{O}(\phi) \\ \mathcal{O}(\phi) & \mathcal{O}\left(\phi^2 + \sigma_{D,2}^2/\sigma_C^2\right) \end{pmatrix}$$

To prove the proposition, it suffices to show that the second column of $\left(\Sigma\Sigma'\right)^{-1/2}$ is zero only in the limit as $\sigma_{D,2}^2$ goes to infinity. To proceed, define

$$\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & \mathcal{O}(\phi) \\ \mathcal{O}(\phi) & \mathcal{O}\left(\phi^2 + \sigma_{D,2}^2/\sigma_C^2\right) \end{pmatrix}^{-1}$$

This yields the following system of linear equations for v_1 and v_2

$$\begin{aligned} v_1 + \mathcal{O}(\phi)\, v_2 &= 0 \\ \mathcal{O}(\phi)\, v_2 + \mathcal{O}\left(\phi^2 + \sigma_{D,2}^2/\sigma_C^2\right) v_2 &= 1 \end{aligned}$$

the solution of which is

$$\begin{aligned} v_1 &= -\mathcal{O}(\phi) / \mathcal{O}\left(\sigma_{D,2}^2/\sigma_C^2\right) \\ v_2 &= 1/\mathcal{O}\left(\sigma_{D,2}^2/\sigma_C^2\right) \end{aligned}$$

Clearly, v_1 and v_2 converge to zero as $\sigma_{D,2}^2/\sigma_C^2$ grows without bounds. Hence,

$$\lim_{\sigma_{D,2}^2 \to \infty} \begin{pmatrix} \sigma_C^2 & \phi \sigma_C^2 \\ \phi \sigma_C^2 & \phi^2 \sigma_C^2 + \sigma_{D,2}^2 \end{pmatrix}^{-1} = \begin{pmatrix} \bar{u}_1 & 0 \\ \bar{u}_2 & 0 \end{pmatrix}$$

Hence,

$$\lim_{\sigma_{D,2}^2 \to \infty} \begin{pmatrix} \sigma_C^2 & \phi \sigma_C^2 \\ \phi \sigma_C^2 & \phi^2 \sigma_C^2 + \sigma_{D,2}^2 \end{pmatrix}^{-1/2} = \begin{pmatrix} \sqrt{\bar{u}_1} & 0 \\ \bar{u}_2 / \sqrt{\bar{u}_1} & 0 \end{pmatrix}$$

Appendix C. Pricing the Lucas (1978) Tree

a. *The Value Function and Hamilton-Jacobi-Bellman (HJB) Equation.* Recall the dynamic budget constraint

$$dW_t = m_W \, dt + W \, \widetilde{\mathbf{g}}_W \, d\widetilde{\mathbf{Z}}_t \tag{C.1}$$

where

$$m_W = a_P \, [m_P - r_t] \, W_t + r_t \, W_t - c_t \tag{C.2}$$

$$\widetilde{\mathbf{g}}_W = a_P \, \widetilde{\mathbf{g}}_P \tag{C.3}$$

The infinitesimal operator $\mathcal{L}$ takes the form

$$\mathcal{L}J = m_W \, \partial_W J + \frac{1}{2} W^2 \left(\widetilde{\mathbf{g}}_W \widetilde{\mathbf{g}}_W' \right) \partial_{WW} J + \mathbf{m}_{\boldsymbol{\pi}}' \partial_{\boldsymbol{\pi}} J + \tag{C.4}$$

$$+ \quad \frac{1}{2} tr \left(\widetilde{\mathbf{H}} \partial^2_{\boldsymbol{\pi}\boldsymbol{\pi}'} J \right) + W \sum_{j=1}^{3} \left(\widetilde{\mathbf{g}}_W \widetilde{\mathbf{h}}_j' \right) \partial_{Wj} \, J \tag{C.5}$$

where I denote $\boldsymbol{\pi} := [\pi_1, \pi_2, \pi_3]'$, the matrix $\widetilde{\mathbf{H}} := \left[\left(\widetilde{\mathbf{h}}_i \widetilde{\mathbf{h}}_j' \right) \right]_{3\times 3}$, the operators $\partial_{\boldsymbol{\pi}} = [\partial_1, \, \partial_2, \, \partial_3]'$ and $\partial^2_{\boldsymbol{\pi}\boldsymbol{\pi}'} = [\partial_{ij}]_{3\times 3}$ with $\partial_i \equiv \partial / \partial \pi_i$, and the vector $\mathbf{m}_{\boldsymbol{\pi}} := [m_1, \, m_2, \, m_3]$.

According to Duffie and Epstein (1992ab), the Hamilton-Jacobi-Bellman (HJB) equation is

$$0 = \sup_{\{c, \, a_P\}} \left\{ f(c, \, J) + \mathcal{L}J \right\} \tag{C.6}$$

I make an educated guess that

$$J(W, \boldsymbol{\pi}) = \alpha^{-1} \, \delta^{\alpha/\rho} \, [\Psi(\boldsymbol{\pi})]^{\frac{1-\rho}{(\rho/\alpha)}} \, W^{\alpha} \tag{C.7}$$

The partial derivatives of the value function are

$$\begin{aligned}
\partial_W \, J &= \frac{\alpha J}{W} \\
\partial_{WW} \, J &= \frac{\alpha \, (\alpha - 1) \, J}{W^2} \\
\partial_i \, J &= \left(\frac{1-\rho}{\rho/\alpha} \right) \left(\frac{\partial_i \Psi}{\Psi} \right) J \\
\partial_{ij} \, J &= \left(\frac{1-\rho}{\rho/\alpha} \right) \left[\frac{\partial_{ij} \Psi}{\Psi} + \left(\frac{1-\rho}{\rho/\alpha} - 1 \right) \left(\frac{\partial_i \Psi \, \partial_j \Psi}{\Psi^2} \right) \right] J \\
\partial_{Wi} \, J &= \left(\frac{1-\rho}{\rho/\alpha} \right) \left(\frac{\partial_i \Psi}{\Psi} \right) \frac{\alpha J}{W}
\end{aligned}$$

and the partial derivative of the normalized aggregator is

$$\partial_c \, f(c, J) = \frac{\delta \, c^{\rho - 1}}{(\alpha J)^{(\rho/\alpha) - 1}}$$

The first-order condition for the consumption rate c yields

$$\begin{aligned}
\partial_C\, f(c,J) &= \partial_W\, J \\
\frac{\delta\, c^{\rho-1}}{(\alpha J)^{(\rho/\alpha)-1}} &= \frac{\alpha J}{W} \\
\delta\, c^{\rho-1} &= (\alpha J)^{(\rho/\alpha)}\, W^{-1} \\
c^{\rho-1} &= \Psi^{1-\rho} W^{\rho-1} \\
\frac{W}{c} &= \Psi(\boldsymbol{\pi})
\end{aligned}$$

In equilibrium, financial markets must clear, $a_P = 1$, and goods markets must clear, $c = C$. Substituting into the budget constraint (C.1) yields

$$dW_t \;=\; (m_P W_t - C_t)\; dt \;+\; W_t\, \widetilde{\mathbf{g}}_P\, d\widetilde{\mathbf{Z}}_t$$

Recalling the dynamics of the Lucas tree

$$dP_t \;=\; (m_P P_t - C_t)\; dt \;+\; P_t\, \widetilde{\mathbf{g}}_P\, d\widetilde{\mathbf{Z}}_t$$

and the condition that the initial wealth W_0 equals the value of the Lucas tree P_0 yields that

$$\forall t \geq 0 \;:\; P_t = W_t \tag{C.8}$$

As a result, the equilibrium wealth-consumption ratio equals the price-endowment ratio of the underlying Lucas tree

$$\frac{P}{C} = \frac{W}{c} = \Psi(\boldsymbol{\pi}) \tag{C.9}$$

Cross-multiplying by the endowment rate, and applying Itô lemma yields

$$\begin{aligned}
P &= \Psi(\boldsymbol{\pi})\, C \\
\frac{dP}{P} &= \frac{d\Psi}{\Psi} + \frac{dC}{C} + \frac{d[\Psi,C]}{\Psi C}
\end{aligned}$$

Furher application of Itô lemma to function $\Psi = \Psi(\boldsymbol{\pi})$ gives

$$\begin{aligned}
d\Psi &= \sum_{i=1}^{3} \partial_i \Psi\, d\pi_i + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \partial_{ij}\Psi\, d[\pi_i,\pi_j] \\
&= \left\{ \sum_{i=1}^{3} m_i \partial_i \Psi \;+\; \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right)\, \partial_{ij}\Psi \right\} dt \;+ \\
&+ \sum_{i=1}^{3} \widetilde{\boldsymbol{h}}_i\, \partial_i \Psi\, d\widetilde{\boldsymbol{Z}}
\end{aligned}$$

In addition, the dynamics of the endowment is

$$\frac{dC}{C} \;=\; m_C\, dt + \widetilde{\mathbf{g}}_C\, d\widetilde{\mathbf{Z}}$$

Finally, the cross-variation between the processes $\Psi$and C is

$$\frac{d[\Psi,C]}{\Psi C} \;=\; \sum_{i=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\mathbf{g}}_C'\right) \left(\frac{\partial_i \Psi}{\Psi}\right) dt$$

Putting all these results together yields

$$
\begin{aligned}
\frac{dP}{P} &= \left(m_P - \frac{C}{P}\right) dt + \widetilde{\mathbf{g}}_P(t)\, d\widetilde{\mathbf{Z}}(t) \\
\frac{dP}{P} &= \left[m_C + \sum_{i=1}^{3}\left(m_i + \left(\widetilde{\boldsymbol{h}}_i \widetilde{\mathbf{g}}_C'\right)\right)\left(\frac{\partial_i \Psi}{\Psi}\right)\right] dt \\
&+ \left[\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_{ij}\Psi}{\Psi}\right)\right] dt + \left[\widetilde{\mathbf{g}}_C + \sum_{i=1}^{3}\widetilde{\boldsymbol{h}}_i \left(\frac{\partial_i \Psi}{\Psi}\right)\right] d\widetilde{\boldsymbol{Z}}
\end{aligned}
$$

I invoke the canonical decomposition for continuous semi-martingales to get the instantaneous conditional expected return of the Lucas tree

$$
\begin{aligned}
m_P &= \Psi^{-1} + m_C + \sum_{i=1}^{3}\left(m_i + \left(\widetilde{\boldsymbol{h}}_i \widetilde{\mathbf{g}}_C'\right)\right)\left(\frac{\partial_i \Psi}{\Psi}\right) \\
&+ \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_{ij}\Psi}{\Psi}\right)
\end{aligned}
\tag{C.10}
$$

and the instantaneous conditional volatility vector of the Lucas tree

$$
\widetilde{\mathbf{g}}_P = \widetilde{\boldsymbol{g}}_C + \sum_{i=1}^{3}\widetilde{\mathbf{h}}_i \left(\frac{\partial_i \Psi}{\Psi}\right) \tag{C.11}
$$

Before we substitute all these intermediate results back into the HJB, note that

$$
\begin{aligned}
\widetilde{\mathbf{g}}_P \widetilde{\mathbf{g}}_P' &= \left[\widetilde{\boldsymbol{g}}_C + \sum_{i=1}^{3}\widetilde{\mathbf{h}}_i \left(\frac{\partial_i \Psi}{\Psi}\right)\right]\left[\widetilde{\boldsymbol{g}}_C + \sum_{j=1}^{3}\widetilde{\mathbf{h}}_j \left(\frac{\partial_j \Psi}{\Psi}\right)\right]' \\
&= \left(\widetilde{\boldsymbol{g}}_C \widetilde{\boldsymbol{g}}_C'\right) + 2\sum_{i=1}^{3}\left(\widetilde{\mathbf{h}}_i \widetilde{\boldsymbol{g}}_C'\right)\left(\frac{\partial_i \Psi}{\Psi}\right) + \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\mathbf{h}}_i \widetilde{\mathbf{h}}_j'\right)\left(\frac{\partial_i \Psi}{\Psi}\right)\left(\frac{\partial_j \Psi}{\Psi}\right)
\end{aligned}
\tag{C.12}
$$

and

$$
\begin{aligned}
\widetilde{\mathbf{g}}_P \widetilde{\mathbf{h}}_j' &= \left[\widetilde{\boldsymbol{g}}_C + \sum_{i=1}^{3}\widetilde{\mathbf{h}}_i \left(\frac{\partial_i \Psi}{\Psi}\right)\right]\widetilde{\mathbf{h}}_j' = \\
&= \left(\widetilde{\boldsymbol{g}}_C \widetilde{\mathbf{h}}_j'\right) + \sum_{i=1}^{3}\left(\widetilde{\mathbf{h}}_i \widetilde{\mathbf{h}}_j'\right)\left(\frac{\partial_i \Psi}{\Psi}\right)
\end{aligned}
\tag{C.13}
$$

Substituting into the HJB equation together with the equilibrium conditions, skipping the quite tedious intermediate steps

$$
\begin{aligned}
0 &= f(C,\,J) + \mathcal{L}J \\
&= \frac{\delta}{\rho}\,\frac{C^{\rho} \,-\, (\alpha\, J)^{\rho/\alpha}}{(\alpha\, J)^{(\rho/\alpha)-1}} + \left(m_P - \Psi^{-1}\right) W\,\partial_W J + \frac{1}{2}\,W^2\,\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right)\,\partial_{WW} J \,+\, \mathbf{m}_{\boldsymbol{\pi}}'\partial_{\boldsymbol{\pi}} J \,+ \\
&+\; \frac{1}{2}\,tr\left(\widetilde{\mathbf{H}}\partial^2_{\boldsymbol{\pi}\boldsymbol{\pi}'} J\right) \,+\, W \sum_{j=1}^{3}\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{h}}_j'\right)\,\partial_{Wj}\, J = \\
&= \left(\frac{1}{\delta\Psi} - 1\right)\frac{\delta}{\rho}\alpha J + \\
&+\; \left[m_C + \sum_{i=1}^{3}\left(m_i + \left(\widetilde{\boldsymbol{h}}_i\widetilde{\mathbf{g}}_C'\right)\right)\left(\frac{\partial_i\Psi}{\Psi}\right) + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{h}}_j'\right)\,\left(\frac{\partial_{ij}\Psi}{\Psi}\right)\right]\alpha J \\
&+\; \frac{1}{2}\left[\left(\widetilde{\boldsymbol{g}}_C\widetilde{\boldsymbol{g}}_C'\right) + 2\sum_{i=1}^{3}\,\left(\widetilde{\mathbf{h}}_i\widetilde{\boldsymbol{g}}_C'\right)\,\left(\frac{\partial_i\Psi}{\Psi}\right) + \sum_{i=1}^{3}\sum_{j=1}^{3}\,\left(\widetilde{\mathbf{h}}_i\widetilde{\mathbf{h}}_j'\right)\,\left(\frac{\partial_i\Psi}{\Psi}\right)\left(\frac{\partial_j\Psi}{\Psi}\right)\right]\alpha\,(\alpha-1)\,J \\
&+\; \left(\frac{1-\rho}{\rho/\alpha}\right)\sum_{i=1}^{3} m_i\left(\frac{\partial_i\Psi}{\Psi}\right)\,J + \\
&+\; \frac{1}{2}\left(\frac{1-\rho}{\rho/\alpha}\right)\sum_{i=1}^{3}\sum_{j=1}^{3}\,\left(\widetilde{\mathbf{h}}_i\widetilde{\mathbf{h}}_j'\right)\,\left[\frac{\partial_{ij}\Psi}{\Psi} + \left(\left(\frac{1-\rho}{\rho/\alpha}\right) - 1\right)\left(\frac{\partial_i\Psi}{\Psi}\right)\left(\frac{\partial_j\Psi}{\Psi}\right)\right]\,J \\
&+\; \left(\frac{1-\rho}{\rho/\alpha}\right)\sum_{j=1}^{3}\left[\left(\widetilde{\boldsymbol{g}}_C\widetilde{\mathbf{h}}_j'\right) \,+\, \sum_{i=1}^{3}\,\left(\widetilde{\mathbf{h}}_i\widetilde{\mathbf{h}}_j'\right)\,\left(\frac{\partial_i\Psi}{\Psi}\right)\right]\,\left(\frac{\partial_j\Psi}{\Psi}\right)\,\alpha J
\end{aligned}
$$

Simplifying and re-arranging the terms yields that the price-endowment ratio $\Psi\,:\,\Delta^3\ \rightarrow \mathbb{R}^+$ solves the nonlinear elliptic PDE

$$0 = \mathcal{L}\,\Psi \,+\, a_0\,\Psi\ \,+1 \tag{C.14}$$

subject to the boundary conditions

$$\lim_{\boldsymbol{\pi}\,\rightarrow\,\partial\Delta^3}\ \Psi(\boldsymbol{\pi})\ \sim\ \text{finite} \tag{C.15}$$

where differential operator $\mathcal{L}$ is given by

$$\mathcal{L}\,\Psi \quad = \quad \frac{1}{2}\,tr\left(\widetilde{\mathbf{H}}\,\partial^2_{\boldsymbol{\pi}\boldsymbol{\pi}'}\Psi\right) - \frac{1}{2\,\Psi}\left(1 - \frac{\alpha}{\rho}\right)\left(\partial_{\boldsymbol{\pi}}\Psi'\,\widetilde{\boldsymbol{H}}\,\partial_{\boldsymbol{\pi}}\Psi\right) + \left(\mathbf{a}_1'\partial_{\boldsymbol{\pi}}\Psi\right) \tag{C.16}$$

I additionally denote the vector $\mathbf{a}_1 := [m_i + \alpha\,(\widetilde{\boldsymbol{g}}_C\mathbf{h}_i')]_{3\times 1}$ and the scalar

$$a_0 := -\delta + \rho\left[\sum_{i=1}^{3}\theta_i\pi_i + \theta_4\,(1 - \pi_1 - \pi_2 - \pi_3)\right] - \frac{1}{2}\,(1-\alpha)\,\rho\,\left(\widetilde{\boldsymbol{g}}_C\widetilde{\boldsymbol{g}}_C'\right).$$

Appendix D. The Real Interest Rate

The first-order condition from the HJB equation with respect to the portfolio weight a_P yields

$$0 \quad = \quad (m_P - r)\,W\partial_W J + W^2 a_P\,\left(\widetilde{\boldsymbol{g}}_P\widetilde{\boldsymbol{g}}_P'\right)\partial_{WW} J + W\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_P'\right)\partial_{Wi}\,J \tag{D.1}$$

To ensure equilibrium in the financial market, we must have

$$a_P = 1 \tag{D.2}$$

Imposing this condition upon (D.1) and solving for the real interest rate gives

$$\begin{aligned} r &= m_P + \frac{W \partial_{WW} J}{\partial_W J} \left(\widetilde{\boldsymbol{g}}_P \widetilde{\boldsymbol{g}}_P'\right) + \sum \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{g}}_P'\right) \left(\frac{\partial_{Wi} J}{\partial_W J}\right) \\ r &= m_P - (1-\alpha)\left(\widetilde{\boldsymbol{g}}_P \widetilde{\boldsymbol{g}}_P'\right) + \left(\frac{1-\rho}{\rho/\alpha}\right) \sum_{i=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{g}}_P'\right) \left(\frac{\partial_i \Psi}{\Psi}\right) \end{aligned}$$

Recalling equations (C.10), (C.12) and (C.13) yields after tedious algebra $r = r(\boldsymbol{\pi})$; that is,

$$\begin{aligned} r &= \delta + (1-\rho)\left[\sum_{i=1}^{3} \theta_i \pi_i + \theta_4 (1 - \pi_1 - \pi_2 - \pi_3)\right] - \\ &- \frac{1}{2}(1-\alpha)(2-\rho)\left(\widetilde{\boldsymbol{g}}_C \widetilde{\boldsymbol{g}}_C'\right) - \left(1 - \frac{\alpha}{\rho}\right) \sum_{i=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{g}}_C'\right) \left(\frac{\partial_i \Psi}{\Psi}\right) - \\ &- \frac{1}{2}\left(1 - \frac{\alpha}{\rho}\right) \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right) \left(\frac{\partial_i \Psi \, \partial_j \Psi}{\Psi^2}\right) \end{aligned}$$

Appendix E. Real Pricing Kernel

In case of the normalized aggregator

$$f(C, J) = \frac{\delta}{\rho} \frac{C^\rho - (\alpha J)^{\rho/\alpha}}{(\alpha J)^{(\rho/\alpha)-1}},$$

Duffie and Epstein (1992ab) show that the equilibrium pricing kernel $\{ M_t, \mathcal{I}_t \}_{t \geq 0}$ takes the form

$$M_t = \exp\left(\int_0^t \partial_J f(C_s, J_s)\, ds\right) \partial_C f(C_t, J_t).$$

Application of Itô lemma gives

$$\begin{aligned} \frac{dM}{M} &= \partial_J f(C, J)\, dt + \frac{d\partial_C f(C, J)}{\partial_C f(E, J)} \\ &= \mathcal{O}(dt) + d \log \partial_C f(C, J) \\ &= \mathcal{O}(dt) + d \log \left[\frac{C^{\rho-1}}{(\alpha J)^{(\rho/\alpha)-1}}\right] \\ &= \mathcal{O}(dt) + (\rho - 1)\, d\log C - \left(\frac{\rho}{\alpha} - 1\right) d \log J \end{aligned}$$

Note that

$$\log J \propto \left(\frac{1-\rho}{\rho/\alpha}\right) \log \Psi + \alpha \log W \tag{E.1}$$

$$\log W = \log C + \log \Psi \tag{E.2}$$

Hence,

$$\begin{aligned} \frac{dM}{M} &= \mathcal{O}(dt) + (\rho-1)\, d\log C + \left(\frac{\rho}{\alpha} - 1\right)\left[\left(\frac{1-\rho}{\rho/\alpha}\right) d\log\Psi + \alpha\left(\log C + \log \Psi\right)\right] \\ &= \mathcal{O}(dt) - (1-\alpha)\, d\log C - \left(1 - \frac{\rho}{\alpha}\right) d \log \Psi \\ &= \mathcal{O}(dt) - \left\{(1-\alpha)\, \widetilde{\boldsymbol{g}}_C + \left(1 - \frac{\alpha}{\rho}\right) \sum_{i=1}^{3} \widetilde{\boldsymbol{h}}_i \left(\frac{\partial_i \Psi}{\Psi}\right)\right\} d\widetilde{\boldsymbol{Z}} \end{aligned}$$

We know that the drift of the real pricing kernel growth rate is equal to the minus of the real interest rate $-r_t dt$. I have found the expression for $r \equiv r(\boldsymbol{\pi})$ previously in a different, and much easier, way (see subsection above);

as a result,

$$\frac{dM_t}{M_t} = -r_t\,dt - \left\{(1-\alpha)\,\widetilde{\boldsymbol{g}}_C + \left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\widetilde{\boldsymbol{h}}_i\left(\frac{\partial_i\Psi}{\Psi}\right)\right\} d\widetilde{\boldsymbol{Z}}_t$$

Appendix F. Expected Excess Return on the Lucas (1978) Tree

The dynamic Euler equation

$$\mathbb{E}\left(\frac{dP_t}{P_t} + \frac{C_t}{P_t}dt\,|\,\mathcal{I}_t\right) - r_t\,dt = -Cov\left(\frac{dM_t}{M_t}, \frac{dP_t}{P_t}\,|\,\mathcal{I}_t\right)$$

yields the formula for the instantaneous conditional expected excess return on the wealth portfolio as

$$m_P - r = (1-\alpha)\left(\widetilde{\boldsymbol{g}}_C\widetilde{\boldsymbol{g}}_P'\right) + \left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_P'\right)\left(\frac{\partial_i\Psi}{\Psi}\right)$$

See a subsection above for the expression for the instantaneous conditional volatility of the Lucas tree.

Appendix G. Finite Horizon Moments

A. **Preliminaries.** The $\mathcal{P}$-dynamics of the value of the Lucas (1978) tree is

$$\frac{dP_t}{P_t} = \left(m_P - \frac{C_t}{P_t}\right)dt + \widetilde{\mathbf{g}}_P\,d\widetilde{\mathbf{Z}}_t$$

Hence,

$$d\log P_t + \frac{C_t}{P_t}dt \quad -r_t dt = \left(m_P - r_t - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right)\right)dt + \widetilde{\mathbf{g}}_P\,d\widetilde{\mathbf{Z}}_t$$

We recognize the term on the left-hand side as the instantaneous excess log-return; that is,

$$d\log R_t^e = \left(m_P - r_t - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right)\right)dt + \widetilde{\mathbf{g}}_P\,d\widetilde{\mathbf{Z}}_t$$

Written in the integral form,

$$\log R_T^e - \log R_t^e = \int_t^T\left[m_P - r_\tau - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right)\right]d\tau + \int_t^T\widetilde{\mathbf{g}}_P\,d\widetilde{\mathbf{Z}}_\tau$$

The left-hand side tells us the cumulative log-return earned over the time interval $[t,T]$. I am interested in computing its first two conditional moments.

B. **Conditional Mean.** Define

$$G(t;T) = \mathbb{E}\left[\log R(T) - \log R(t)\,|\,\mathcal{I}(t)\right]$$

As before, the vector of posterior probabilities $\boldsymbol{\pi}$ is a Markov process, and hence, with a slight abuse of notation,

$$\mathbb{E}\left[\log R_T - \log R_t\,|\,\mathcal{I}_t\right] = \mathbb{E}\left[\log R_T - \log R_t\,|\,\boldsymbol{\pi}_t\right] =: G(\boldsymbol{\pi}_t, t; T)$$

As a result,

$$G(\boldsymbol{\pi}_t, t; T) = E^{\boldsymbol{\pi},t}\left\{\int_t^T\left[m_P - r - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right)\right]d\tau\,|\,\mathcal{I}_t\right\} + E^{\boldsymbol{\pi},t}\left\{\int_t^T\widetilde{\mathbf{g}}_P\,d\widetilde{\mathbf{Z}}\,|\,\mathcal{I}_t\right\}$$

$$G(\boldsymbol{\pi}, t; T) = \mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T\left[m_P - r - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right)\right]d\tau\right\}$$

where the Itô integral term is a zero-mean local martingale, and under integrability conditions a martingale, and hence it is zero.

Feynman-Kač theorem tells us that the long-horizon expected return $G(\boldsymbol{\pi}, t; T)$ is the solution of the following backward parabolic partial differential equation

$$0 = \partial_t \mathrm{G} + \mathcal{L}^{\mathcal{G}} G + \left[m_P(\boldsymbol{\pi}) - r(\boldsymbol{\pi}) - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P(\boldsymbol{\pi})\, \widetilde{\mathbf{g}}_P(\boldsymbol{\pi})' \right) \right]$$

subject to the final condition

$$G(\boldsymbol{\pi}, T; T) = 0 \tag{G.1}$$

and the boundary conditions

$$\lim_{\boldsymbol{\pi} \to \partial\Delta^3} G(\boldsymbol{\pi}, t; T) \sim \text{finite for } \forall t \in [0, T] \tag{G.2}$$

where the differential operator

$$\mathcal{L}^{\mathcal{G}} G = \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j' \right) \partial_{ij} G + \sum_{i=1}^{3} m_i\, \partial_i G$$

C. **Conditional Variance.** In the notation from the previous subsection, define the long-horizon variance $Var(t; T)$ as

$$\begin{aligned} Var(t; T) &= Var\left[\log R_T - \log R_t \,|\, \mathcal{I}_t \right] \\ &= \mathbb{E}\left\{ \left[\log R_T - \log R_t \right]^2 \,|\, \mathcal{I}(t) \right\} - \left\{ \mathbb{E}\left[\log R_T - \log R_t \,|\, \mathcal{I}_t \right] \right\}^2 \\ &= H(\boldsymbol{\pi}_t, t; T) - G^2(\boldsymbol{\pi_t}, t; T) \end{aligned}$$

where I denote the conditional second moment as

$$H(\boldsymbol{\pi}, t; T) := \mathbb{E}^{\boldsymbol{\pi}, \mathrm{t}}\left\{ \left[\log R_T - \log R_t \right]^2 \right\}$$

In order to simplify the notation further, define

$$x_T = \log R_T - \log R_t$$

Itô lemma tells us that

$$dx = \left[m_P - r - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P \widetilde{\mathbf{g}}_P' \right) \right] dt + \widetilde{\mathbf{g}}_P\, d\widetilde{\mathbf{Z}}$$

and

$$d\left(x^2\right) = 2x dx + d[x, x] = 2x dx + \left(\widetilde{\mathbf{g}}_P \widetilde{\mathbf{g}}_P' \right) dt$$

In integral form

$$\begin{aligned} x_T^2 - x_t^2 &= \int_t^T 2x dx + \int_t^T \left(\widetilde{\mathbf{g}}_P \widetilde{\mathbf{g}}_P' \right) d\tau \\ &= \int_t^T 2x \left[m_P - r - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P \widetilde{\mathbf{g}}_P' \right) \right] d\tau + \int_t^T 2x \widetilde{\mathbf{g}}_P\, d\widetilde{\mathbf{Z}} + \int_t^T \left(\widetilde{\mathbf{g}}_P \widetilde{\mathbf{g}}_P' \right) d\tau \end{aligned}$$

Taking conditional expectation, the Ito term drops out as it is a zero-mean martingale, and we obtain

$$
\begin{aligned}
H(\boldsymbol{\pi}, t; T) &= \mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T 2x\left[m_P - r - \frac{1}{2}\left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right)\right] d\tau\right\} + \\
&+ \mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T \left(\widetilde{\mathbf{g}}_P\widetilde{\mathbf{g}}_P'\right) d\tau\right\} \\
&= \text{Integral}_1 + \text{Integral}_2
\end{aligned}
$$

Define

$$
f(\boldsymbol{\pi}) = m_P(\boldsymbol{\pi}) - r(\boldsymbol{\pi}) - \frac{1}{2}\left[\widetilde{\mathbf{g}}_P(\boldsymbol{\pi})\widetilde{\mathbf{g}}_P(\boldsymbol{\pi})'\right] \tag{G.3}
$$

The first integral is easily simplified using Fubini Theorem and the law of iterated expectations as follows

$$
\begin{aligned}
\text{Integral}_1 &= 2\,\mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T x_\tau\, f\left(\pi_\tau\right)\, d\tau\right\} \\
&= 2\,\mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T d\tau \int_t^\tau du\, f\left(\pi_u\right) f\left(\pi_\tau\right)\right\} \\
&= 2\,\mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T du \int_u^T d\tau\, f\left(\pi_u\right) f\left(\pi_\tau\right)\right\} \\
&= 2\,\mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T du\, f\left(\pi_u\right)\, \mathbb{E}^{\boldsymbol{\pi},u}\left[\int_u^T f\left(\pi_\tau\right) d\tau\right]\right\} \\
&= 2\,\mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T f\left(\pi_u\right)\, G(\pi_u, u; T)\, du\right\}
\end{aligned}
$$

Merging the two integral yields

$$
H(\boldsymbol{\pi}, t; T) = \mathbb{E}^{\boldsymbol{\pi},t}\left\{\int_t^T 2\, f(\pi_\tau)\, G(\pi_\tau, \tau; T) + \left[\widetilde{\mathbf{g}}_P\left(\pi_\tau\right)\widetilde{\mathbf{g}}_P\left(\pi_\tau\right)'\right]\, d\tau\right\}
$$

Feynman-Kač theorem tells us that $H(\boldsymbol{\pi}, t; T)$ is the solution of the following backward parabolic partial differential equation

$$
0 = \partial_t \mathrm{H} + \mathcal{L}^{\mathcal{H}} H + \left[2\, f(\pi)\, G(\pi, t; T) + \left(\widetilde{\mathbf{g}}_P(\pi)\widetilde{\mathbf{g}}_P(\pi)'\right)\right]
$$

subject to the final condition

$$
H(\boldsymbol{\pi}, T; T) = 0 \tag{G.4}
$$

and the boundary conditions

$$
\lim_{\boldsymbol{\pi} \to \partial\Delta^3} H\left(\boldsymbol{\pi}, t; T\right) \sim \text{finite for } \forall t \in [0, T] \tag{G.5}
$$

where the differential operator

$$
\mathcal{L}^{\mathcal{H}} H = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{h}}_j'\right)\partial_{ij} H + \sum_{i=1}^{3} m_i\, \partial_i H
$$

Finally, the long-horizon variance is then easily calculated as

$$Var(\pi,t;T) = H(\pi,t;T) - G^2(\pi,t;T)$$

Appendix H. Pricing the Leveraged Claim

In our economy, prices are not stationary but price-dividend ratio is. Hence, denote the equilibrium price-dividend ratio as

$$\frac{S}{D} = \Phi(\boldsymbol{\pi})$$

The functional relationship is due to the preference homotheticity; prices (price-dividend ratios) do not depend on the wealth W. Cross-multiplying by the dividend rate yields $S = D\,\Phi(\boldsymbol{\pi})$. Application of Itô lemma gives

$$\frac{dS}{S} = \frac{dD}{D} + \frac{d\Phi}{\Phi} + \frac{d[D,\Phi]}{D\,\Phi}$$

Recall

$$\frac{dD}{D} = m_D\,dt + \widetilde{\mathbf{g}}_D\,d\widetilde{\mathbf{Z}}$$

Furher application of Itô lemma to function $\Phi = \Phi(\boldsymbol{\pi})$ gives

$$\begin{aligned} d\Phi &= \sum_{i=1}^{3} \partial_i \Phi\, d\pi_i + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \partial_{ij}\Phi\, d[\pi_i,\pi_j] \\ &= \left\{ \sum_{i=1}^{3} m_i \partial_i \Phi + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right) \partial_{ij}\Phi \right\} dt + \\ &+ \sum_{i=1}^{3} \widetilde{\boldsymbol{h}}_i\, \partial_i \Phi\, d\widetilde{\boldsymbol{Z}} \end{aligned}$$

Finally, the cross-variation between the processes D and Φ is

$$\frac{d[D,\Phi]}{\Phi D} = \sum_{i=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\mathbf{g}}_D'\right)\left(\frac{\partial_i \Phi}{\Phi}\right) dt$$

Putting all these results together yields

$$\begin{aligned} \frac{dS}{S} &= \left(m_S - \frac{D}{S}\right) dt + \widetilde{\mathbf{g}}_S(t)\, d\widetilde{\mathbf{Z}}(t) \\ \frac{dS}{S} &= \left[m_D + \sum_{i=1}^{3} \left(m_i + \left(\widetilde{\boldsymbol{h}}_i \widetilde{\mathbf{g}}_D'\right)\right)\left(\frac{\partial_i \Phi}{\Phi}\right)\right] dt \\ &+ \left[\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_{ij}\Phi}{\Phi}\right)\right] dt + \left[\widetilde{\mathbf{g}}_D + \sum_{i=1}^{3} \widetilde{\boldsymbol{h}}_i \left(\frac{\partial_i \Phi}{\Phi}\right)\right] d\widetilde{\boldsymbol{Z}} \end{aligned}$$

I invoke the canonical decomposition for continuous semi-martingales to get the instantaneous conditional expected return of the leveraged claim

$$\begin{aligned} m_S &= \Phi^{-1} + m_D + \sum_{i=1}^{3} \left(m_i + \left(\widetilde{\boldsymbol{h}}_i \widetilde{\mathbf{g}}_D'\right)\right)\left(\frac{\partial_i \Phi}{\Phi}\right) \\ &+ \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \left(\widetilde{\boldsymbol{h}}_i \widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_{ij}\Phi}{\Phi}\right) \end{aligned}$$

and the instantaneous conditional volatility vector of the leveraged claim

$$\widetilde{\mathbf{g}}_S = \widetilde{\boldsymbol{g}}_D + \sum_{i=1}^{3} \widetilde{\mathbf{h}}_i \left(\frac{\partial_i \Phi}{\Phi}\right) \tag{H.1}$$

The dynamic Euler equation is

$$\begin{aligned}\mathbb{E}_t\left(\frac{dS_t}{S_t}+\frac{D_t}{S_t}\,dt\right)-r_t\,dt &= -Cov_t\left(\frac{dM_t}{M_t},\frac{dS_t}{S_t}\right)\\ m_S \quad -r = & \;(1-\alpha)\left(\widetilde{\boldsymbol{g}}_C\widetilde{\boldsymbol{g}}_S'\right)+\left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_S'\right)\left(\frac{\partial_i\Psi}{\Psi}\right)\end{aligned}$$

In detail, after relevant substitutions,

$$\begin{aligned}0 &= \Phi^{-1}+m_D+\sum_{i=1}^{3}\left(m_i+\left(\widetilde{\boldsymbol{h}}_i\widetilde{\mathbf{g}}_D'\right)\right)\left(\frac{\partial_i\Phi}{\Phi}\right)+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_{ij}\Phi}{\Phi}\right)\\ &- \quad r-(1-\alpha)\left(\widetilde{\boldsymbol{g}}_C\widetilde{\boldsymbol{g}}_S'\right)-\left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_S'\right)\left(\frac{\partial_i\Psi}{\Psi}\right)\end{aligned}$$

For future reference, note that

$$\begin{aligned}\widetilde{\mathbf{g}}_S\widetilde{\mathbf{g}}_S' &= \left[\widetilde{\boldsymbol{g}}_D+\sum_{i=1}^{3}\widetilde{\mathbf{h}}_i\left(\frac{\partial_i\Phi}{\Phi}\right)\right]\left[\widetilde{\boldsymbol{g}}_D+\sum_{j=1}^{3}\widetilde{\mathbf{h}}_j\left(\frac{\partial_j\Phi}{\Phi}\right)\right]'\\ &= \left(\widetilde{\boldsymbol{g}}_D\widetilde{\boldsymbol{g}}_D'\right)+2\sum_{i=1}^{3}\left(\widetilde{\mathbf{h}}_i\widetilde{\boldsymbol{g}}_D'\right)\left(\frac{\partial_i\Phi}{\Phi}\right)+\\ &+ \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\mathbf{h}}_i\widetilde{\mathbf{h}}_j'\right)\left(\frac{\partial_i\Phi}{\Phi}\right)\left(\frac{\partial_j\Phi}{\Phi}\right)\end{aligned}$$

further

$$\begin{aligned}\widetilde{\mathbf{g}}_S\widetilde{\mathbf{h}}_j' &= \left[\widetilde{\boldsymbol{g}}_D+\sum_{i=1}^{3}\widetilde{\mathbf{h}}_i\left(\frac{\partial_i\Phi}{\Phi}\right)\right]\widetilde{\mathbf{h}}_j'=\\ &= \left(\widetilde{\boldsymbol{g}}_D\widetilde{\mathbf{h}}_j'\right)+\sum_{i=1}^{3}\left(\widetilde{\mathbf{h}}_i\widetilde{\mathbf{h}}_j'\right)\left(\frac{\partial_i\Phi}{\Phi}\right)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathbf{g}}_C\widetilde{\mathbf{g}}_S' &= \left[\widetilde{\boldsymbol{g}}_D+\sum_{i=1}^{3}\widetilde{\mathbf{h}}_i\left(\frac{\partial_i\Phi}{\Phi}\right)\right]\widetilde{\boldsymbol{g}}_C'\\ &= \left(\widetilde{\boldsymbol{g}}_D\widetilde{\boldsymbol{g}}_C'\right)+\sum_{i=1}^{3}\left(\widetilde{\mathbf{h}}_i\widetilde{\boldsymbol{g}}_C'\right)\left(\frac{\partial_i\Phi}{\Phi}\right)\end{aligned}$$

Combining these results with the Euler equation yields, after tedious algebra,

$$\begin{aligned}0 &= \Phi^{-1}+m_D+\sum_{i=1}^{3}\left(m_i+\left(\widetilde{\boldsymbol{h}}_i\widetilde{\mathbf{g}}_D'\right)\right)\left(\frac{\partial_i\Phi}{\Phi}\right)+\\ &+ \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_{ij}\Phi}{\Phi}\right)-r-(1-\alpha)\left(\widetilde{\boldsymbol{g}}_E\widetilde{\boldsymbol{g}}_D'\right)-\\ &- (1-\alpha)\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_C'\right)\left(\frac{\partial_i\Phi}{\Phi}\right)\\ &- \left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\left\{\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_D'\right)+\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{h}}_j'\right)\left(\frac{\partial_j\Phi}{\Phi}\right)\right\}\left(\frac{\partial_i\Psi}{\Psi}\right)\end{aligned}$$

Rearranging yields the linear elliptic partial diferential equation for the price-dividend ratio $\Phi:\Delta^3\to\mathbb{R}^+$ solves the linear elliptic PDE

(H.2) $$0=\mathcal{L}\,\Phi+b_0\,\Phi+1$$

subject to the boundary conditions

(H.3) $$\lim_{\boldsymbol{\pi} \to \partial\Delta^3} \Phi(\boldsymbol{\pi}) \sim \text{finite}$$

where differential operator $\mathcal{L}$ is given by

(H.4) $$\mathcal{L}\,\Phi = \frac{1}{2} tr\left(\widetilde{\mathbf{H}}\,\partial^2_{\boldsymbol{\pi}\boldsymbol{\pi}'}\Phi\right) + \left(\mathbf{b}_1'\partial_{\boldsymbol{\pi}}\Phi\right)$$

I additionally denote the vector $\mathbf{b}_1 := [b_1, b_2, b_3]_{3\times 1}$ and define

$$b_i = m_i + \left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_D\right) - (1-\alpha)\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{g}}_C'\right) - \left(1-\frac{\alpha}{\rho}\right)\sum_{j=1}^{3}\left(\widetilde{\boldsymbol{h}}_i\widetilde{\boldsymbol{h}}_j'\right)\frac{\partial_j\Psi}{\Psi}$$

Finally, the scalar

$$b_0 := m_D - r - (1-\alpha)\left(\widetilde{\boldsymbol{g}}_C\widetilde{\boldsymbol{g}}_D'\right) - \left(1-\frac{\alpha}{\rho}\right)\sum_{i=1}^{3}\left(\widetilde{\boldsymbol{h}_i\boldsymbol{g}_D'}\right)\frac{\partial_i\Psi}{\Psi}$$

Appendix I. Aggregation From Instantaneous to Discrete

As the model is set up in continuous time, many variables are instantaneous. For example, Ψ is the wealth-consumption rate ratio, not wealth-consumption ratio. Similarly, the conditional volatilities $\widetilde{\boldsymbol{g}}_P$, $\widetilde{\boldsymbol{g}}_S$ and $\widetilde{\boldsymbol{g}}_B$ are also instantaneous quantities. Therefore, to obtain correct discrete time-magnitudes, I proceed as follows. First, I compute the wealth-consumption ratio as[39,40]

$$\frac{W_t}{\int_{t-1}^{t} C_\tau d\tau} = \frac{W_t}{C_t} \times \frac{C_t}{\int_{t-1}^{t} C_\tau d\tau}$$

I solve for the 'correction factor' $\int_{t-1}^{t} C_\tau d\tau / C_t$ by means of simulation as follows

$$\begin{aligned}
\frac{1}{C_t}\int_{t-1}^{t} C_\tau d\tau &= \frac{C_{t-1}}{C_t} \times \int_{t-1}^{t} \exp\left(\log(C_\tau) - \log(C_{t-1})\right) d\tau \\
&= \frac{C_{t-1}}{C_t} \times \int_{t-1}^{t} \exp\left(\int_{t-1}^{\tau} d\log C_s\right) d\tau \\
&\approx \frac{C_{t-1}}{C_t} \times \sum_s \exp\left(\Delta \log C_s\right)\Delta s
\end{aligned}$$

and then jointly simulate it with the vector of the posterior probabilities when I simulate the whole economy because it is much simpler. Technically, I simulate jointly the posterior probabilities, the consumption growth, the dividend growth, and the respective correction factors for 400,000 quarters at the frequency of $\Delta t = 0.001$ of a year. Subsequently, I assign the relevant variables, such as price-dividend ratio, that corresponds to these simulated series.

Second, I construct the quarterly returns on the aggregate wealth R_t^W and aggregate equity R_t^S as

$$\begin{aligned}
R_{t+1}^W &= \frac{W_{t+1} + \int_t^{t+1} C_\tau d\tau}{W_t} \\
&= \frac{\left(W_{t+1}/\int_t^{t+1} C_\tau d\tau\right) + 1}{\left(W_t/\int_{t-1}^{t} C_\tau d\tau\right)} \times \frac{\int_t^{t+1} C_\tau d\tau}{\int_{t-1}^{t} C_\tau d\tau} \\
&= \frac{\left(W_{t+1}/\int_t^{t+1} C_\tau d\tau\right) + 1}{\left(W_t/\int_{t-1}^{t} C_\tau d\tau\right)} \times \frac{\frac{1}{C_t}\int_t^{t+1} C_\tau d\tau}{\frac{1}{C_{t-1}}\int_{t-1}^{t} C_\tau d\tau} \times \frac{C_t}{C_{t-1}}
\end{aligned}$$

[39]Time is measured in quarters.

[40]For presentation reasons, I annualize the ratio by multiplying it by four.

where C_t denotes the endowment rate. I find the "correction factor" again by means of simulation as described above. Similar formulas apply for the construction of the return on the equity portfolio

$$
\begin{aligned}
R^S_{t+1} &= \frac{S_{t+1} + \int_t^{t+1} D_\tau d\tau}{S_t} \\
&= \frac{\left(S_{t+1}/\int_t^{t+1} D_\tau d\tau\right) + 1}{\left(S_t/\int_{t-1}^{t} C_\tau d\tau\right)} \times \frac{\int_t^{t+1} D_\tau d\tau}{\int_{t-1}^{t} D_\tau d\tau} \\
&= \frac{\left(S_{t+1}/\int_t^{t+1} D_\tau d\tau\right) + 1}{\left(S_t/\int_{t-1}^{t} D_\tau d\tau\right)} \times \frac{\frac{1}{D_t}\int_t^{t+1} D_\tau d\tau}{\frac{1}{D_{t-1}}\int_{t-1}^{t} D_\tau d\tau} \times \frac{D_t}{D_{t-1}}
\end{aligned}
$$

I compute the excess return by subtracting the yield-to-maturity on an indexed zero-coupon bond with maturity one quarter. Finally, I construct the quarterly consumption growth rate as

$$
\frac{\int_t^{t+1} C_\tau d\tau}{\int_{t-1}^{t} C_\tau d\tau} = \frac{\frac{1}{C_t}\int_t^{t+1} C_\tau d\tau}{\frac{1}{C_{t-1}}\int_{t-1}^{t} C_\tau d\tau} \times \frac{C_t}{C_{t-1}}
$$

and the quarterly dividend growth rate

$$
\frac{\int_t^{t+1} D_\tau d\tau}{\int_{t-1}^{t} D_\tau d\tau} = \frac{\frac{1}{D_t}\int_t^{t+1} D_\tau d\tau}{\frac{1}{D_{t-1}}\int_{t-1}^{t} D_\tau d\tau} \times \frac{D_t}{D_{t-1}}
$$

Appendix J. Bayesian Analysis of the Four-State Univariate Regime-Switching Model for the Log-Consumption Growth Rate

This appendix details the estimation of the Bayesian univariate regime-switching model, as applied to the simple growth rate of real per-capita nondurable consumption and services, denoted C. Let us recall the dynamics of the consumption growth rate

(J.1) $$\frac{dC_t}{C_t} = \kappa_t\, dt + \sigma_C\, dZ_{1t}$$

where Z_{1t} is a standard Wiener process. I discretize the above equation as follows

(J.2) $$\frac{\Delta C_t}{C_t} = \kappa(s_t) + \sigma_C\, E_t$$

where $E_t \sim$ i.i.d. $N(0,1)$, s_t follows a four-state hidden Markov chain, and $\kappa(s_t) \in \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$. Furthermore, let

(J.3) $$\mathbf{Y} = \left[\frac{\Delta C_{1947:Q2}}{C_{1947:Q2}}, \ldots, \frac{\Delta C_{2007:Q2}}{C_{2007:Q2}}\right]'_{T\times 1}$$

(J.4) $$\mathbf{s}_1 = [1(s_{1947:Q2} = 1), \ldots, 1(s_{2007:Q2} = 1)]'_{T\times 1}$$

(J.5) $$\mathbf{s}_2 = [1(s_{1947:Q2} = 2), \ldots, 1(s_{2007:Q2} = 2)]'_{T\times 1}$$

(J.6) $$\mathbf{s}_3 = [1(s_{1947:Q2} = 3), \ldots, 1(s_{2007:Q2} = 3)]'_{T\times 1}$$

(J.7) $$\mathbf{s}_4 = [1(s_{1947:Q2} = 4), \ldots, 1(s_{2007:Q2} = 4)]'_{T\times 1}$$

(J.8) $$\mathbf{X} = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4]_{T\times 4}$$

where $1(\bullet)$ is an indicator function for the simulated state s_t at time t. In addition, let the matrix $\mathbf{B}_{4\times 1}$ of regression coefficients be given by

(J.9) $$\mathbf{B} = [\kappa_1, \kappa_2, \kappa_3, \kappa_4]'_{4\times 1}$$

Stacking the data into matrices, the discretized model may be written as

(J.10) $$\mathbf{Y} = \mathbf{X}\mathbf{B} + \sigma_C\, \mathbf{E}$$

A. **Generating B and Σ Conditional on the Rest of the Parameters and $\mathbf{s}_T$.** Under the assumptions made above, the likelihood function $l(\mathbf{B}, \sigma_C^2 \,|\, \mathbf{Y}, \mathbf{X})$ is

$$l(\mathbf{B}, \sigma_C^2) \,|\, \mathbf{y}, \mathbf{x}) \quad \propto \quad \left(\sigma_C^2\right)^{-T/2} \times \exp\left\{ -\frac{1}{2\,\sigma_C^2} \,(\mathbf{Y} - \mathbf{XB})' \,(\mathbf{Y} - \mathbf{XB}) \right\} \tag{J.11}$$

Noting that

$$(\mathbf{Y} \,-\, \mathbf{X}\,\mathbf{B})'\,(\mathbf{Y} \,-\, \mathbf{X}\,\mathbf{B}) \,=\, \mathbf{S} \,+\, (\mathbf{B} \,-\, \hat{\mathbf{B}})'\,\mathbf{X}'\,\mathbf{X}\,(\mathbf{B} \,-\, \hat{\mathbf{B}}) \tag{J.12}$$

where $\hat{\mathbf{B}}$ is the OLS estimate

$$\hat{\mathbf{B}} \,=\, (\mathbf{X}'\,\mathbf{X})^{-1}\,\mathbf{X}'\,\mathbf{Y} \tag{J.13}$$

and

$$S \,=\, (\mathbf{Y} \,-\, \mathbf{X}\,\hat{\mathbf{B}})'\,(\mathbf{Y} \,-\, \mathbf{X}\,\hat{\mathbf{B}}) \tag{J.14}$$

I find that the likelihood function takes the form

$$l(\mathbf{B}, \sigma_C^2 \,|\, \mathbf{Y}, \mathbf{X}) \quad \propto \quad \left(\sigma_C^2\right)^{-T/2} \times \exp\left\{ -\frac{1}{2\,\sigma_C^2}\, S \,-\, \frac{1}{2\,\sigma_C^2}\,(\mathbf{B} \,-\, \hat{\mathbf{B}})'\,\mathbf{X}'\,\mathbf{X}\,(\mathbf{B} \,-\, \hat{\mathbf{B}}) \right\}$$

The prior distribution of the consumption variance is assumed to be inverse Gamma $\mathrm{I\Gamma}(\nu_0/2,\, \delta_0/2)$, with the density

$$p\left(\frac{1}{\sigma_C^2}\right) \,\propto\, \left(\frac{1}{\sigma_C^2}\right)^{\frac{\nu_0}{2}-1} \exp\left\{ -\frac{\delta_0}{2\,\sigma_C^2} \right\} \tag{J.15}$$

and the prior for the consumption growth rates $\mathbf{B}$ is multivariate Normal $N(\bar{\mathbf{B}},\, \mathbf{C})$, having the density

$$p(\mathbf{B}) \,\propto\, \exp\left\{ -\frac{1}{2}\,(\mathbf{B} \,-\, \bar{\mathbf{B}})'\,\mathbf{C}^{-1}\,(\mathbf{B} \,-\, \bar{\mathbf{B}}) \right\} \tag{J.16}$$

In addition, I assume that the priors for $\mathbf{B}$ and $1\,/\,\sigma_C^2$ are statistically independent, that is,

$$p\left(\frac{1}{\sigma_C^2},\, \mathbf{B}\right) \quad = \quad p\left(\frac{1}{\sigma_C^2}\right) \,\times\, p(\mathbf{B}) \tag{J.17}$$

$$\propto \quad \left(\frac{1}{\sigma_C^2}\right)^{\frac{\nu_0}{2}-1} \exp\left\{ -\frac{\delta_0}{2\,\sigma_C^2} \,-\, \frac{1}{2}\,(\mathbf{B} \,-\, \bar{\mathbf{B}})'\,\mathbf{C}^{-1}\,(\mathbf{B} \,-\, \bar{\mathbf{B}}) \right\} \tag{J.18}$$

The joint posterior distribution is then as follows

$$p\left(\mathbf{B},\, \frac{1}{\sigma_C^2} \,|\, \mathbf{Y}, \mathbf{X}\right) \quad \propto \quad \left(\frac{1}{\sigma_C^2}\right)^{\frac{\nu_0+T}{2}-1} \quad \times \tag{J.19}$$

$$\times \quad \exp\left\{ -\frac{\delta_0}{2\,\sigma_C^2} \,-\, \frac{1}{2}\,(\mathbf{B} \,-\, \bar{\mathbf{B}})'\,\mathbf{C}^{-1}\,(\mathbf{B} \,-\, \bar{\mathbf{B}}) \right\} \,\times \tag{J.20}$$

$$\times \quad \exp\left\{ -\frac{1}{2\,\sigma_C^2}\, S \,-\, \frac{1}{2\,\sigma_C^2}\,(\mathbf{B} \,-\, \hat{\mathbf{B}})'\,\mathbf{X}'\,\mathbf{X}\,(\mathbf{B} \,-\, \hat{\mathbf{B}}) \right\} \tag{J.21}$$

Merging the two quadratic forms in the exponent yields

$$p\left(\mathbf{B},\, \frac{1}{\sigma_C^2} \,|\, \mathbf{Y}, \mathbf{X}\right) \quad \propto \quad \left(\frac{1}{\sigma_C^2}\right)^{\frac{\nu_0+T}{2}-1} \times \exp\left\{ -\,\frac{\delta_0 \,+\, S}{2\,\sigma_C^2} \,-\, \frac{1}{2}\,(\mathbf{B} \,-\, \mathbf{d})'\,\mathbf{D}^{-1}\,(\mathbf{B} \,-\, \mathbf{d}) \right\}$$

where

$$\mathbf{D} \quad = \quad \left(\frac{1}{\sigma_C^2}\mathbf{X}'\,\mathbf{X} \,+\, \mathbf{C}^{-1}\right)^{-1} \tag{J.22}$$

$$\mathbf{d} \quad = \quad \mathbf{D} \times \left(\frac{1}{\sigma_C^2}\,\mathbf{X}'\,\mathbf{X}\,\hat{\mathbf{B}} \,+\, \mathbf{C}^{-1}\,\bar{\mathbf{B}}\right) \,=\, \mathbf{D} \times \left(\frac{1}{\sigma_C^2}\,\mathbf{X}'\,\mathbf{Y} \,+\, \mathbf{C}^{-1}\,\bar{\mathbf{B}}\right) \tag{J.23}$$

Note that I can factor out the joint posterior as

$$p\left(\mathbf{B},\, \frac{1}{\sigma_C^2},\, |\, \mathbf{Y},\, \mathbf{X}\right) = p\left(\mathbf{B}\,|\, \frac{1}{\sigma_C^2},\, \mathbf{Y},\, \mathbf{X}\right) \times p\left(\frac{1}{\sigma_C^2}\,|\, \mathbf{Y},\, \mathbf{X}\right), \tag{J.24}$$

so that the conditional posterior $p\left(\mathbf{B}\,|\,\frac{1}{\sigma_C^2},\,\mathbf{Y},\,\mathbf{X}\right)$, which in fact I do need for the Gibbs sampler, is

$$p(\mathbf{B}\,|\,\frac{1}{\sigma_C^2},\,\mathbf{Y},\,\mathbf{X}) \quad \propto \quad \exp\left\{-\frac{1}{2}\,(\mathbf{B}-\mathbf{d})'\,\mathbf{D}^{-1}\,(\mathbf{B}-\mathbf{d})\right\} \tag{J.25}$$

which I recognize as the density of the multivariate Normal distribution

$$\mathbf{B}\,|\,\frac{1}{\sigma_C^2},\,\mathbf{Y},\,\mathbf{X} \;\sim\; N(\mathbf{d},\,\mathbf{D}) \tag{J.26}$$

With respect to the conditional posterior of the inverse of the consumption growth rate variance σ_C^2, it is helpful to recall eq. (J.12) and rewrite the joint posterior in eq. (J.19) as

$$p\left(\mathbf{B},\frac{1}{\sigma_C^2}\,|\,\mathbf{Y},\mathbf{X}\right) \quad \propto \quad \left(\frac{1}{\sigma_C^2}\right)^{\frac{\nu_0+T}{2}-1} \quad \times \tag{J.27}$$

$$\times \quad \exp\left\{-\frac{\delta_0}{2\,\sigma_C^2} \;-\; \frac{1}{2}\,(\mathbf{B}-\bar{\mathbf{B}})'\,\mathbf{C}^{-1}\,(\mathbf{B}-\bar{\mathbf{B}})\right\} \times \tag{J.28}$$

$$\times \quad \exp\left\{-\frac{1}{2\,\sigma_C^2}\;(\mathbf{Y}-\mathbf{XB})'\,(\mathbf{Y}-\mathbf{XB})\right\} \tag{J.29}$$

or

$$p\left(\mathbf{B},\frac{1}{\sigma_C^2}\,|\,\mathbf{Y},\mathbf{X}\right) \quad \propto \quad \left(\frac{1}{\sigma_C^2}\right)^{\frac{\nu_0+T}{2}-1} \quad \times \tag{J.30}$$

$$\times \quad \exp\left\{-\frac{1}{2\,\sigma_C^2}\;\left[\delta_0 \,+\, (\mathbf{Y}-\mathbf{XB})'\,(\mathbf{Y}-\mathbf{XB})\right]\right\} \times \tag{J.31}$$

$$\times \quad \exp\left\{-\frac{1}{2}\,(\mathbf{B}-\bar{\mathbf{B}})'\,\mathbf{C}^{-1}\,(\mathbf{B}-\bar{\mathbf{B}})\right\} \tag{J.32}$$

Note that I may factorize the joint posterior for $\left(\mathbf{B},\,\frac{1}{\sigma_C^2}\right)$ also as

$$p\left(\mathbf{B},\,\frac{1}{\sigma_C^2}\,|\,\mathbf{Y},\,\mathbf{X}\right) \;=\; p\left(\frac{1}{\sigma_C^2}\,|\,\mathbf{B},\,\mathbf{Y},\,\mathbf{X}\right) \;\times\; p(\mathbf{B}\,|\,\mathbf{Y},\,\mathbf{X}) \tag{J.33}$$

This gives me the conditional distribution for $1\,/\,\sigma_C^2$ as

$$p\left(\frac{1}{\sigma_C^2}\,|\,\mathbf{B},\,\mathbf{Y},\,\mathbf{X}\right) \quad \propto \quad \left(\frac{1}{\sigma_C^2}\right)^{\frac{\nu_0+T}{2}-1} \exp\left\{-\frac{1}{2\,\sigma_C^2}\;\left[\delta_0 \,+\, (\mathbf{Y}-\mathbf{XB})'\,(\mathbf{Y}-\mathbf{XB})\right]\right\}$$

which may be recognized as the density of the inverse Gamma distribution

$$\frac{1}{\sigma_C^2}\,|\,\mathbf{B},\,\mathbf{Y},\,\mathbf{X} \;\sim\; I\Gamma\left[\frac{\nu_0\,+\,T}{2},\;\frac{\delta_0\,+\,(\mathbf{Y}-\mathbf{XB})'(\mathbf{Y}-\mathbf{XB})}{2}\right] \tag{J.34}$$

B. Generating Transition Probabilities Conditional on $\mathbf{s}_T$.

a. *Finding the Posterior Distribution.* Observe that conditional on $\mathbf{s}_T$, the transition probabilities $(\pi_{i,j})_{4x4}$ are independent of the data set $\mathbf{Y}$ and the model's other parameters. I assume that the rows of the transition probability matrix are a priori independent, each having the Dirichlet distribution

$$(\pi_{i,1},\,\pi_{i,2},\,\pi_{i,3},\,\pi_{i,4}) \quad \sim \quad Dir(\delta_{i,1},\,\delta_{i,2},\,\delta_{i,3},\,\delta_{i,4};\,4), \quad i \,=\, 1,2,3,4 \tag{J.35}$$

$$\propto \quad \pi_{i,1}^{\delta_{i,1}} \times \pi_{i,2}^{\delta_{i,2}} \times \pi_{i,3}^{\delta_{i,3}} \times (1-\pi_{i,1}-\pi_{i,2}-\pi_{i,3})^{\delta_{i,4}} \tag{J.36}$$

The likelihood function is

$$l\left(\{\pi_{ij}\}_{i,j=1}^{4}\,|\,\mathbf{s}_T\right) \;=\; \prod_{i,j=1}^{4} \pi_{i,j}^{n_{i,j}} \tag{J.37}$$

where $n_{i,j}$ is the total number of transitions from state i to state j. Hence, the posterior distribution takes the form

$$(\pi_{i,1},\,\pi_{i,2},\,\pi_{i,3},\,\pi_{i,4}\,|\,\mathbf{s}_T) \;\sim\; Dir(\delta_{i,1}\,+\,n_{i,1},\,\delta_{i,2},+n_{i,2},\,\delta_{i,3},+n_{i,3},\,\delta_{i,4},+n_{i,4};\,4), \quad i \,=\, 1,2,3,4 \tag{J.38}$$

Clearly, the rows of the transition matrix are conditionally independent.

Sampling from the Dirichlet distribution $Dir(\delta_1, ..., \delta_4; 4)$ is relatively simple. One first generates independent ξ_1,...,ξ_4 from the Gamma distribution $\Gamma(\delta_i, 1)$. Then, the four-tuple

(J.39)
$$\left(\frac{\xi_i}{\sum_{j=1}^{4} \xi_j}\right)_{i=1,2,3,4}$$

has the desired Dirichlet distribution.

b. *Testing the Embeddability Conditions of the Sampled Transition Probability Matrix.* Although the sufficient conditions for the existence of an infinitesimal generator that corresponds to a given sampled 4×4 transition probability matrix have yet to be found, we do test several necessary conditions. These are summarized in the following lemmas for the convenience of the reader.

Lemma 5. *Let $P = \{p_{ij}\}_{i,j=1}^{4}$ be a transition probability matrix. Suppose that any of the following conditions*

(1) *det(P) ≤ 0*
(2) *det(P) $> \prod_{i=1}^{4} p_{ii}$*
(3) *States i and j communicate but $p_{ij} = 0$*

hold. Then, there does not exist an infinitesimal generator for P.

Proof. See Kingman (1962), Theorem 6.1 in Goodman (1970), Chung (1992), Grimmett and Stirzaker(1992) □

Lemma 6. *Let $P = \{p_{ij}\}_{i,j=1}^{4}$ be a transition probability matrix with a real spectrum. If the eigenvalues are distinct but some are negative, then there is no real matrix Q satisfying* $\exp(Q) = P$.

Proof. See Singer and Spilerman (1976). □

C. **Generating $\mathbf{s}_T$ Conditional on $\mathbf{B}$, $\mathbf{\Sigma}$, $(\pi_{i,j})_{4x4}$ and $\mathbf{Y}$.** I follow Kim and Nelson (1998) and use multi-move Gibbs sampling, simulating s_t, t=1,...,T, as a block from the joint conditional distribution

(J.40)
$$p\left(\mathbf{s}_T \,|\, \mathbf{B},\, \mathbf{\Sigma},\, (\pi_{i,j})_{4x4},\, \mathbf{Y}\right)$$

which may be derived as follows, suppressing the conditioning on the model's parameters

(J.41)
$$p(\mathbf{s}_T \,|\, \mathbf{Y}) \;=\; p(s_1, ..., s_T \,|\, \mathbf{Y}) \;=\; p(s_T \,|\, \mathbf{Y}) \times \prod_{t=1}^{T-1} p(s_t \,|\, s_{t+1},\, \mathbf{Y})$$

where I invoke the Markov property of the chain. Then, observe that

(J.42)
$$p(s_t \,|\, s_{t+1},\, \mathbf{Y}) \;=\; \frac{p(s_t,\, s_{t+1} \,|\, \mathbf{Y})}{p(s_{t+1} \,|\, \mathbf{Y})} \propto p(s_{t+1} \,|\, s_t) \times p(s_t \,|\, \mathbf{Y})$$

We find $p(s_t \,|\, \mathbf{Y})$ by running Hamilton's (1989) filter.

I find s_T by sampling from $p(s_T \,|\, \mathbf{Y})$ as follows. I generate a random number from the uniform distribution. If it is less or equal to $p(s_T = 1 \,|\, \mathbf{Y})$, I set $s_T = 1$. If not, I calculate

(J.43)
$$p(s_T = 2 \,|\, s_T \neq 1,\, \mathbf{Y}) \;=\; \frac{p(s_T = 2 \,|\, \mathbf{Y})}{\sum_{j=2}^{4} p(s_T = j \,|\, \mathbf{Y})}$$

If a new generated number from the uniform distribution is less or equal to $p(s_T = 2 \,|\, s_T \neq 1,\, \mathbf{Y})$, I set $s_T = 2$. If not, I compute

(J.44)
$$p(s_T = 3 \,|\, s_T \neq 1, 2,\, \mathbf{Y}) \;=\; \frac{p(s_T = 3 \,|\, \mathbf{Y})}{\sum_{j=3}^{4} p(s_T = j \,|\, \mathbf{Y})}$$

If a yet new generated number from the uniform distribution is less or equal to $p(s_T = 3 \,|\, s_T \neq 1, 2, \mathbf{Y})$, I set $s_T = 3$. If not, I set $s_T = 4$.

I generate $\{s_t\}_{t=1,\dots,T-1}$ using sampling from the uniform distribution as follows. First, I compute

$$p(s_t = 1 \,|\, s_{t+1}, \mathbf{Y}) = \frac{p(s_{t+1} \,|\, s_t = 1) \times p(s_t = 1 \,|\, \mathbf{Y})}{\sum_{j=1}^{4} p(s_{t+1} \,|\, s_t = j) \times p(s_t = j \,|\, \mathbf{Y})} \tag{J.45}$$

If the generated number from the uniform distribution is less or equal to $p(s_t = 1 \,|\, s_{t+1}, \mathbf{Y})$, I set $s_t = 1$. If not, I compute

$$p(s_t = 2 \,|\, s_{t+1}, s_t \neq 1, \mathbf{Y}) = \frac{p(s_{t+1} \,|\, s_t = 2) \times p(s_t = 2 \,|\, \mathbf{Y})}{\sum_{j=2}^{4} p(s_{t+1} \,|\, s_t = j) \times p(s_t = j \,|\, \mathbf{Y})} \tag{J.46}$$

If the new generated number from the uniform distribution is less or equal to $p(s_t = 2 \,|\, s_{t+1}, s_t \neq 1, \mathbf{Y})$, I set $s_t = 2$. If not, I compute

$$p(s_t = 3 \,|\, s_{t+1}, s_t \neq 1, 2, \mathbf{Y}) = \frac{p(s_{t+1} \,|\, s_t = 3) \times p(s_t = 3 \,|\, \mathbf{Y})}{\sum_{j=3}^{4} p(s_{t+1} \,|\, s_t = j) \times p(s_t = j \,|\, \mathbf{Y})} \tag{J.47}$$

If the yet new generated number from the uniform distribution is less or equal to $p(s_t = 3 \,|\, s_{t+1}, s_t \neq 1, \mathbf{Y})$, I set $s_t = 3$. If not, I set $s_t = 4$.

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