Bidding and Searching for the Best Deal: Strategic Behavior in Internet Auctions.

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Abstract

In Internet auctions bidders frequently bid in one of two ways: either they bid only late (late bidding) or, initially, they bid early and then revise their bids just before the auction closes (early bidding). In this paper we build a model of a dynamic auction with two bidders who can search for outside prices while bidding in the auction. For the case of common price draws (realized value is the same for both bidders) we characterize an equilibrium in which both bidders bid early and then coordinate who searches for the outside price, i.e., the bidder with the lower early bid searches and the other bidder does not. When outside prices are independent and identically distributed, then, there is an equilibrium in which bidders with the low search costs bid only late and always search, while the remaining bidders bid early and then coordinate searching. A contribution of our model is in explaining both frequently occurring bidding patterns (late and early bidding) within a single equilibrium. In terms of total welfare the equilibrium in which buyers coordinate their searching is always better than the equilibrium in when they don't coordinate. However, buyers are not always better off in the coordinating equilibrium. When outside prices are iid buyers could collectively improve their surplus if they only bid late.

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1 Introduction

Understanding bidding behavior is the first step in building efficient electronic markets. Bidding in online auctions, however, is no easy task. Before making a bid each bidder must make several tough decisions regarding the amount and the timing of her bid. Over time, many bidders find bidding strategies that are successful and continue to use them. A few recent studies (i.e., Shah *et al.,* 2004, Gonzales *et al.*, 2004, and Morgan and Hossain, 2003) collect data on bids from several Internet auction sites and identify two bidding patterns that occur more frequently than others. The first is formed by bidders who make only a single bid in the last moments of auction - a practice called "late bidding" or "sniping." The second bidding pattern involves multiple bids by the same bidder in the same auction, i.e., she makes her initial bid early and then revises it later as the auction nears its end. We will call this pattern of behavior "multiple bidding." Most of the current literature has focused on late bidding which is commonly observed among experienced traders. Roth and Ockenfels (2002) argue that one of the main advantages of late bidding is that it is a best response to myopic behavior by bidders who increase their bids by an increment every time they are outbid. Furthermore, in auctions that close at an exact specific time (e.g., eBay), bidders might tacitly collude by bidding only in the last few seconds of the auction. In light of this support for late bidding it seems puzzling that many bidders in fact choose to bid early. What motivates multiple bidding and why is it that in the data multiple bidding appears almost as frequently as the late bidding remains an open question. In this paper we propose a framework in which late and multiple bidding emerge together as a part of a single equilibrium.

Late and multiple bidding are common behaviors. Shah *et al.* (2004) analyzed the bidding histories of eBay auctions for Sony Playstation 2 and Nintendo consoles. In their sample, 38% of all bidders placed a single bid in the last hour of the auction while 34% of bidders made multiple bids. In another study, Gonzales *et al.* (2004) looked at computer monitor auctions on eBay. They found that there is at least one bidder who bids more than once in 77% of all auctions in their sample. Similarly, Morgan and Hossain (2003) conducted a field experiment and observed multiple bidding in 75% of their auctions. More evidence of late and multiple bidding is found by Roth and Ockenfels (2002), Bajari and Hortacsu (2003) and Yang *et al.* (2003). Given that so many bidders use multiple bidding strategy, we must ask a following question: What is the merit to placing multiple bids? In this paper we will construct an environment in which multiple bidding is a product of optimal behavior by rational bidders. We further demonstrate that in this environment, incentives for late bidding (or sniping) do not entirely disappear. Thus, in the equilibrium of our model some bidders find it optimal to bid several times while others will only bid late. Most importantly, our equilibrium has a descriptive as well as intuitive appeal since it gives rational interpretation to both bidding patterns.

When could one benefit from bidding early? Our argument is based on the idea of price searching. Bidders in Internet auctions are not necessarily interested in buying the object at any price, but rather are looking for a good deal. In the auction, if the price rises too high, they might decide to look for a better deal elsewhere. What complicates the matter is that price searching is not easy and one often has to go to considerable effort in order to find a good price. The cost of this effort varies across bidders. It might, for example, depend on the buyer's location or individual time constraints. A buyer looking for a car in Los Angeles is going to have many opportunities to shop around at different dealers compared to a buyer in Montana. Therefore, a buyer in Montana will find it much more difficult to check different dealers and her only chance of finding a car she wants could be on the Internet. She can benefit from signaling her high search costs to the other bidders by making a high early bid. Then, when the bidder in Los Angeles is outbid, she will realize that she faces an opponent who is not very flexible in terms of her alternatives and she will respond by intensifying her search efforts. As a result, both bidders benefit. The flexible bidder benefits from a low price that she finds and the inflexible bidder benefits from reduced competition in the auction. The model we propose in this paper is a simplification of this story.

In modeling the bidding behavior in Internet auctions it is important to recognize two distinguishing features of the Internet auctions mechanism. The first is the dynamic structure of the auction. Typical Internet auction normally last for several days. However, bidders do not permanently sit at their computers; they revise their bids in discrete time intervals. Therefore, an online auction could be modeled as a sequence of discrete time intervals during which bidders submit their bids simultaneously. The second feature is proxy bidding, which is implemented at numerous sites, including eBay. When a bidder enters a proxy bid, the auction automatically bids for the bidder up to the minimum of either the amount which is necessary to make her a winning bidder or the amount of her proxy bid. Thus, proxy bidding effectively gives the auction the properties of a dynamic, second-price auction.

The auction format that we consider is the simplest version of a dynamic second-price auction. It has two bidding rounds: early and late. In both rounds bidders submit their bids simultaneously. On the demand side we consider two bidders who have a common valuation equal to one and independent and private search costs that are distributed uniformly on the unit interval. The game unfolds in three stages. In the first stage both bidders place bids simultaneously. In the second stage both bidders first observe the standing price, which equals the second highest bid from the initial round, and then decide whether to search for the outside price-offer or not. The outside price-offer is a random draw from the unit interval. If a bidder decides to search she gets the price-offer but has to pay her private search cost. A bidder who doesn't search, moves directly into the third stage. In the third stage both bidders submit another round of bids.

We characterize two equilibria of this game. In the first equilibrium, all bidders bid only in the late bidding round and use threshold strategies for their search decisions. This means that bidders with lower search costs will always search in the equilibrium and the remaining bidders with high search costs will never search. The outcome of this equilibrium is consistent with the late bidding pattern but fails to explain multiple bidding.

The second equilibrium is more interesting and includes bidding activity in both bidding rounds. Whether a bidder only bids late or places multiple bids depends on her search cost. Bidders who have sufficiently high search costs bid multiple times, i.e., early and late. Their early bids are increasing in their search costs and allow implicit coordination their searching decisions. In particular, when a bidder submits the highest bid in the early bidding round, she can infer that she has higher search cost than the other bidder. Therefore, she does not search and bids her valuation in the final bidding round, i.e., she bids one. On the other hand, the low bidder will search in the second stage because she can infer that her opponent is not searching; therefore, she can only win the auction at the price of one, which would give her a payoff of zero. Thus, she can improve her payoff by finding an outside price offer which is almost surely less than one[1]. In this part of the equilibrium bidders with sufficiently high search costs generate multiple bidding.

[1]In our model the search costs are sufficiently low so that searching will almost always generate positive expected payoff.

The second part of this equilibrium is the late bidding part. Suppose that a bidder has a very low search cost, e.g., zero. For example a bidder lives right next to five different car dealerships and can easily check out the prices on her way home from work. Then, she might want to search irrespective of whether she expects her opponent to search as well. In that case, it makes no sense for her to bid early in the auction, because than she would be risking winning the auction at a higher price than what her best outside price-offer could be. Therefore, she will check the prices at the car dealers first and then she will bid her best price-offer in the final bidding round of the auction. In this part of the equilibrium bidders with low search costs generate late bidding.

To summarize, in this equilibrium bidders for whom searching is relatively effortless always search and therefore always bid only late in the auction. On the other hand, bidders for whom searching is more costly bid in both bidding rounds and search only if they were outbid in the early round. In equilibrium the early-round bidding induces bidders to act as if they coordinated their searching.

A paper which is most closely related to ours is by Hossain (2003). He constructs a model in which some bidders are fully informed about their private valuations for the object and the remaining bidders have to learn their valuations from signals they receive throughout the auction. Bidding takes place in a dynamic second-price auction with discrete bidding rounds - an environment very much like ours. In each round the uninformed bidder observes the standing price and receives a private signal about whether her true valuation is above (positive signal) or below (negative signal) the standing price. In the equilibrium the uninformed bidder revises her bid in every round as long as her signal is positive and quits bidding when the signal turns negative. Informed bidders bid just once, i.e., early, if their valuations are sufficiently low, and late, if they are sufficiently high. Gradual learning of her value by the uninformed bidder drives the bidding in the equilibrium of this model. Our model differs in several respects: it is symmetric, it does not rely on behavioral assumptions and it is driven by price-searching instead of value discovery.

A similar framework to ours was proposed by Rasmussen (2001). In his model, two bidders, one informed and one uninformed, bid in a dynamic auction. The uninformed bidder can learn her private value but has to pay a discovery cost. In equilibrium the uninformed bidder has an incentive to place an early, "preemptive bid" which increases in her cost and which allows her to avoid paying the discovery cost whenever she wins the auction with that bid. Once she is outbid then she invests in discovering her true value. The information acquisition is endogenous in the same spirit as it is in our case and the early bidding reflects the trade-off between the cost of information versus the benefit of winning the auction when it's profitable. Our framework differs Rasmussen's in that he focuses on the process of individual value discovery which causes bidders to bid late in the equilibrium. In our case we model the process of price searching which motivates bidders to bid early. Furthermore, we consider symmetric bidders and our auction mechanism is explicitly modeled.

The rest of the paper proceeds as follows. In the next section we set up our model. Then, in section two we first characterize two equilibria that exist in the environment with common price draws and then we do the same for the the environmemnt with independent and private price draws. Section four then looks at welfare properties of both equilibria. Lastly, we conclude by a short discussion of our results.

2 The Model

We construct a model of an auction which is similar to auctions commonly found on the Internet. The prime differences between the Internet-type auctions and standard "textbook" auction formats are multiple bidding rounds, availability of outside buying opportunities and proxy-bidding. In this section we integrate these features into our model.

There are two identical units of the same object for sale. Both objects are worthless to the seller(s), i.e., $v_0 = 0$, where the zero-index indicates the seller(s). The first unit is sold in the auction. The second unit is sold outside of the auction for a posted price. There are two buyers, $i \in \{A, B\}$, who bid in the auction. Both are risk neutral and have common valuation, i.e. $v_A = v_B = 1$. Before the auction closes, both buyers may invoke a private price-offer, o_i, on a second unit. The price-offer is a random draw from the uniform distribution, i.e., $o_i \backsim U[0, 1]$. To obtain the price-offer, a bidder i has to pay a search cost $c_i \backsim U[0, 1/2]$, which represents her private type.

The bidding format is a dynamic (multi-round), second-price auction. It has two bidding rounds, $r \in \{1, 2\}$. Each round, r, begins with both bidders simultaneously placing their bid $b_{i,r} \in [0, 1]$. In the second round, let $B_i = \max\limits_{k\in\{1,2\}} b_{i,k}$ be the highest bid submitted by bidder i. For example, the highest bid of bidder A in the second bidding round is $B_A = \max[b_{A,1}, b_{A,2}]$. In the auction we define three types of prices: the starting price, p_0, the standing price, p_1, and the final price p_2. The starting price is set to zero, i.e., $p_0 = 0$. The standing price, p_1, equals to $\min[b_{A,1}, b_{B,1}]$ and the final auction price, p_2, equals to $\min[B_A, B_B]$.

The game unfolds in three stages: *the initial bidding round* ($s = 0$), *the searching round* ($s = 1$), and *the final bidding round* ($s = 2$). The auction begins with the initial bidding round. All bidders observe $p_0 = 0$ and then simultaneously place bids in the auction. In the searching round, each bidder first observes the standing price, p_1, and whether she is the current high bidder.[2] Then, both bidders decide whether they want to search for the outside price-offer, o_i. If bidder i searches, then she incurs a search cost, c_i, and gets a price-draw, o_i, in return, where $o_i \backsim U[0, 1]$. In the final bidding round bidders submit another round of bids. After all bids are submitted the auction closes and any bidder who has searched can purchase the object for her outside price. Then the game ends and payoffs are realized.

Denote H_s a set of all histories in stage s. Then, H_0 contains a single element $h_0 = p_0 = 0$. H_1 contains all histories $h_1 \equiv \{p_1, W\}$, where the first element is $p_1 \in [0, 1]$, the current standing price after the initial bidding-round. The second element, $W \in \{A, B\}$, gives the index of the current high bidder, i.e.,

$$W(b_{A,1}, b_{B,1}; \rho) = \begin{cases} A & \text{if} \quad \begin{array}{l} b_{A,1} > b_{B,1} \text{ or} \\ b_{A,1} = b_{B,1} \text{ and } I = A \end{array} \\ \oslash & \text{if} \quad b_{A,1} = b_{B,1} = 0 \\ B & \text{if} \quad \textit{otherwise} \end{cases} .$$

Ties are broken with positive probability and $I \in \{A, B\}$ indicates the bidder in who's favor the tie was broken. For the purposes of our discussion tiebreaking rule is irrelevant. Notice that if $b_{i,1} = 0$, then $W \neq i$, i.e., in our framework bidding zero has exactly the same effect as not bidding at all. Finally, H_2 contains all elements $h_2 = \{h_1, o_i\}$. If no price-offer was drawn then we set $o_i = 1$.

To examine the payoffs let $\tau_i \in \{0, 1\}$ be an indicator of whether bidder i has searched, ($\tau_i = 1$), or passed, ($\tau_i = 0$). Similarly, let $\gamma_i \in \{0, 1\}$ be the indicator of whether i has bought the unit for the offered price o_i, ($\gamma_i = 1$)

[2] The current high bidder is the one who would be awarded the object if the auction ended in that instance.

or not, ($\gamma_i = 0$). The ex-post payoff to bidder i if she had won the auction is given by

$$V_i(\tau_i, \gamma_i) = \begin{cases} 1 - p_2 - c_i - o_i & \text{if} \quad \tau_i = 1 \text{ and } \gamma_i = 1 \\ 1 - p_2 - c_i & \text{if} \quad \tau_i = 1 \text{ and } \gamma_i = 0 \\ 1 - p_2 & \text{if} \quad \tau_i = 0 \end{cases} .$$

Her payoff in case she had lost the auction is given by

$$V_i(\tau_i, \gamma_i) = \begin{cases} 1 - c_i - o_i & \text{if} \quad \tau_i = 1 \text{ and } \gamma_i = 1 \\ -c_i & \text{if} \quad \tau_i = 1 \text{ and } \gamma_i = 0 \\ 0 & \text{if} \quad \tau_i = 0 \end{cases} .$$

A strategy for a bidder i is a triple

$$(\beta_i^1(c_i),\ \tau_i(c_i; h_1),\ \beta_i^2(c_i, h_2)),$$

where $\beta_i^1 : [0, 1/2] \to [0, 1]$ is the initial-round bidding function; $\tau_i : [0, 1/2] \times \{[0, 1] \times \{1, 2\}\} \to [0, 1]$ is the probability of searching in the middle stage and $\beta_i^2 : [0, 1/2] \times \{[0, 1] \times \{1, 2\} \times [0, 1]\} \to [0, 1]$ is the bidding function in the final round.

3 Equilibrium

In this section we characterize the equilibria of this model. The concept we use is the Perfect Bayesian Equilibrium. We also restrict our attention to symmetric, pure and undominated strategies. Notice, that due to symmetry restriction, a strategy which could be considered an equilibrium candidate has no bidder subscript, i.e., $(\beta^1(c_i), \tau(c_i; h_1), \beta^2(c_i, h_2))$. Furthermore, considering only pure strategies implies that in the searching round each bidder either searches or passes with probability one, i.e. $\tau : [0, 1/2] \times \{[0, 1] \times \{1, 2\}\} \to \{0, 1\}$.

Finally, we disallow the use of dominated strategies in equilibrium. The standard argument, due to Vickrey (1961), by which value bidding is a (weakly) dominant strategy in the second-price sealed bid auction applies to our context as well. In the final bidding round, if the bidder did not search, then if she loses the auction she gets zero payoff. This is equivalent to setting her outside price to 1, i.e., $o_i = 1$. If she did search then her outside price is drawn from the unit interval, i.e., $o_i \in [0, 1]$. The value of the outside price-offer completely defines bidder i's maximum willingness to pay for the object in the final round. This puts us to Vickrey's world in which bidding o_i is an undominated strategy. In the rest of the paper we will assume that bidders bid their outside price-offers in the final round.

Assumption: Bidders bid their outside price in the final round, i.e.

$$\beta^2(c; h_2) = o.$$

Bidding outside price (weakly) dominates any other bid in the final bidding round. We will use this assumption to characterize bidders' behavior in the final bidding round. This implies that our equilibria in this section differ in their prescribed behavior solely in the initial and middle stages of the game, i.e. initial round bids and searching decisions.

3.1 Common Price Draws.

In this section we present a simplified version of our model in which price draws are common, i.e., $o_A = o_B = o$, where $o \backsim U[0, 1]$. Both bidders get the same draw if they decide to search. We focus our attention on two classes of equilibria: bid-separating and bid-pooling.

Definition: Any symmetric equilibrium in which bidders use the increasing bidding function in the initial bidding round, i.e., bidding function β^1 is strictly increasing, we *bid-separating* equilibrium. Any symmetric equilibrium in which bidders bid the same amount in the initial bidding round we call a *bid-pooling* equilibrium.

Below we characterize two equilibria. The first is bid-separating, i.e., all bidders use increasing bidding function in the early bidding round. The second is bid-pooling, i.e., all bidders bid zero in the early bidding round.

We start by looking at the expected payoffs in the searching stage. In this stage, a bidder's payoff from searching or passing will depend on whether she was outbid after the initial bidding round or not. There are two cases: $W = A$ and $W = B$. We look at these two cases in turn.

Suppose first that in the searching stage bidder A is the high bidder, i.e., $b_A \geq b_B$ and the history is $h_1 \in \{b_B, A\}$. A's expected payoff will depend on whether she searches or not and whether her opponent, bidder B, searches or not. If A searches then she will pay her search cost c_A, get the outside price o and bid it the final bidding round. Hence, A's final round bid B_A will be the maximum of her initial round bid and the outside price, i.e., $B_A = \max[b_A, o]$. However, since A is the high bidder after the initial bidding round, she cannot use her outside price unless she is outbid in the final bidding round, i.e., unless $B_B \geq B_A = \max[b_A, o]$. Thus, if bidder B searches, then $B_B = o$ and A will win the auction with certainty when $o \leq b_A$. The final auction price will equal to the standing price b_B if $o \leq b_B$ and it will equal to o otherwise. When $o > b_A$ then the final round bids of both bidders are tied, i.e., $B_A = B_B = o$, and A will pay o whether she wins or loses the auction. The second case is when B does not search. Then, $B_B = 1 \geq B_A = \max[b_A, o]$. Now A is outbid almost surely which means that she will end up paying the outside price o. The following expression gives A's expected payoff from searching

$$\begin{aligned} V^S(c_A; h_1) &= \tau_B \left(\int_0^{b_B} (1 - b_B) do + \int_{b_B}^1 (1 - o) do \right) + \\ &\quad (1 - \tau_B) \left(\int_o^1 (1 - o) do \right) - c_A \\ &= \frac{1 - \tau_B}{2} + \frac{\tau_B}{2}(1 - b_B^2) - c_A, \end{aligned} \tag{1}$$

where τ_B denotes the probability with which B searches and $h_1 = \{b_B, A\}$.

Alternatively, suppose that A passes. Then, $B_A = 1$. If bidder B has searched, then she will bid the outside price in the final bidding round, i.e., $B_B = o$ and A will win the auction almost surely. She will pay the standing pice b_B if $o \leq b_B$ and she will pay the outside price o if $o > b_B$. On the other hand, if B hasn't searched, then $B_B = 1$, and because neither bidder has searched and $B_A = B_B = 1$, both bidders earn zero payoff. The following expression gives A's expected payoff from passing, i.e.,

$$\begin{aligned} V^P(c_A; h_1) &= \tau_B \left(\int_0^{b_B} (1 - b_B) do + \int_{b_B}^1 (1 - o) do \right) \\ &= \frac{\tau_B}{2}(1 - b_B^2). \end{aligned} \tag{2}$$

Now we turn to the second case and suppose that bidder B is the high bidder after the initial round, i.e., $b_A \leq b_B$ and $h_1 = \{b_A, B\}$. Notice that in second price auction bidder A does not observe b_B. However, for the purposes of clarity let us suppose for a moment that b_B were observable to both bidders. This will simplify our expressions considerably and illustrate the intuition behind the payoffs.

As before, suppose first that A has searched. Then, she pays her search cost c_A and gets the outside price. Her final round bid is $B_A = o$. If B searches, then $B_B = o$ and if B passes, then $B_B = 1$. Hence, when A wins the auction, she will pays the final auction price, o, and if she loses the auction, then she pays the outside price, o, i.e., she ends up paying o whether she wins or loses the auction. Her expected payoff from searches is

$$V^S(c_A; h_1) = \int_0^1 (1-o)do - c_A = \frac{1}{2} - c_A, \tag{3}$$

where $h_1 = \{b_A, B\}$.

Finally, when A does not search then she bids in the final bidding round, i.e., $B_A = 1$. Bidder B's final round bid is either $B_B = o$ if she searched or $B_B = 1$ if she hasn't searched. In the former case A wins the auction and pays b_B if $o < b_B$, and o if $o > b_B$. In the latter case both bidders get zero. Hence, conditional on b_B, A's payoff from passing is[3]

$$\begin{aligned} V^P(c_A; h_1) &= \tau_B \left(\int_0^{b_B} (1-b_B)do + \int_{b_B}^1 (1-o)do \right) \\ &= \frac{\tau_B}{2}(1-b_B^2). \end{aligned} \tag{4}$$

Notice that (2) and (4) are exactly equivalent. When bidder A does not search she rises her bid to the maximum, 1, and wins the auction at the price of the second highest bid, i.e., $\max[b_B, o]$, no matter whether she was the high or the low bidder after the initial bidding round.

The following strategy profile is the *early-bidding* equilibrium.

Proposition 1: There is an *early-bidding* (EB)[4] equilibrium in which each bidder bids according to the increasing and concave bidding function in the initial bidding round and bids her price-offer, i.e., $\min[o, 1]$, in the final round. In the middle stage the low bidder searches and the high bidder passes. Bidder i uses

$$\beta^1(c_i) = \sqrt{2c_i}$$

[3] Here we make use of our temporary assumption that A is able to observe the winning bid b_B. Then, A's expected payoff is largely simplified. Without this assumption we would have to integrate over the relevant types of bidder B.

[4] where the abbreviation EBcv stands for early-bidding (equilibrium) common prices case.

in the first round,

$$\tau(c_i; h_1) = \begin{cases} 0 & \text{if} \quad W = i \\ 1 & \text{if} \quad otherwise \end{cases}$$

in the searching round and

$$\beta^2(c_i; h_2) = o$$

in the final round. All information sets are on equilibrium path and hence beliefs are determined by the equilibrium strategies.

Proof: Appendix.

An interesting feature of this equilibrium is that bidders implicitly coordinate[5] their searching decisions in the searching round. The high bidder passes while the low bidder searches. This strategy is important since as we will see later this type of coordinated searching has an impact on the efficiency of the market. We will call this searching-when-loosing strategy.

Definition: Any strategy by a bidder "*searches* when she is losing (after the initial biding round) and *pass* otherwise" we call the *searching-when-losing* strategy.

The searching-when-losing strategy makes the equilibrium strong in the following sense: when bidders use the searching-when-losing strategy in the continuation game (searching stage), then bidding $\sqrt{2c_i}$ is a weakly dominant strategy for bidder $i \in \{A, B\}$, i.e., it is the best response to any bid by the other bidder. To see the intuition behind this consider what happens when A bids slightly more or slightly less, i.e., when $\sqrt{2c_A} \pm \Delta$ for an arbitrarily small and positive Δ. We will only illustrate the upward deviation, $\sqrt{2c_A} + \Delta$, but the same logic applies to the opposite case of $\sqrt{2c_A} - \Delta$. Take an arbitrary bid b_B, by bidder B. If $b_B < \sqrt{2c_A}$, then both bids, $\sqrt{2c_A}$ and $\sqrt{2c_A} + \Delta$, win the auction and in both cases A gets the same payoff (2) where $\tau_B = 1$. Alternatively, if $\sqrt{2c_A} + \Delta < b_B$, then A loses the auction in both cases and her payoffs from both bids are the same, i.e., given by (3) where $\tau_B = 0$.

[5] The word coordinate implies some sort of communication on the part of the bidders. Since our environment is purely non-cooperative we want to emphasize that the coordination in our case is implicit - or, in other words, in the equilibrium bidders act as if they coordinated.

The only case in which payoffs from bidding $\sqrt{2c_A}$ versus $\sqrt{2c_A}+\Delta$ differ is when $\sqrt{2c_A} \leq b_B < \sqrt{2c_A}+\Delta$. In this case, bidding b_A causes A to become the low bidder after the initial bidding round, $W = B$, and her expected payoff is given by (3) where $\tau_B = 0$, i.e.,

$$\frac{1}{2} - c_A = \frac{1}{2} - \frac{(\sqrt{2c_A})^2}{2}. \tag{5}$$

Now consider what happens when A bids $\sqrt{2c_A} + \Delta$ instead. Then, she becomes the high bidder, $W = A$, and her payoff is given by (2), where $\tau_B = 1$, i.e.,

$$\frac{1}{2} - \frac{b_B^2}{2}. \tag{6}$$

The optimality of b_A requires that (5) is as least as big as (6) which is true since $\sqrt{2c_A} \leq b_B$. Similar arguments apply for the deviation in the opposite direction, $b_A - \Delta$.

The EB equilibrium is fairly intuitive. The *searching-when-losing* strategy allows the implicit coordination of searching decisions. Notice that in this equilibrium there is always precisely one bidder who searches while the other bidder passes. This type of coordination is facilitated by the initial round bidding which is intuitively appealing.

A question one could be asking is whether there are any other increasing bidding functions that could support the EB equilibrium. The answer is no as established by the following corollary

Corollary 1: *Early-bidding* equilibrium given in Proposition 1 is a bid-separating equilibrium which is unique in symmetric, undominated and pure strategies.

Proof: Appendix.

Next, we discuss another equilibrium which has empirical as well as intuitive appeal. It is a *late-bidding* equilibrium. Recall that in the initial bidding round a zero bid, i.e. $b_i = 0$, has the same effect as if the bidder did not bid at all – a bidder cannot become the high bidder with a zero bid just as she cannot become the high bidder if she hasn't bid at all. There exists an equilibrium in which all bidders bid zero in the initial bidding round and then use a *threshold* strategy in the searching round. A *threshold* strategy is a value $\hat{c} \in [0, 1/2]$ such that all bidders with $c_i < \hat{c}$ search and the rest pass.

The next proposition gives the equilibrium.

Proposition 2: There is a *late-bidding* (LBcp) equilibrium in which all bidders bid zero in the initial bidding round and bid their price-offers in the final round. Searching decisions follow cut-off strategies, such that all types with low search costs, i.e., $c_i < 1/4$, search and types with high search costs, i.e., $c_i > 1/4$, pass. Bidder i uses

$$\beta^1(c_i) = 0$$

in the first round,

$$\tau(c_i; h_1) = \begin{cases} 1 & \text{if} \quad c_i < 1/4 \\ 0 & \text{if} \quad \textit{otherwise} \end{cases}$$

in the searching round and

$$\beta^2(c_i; h_2) = o$$

in the final round. For all histories $\{h_1 \in H_1 \mid p_1 \in h_1 \text{ and } p_1 > 0\}$ bidder i believes that $c_j \sim U[0, 1/4]$.

Proof: Appendix.

In the equilibrium above all bidders pass in the initial bidding round. In that case the history h_1 is $\{0, \oslash\}$, i.e., the standing price equals to the starting price and neither of the bidders is the high bidder after the initial bidding round. The main difference between this equilibrium and the early-bidding equilibrium is that in this case the history will reveal nothing about bidders' respective types (search costs), and hence, the bidders will remain uncertain about whether their opponent is going to be searching or passing. This causes all bidders for whom searching is relatively cheap ($c < \hat{c}$) to search and the others for whom searching is costly ($c > \hat{c}$) to pass.

The pattern of behavior which appears in the LBcp equilibrium is consistent with the empirical evidence of late bidding that we outlined in the introduction. The equilibrium, however, suffers from the coordination problem. Since bidders use threshold strategies in the searching round it will be the case that with positive probability both bidders search. This is wasteful and cannot be a Pareto efficient outcome as one could for example generate extra surplus by having one of the bidders pass instead and save the search cost.

3.2 Independent Price Draws

In this section we relax the assumption of common price draws and consider the case in which price draws are identically and independently distributed with the uniform distribution, i.e., $o_i \sim U[0,1]$ for $i \in \{A, B\}$. This assumption seems more realistic. One could argue that people do not shop in the same store which was one of the interpretations of the common price draws. We will illustrate the impact of independent price draws on the behaviors in our two equilibria from the previous section. A remarkable result of this section is that the heterogeneity of price draws causes some late bidding in the early bidding equilibrium. In what follows we characterize two equilibra that are very similar to our equlibria from the previous section. The first is late-bidding equilibrium and the second is early-bidding equilibrium.

Next proposition gives the late bidding equilibrium.

Proposition 3: There is a *late-bidding* (LBip)[6] equilibrium in which all bidders bid zero in the initial round and bid their outside prices in the final round. Searching decisions follow cut-off strategies, such that all types with low search costs, i.e., $c_i < 3/10$, search and types with high search costs, i.e., $c_i > 3/10$, pass. Bidder i uses

$$\beta^1(c_i) = 0$$

in the first round

$$\tau(c_i; h_1) = \begin{cases} 1 & \text{if} \quad c_i < 3/10 \\ 0 & \text{if} \quad otherwise \end{cases}$$

in the searching round and

$$\beta^2(c_i; h_2) = o$$

in the final round. For all histories $\{h_1 \in H_1 \mid p_1 \in h_1 \text{ and } p_1 > 0\}$ bidder i believes that $c_j \sim U[0, 3/10]$.

Proof: Appendix.

The equilibrium is almost the same as the LBcp equilibrium from the previous section. The only element that changes is the value of the threshold, $\hat{c}$. In the LBcp equilibrium it was the case that when both bidders searched

[6] where LBip refers to late-bidding equilibrium with independent price draws.

they both received the same price draw from the unit interval. Hence, on average, the final auction price was just equal to the average of the unit interval, i.e., 1/2. With independent private price draws, however, if both bidders search, then each bidder gets her own draw. Then, after bidders have bid their outside prices in the auction, the final auction price will become the lower of the two draws. Hence, the expected final auction price in this case is the value of the second order statistic from a sample of two, i.e., 1/3. The extra gain that the additional draw brings to the table causes the threshold value to decrease in the LBip equilibrium.

Now we turn to early-bidding equilibrium – EB. Here the independent price draws have more serious consequences than it was in the case of late bidding. In fact we will illustrate that with independent price draws the EB equilibrium in which all bidders use a strictly increasing bidding function fails: some low search cost types will profitably deviate to late bidding. To see how this happens consider the following example. Assume for a moment that price draws are common and that bidders use strategies given by EB equilibrium. Further suppose that $c_B = 0$ and $c_A = \varepsilon$ for some very small ε. Then, in equilibrium, B bids zero and A bids $\sqrt{2\varepsilon}$ (which is very small but positive). Bidder B is the low bidder, and hence, searches – costlessly. Since the standing price is $p_1 = 0$, both bidders pay the price-offer that B draws, o. Had A searched as well it would have cost her very little, i.e., ε, but the marginal benefit from searching would have been zero as her price-offer would also be o. Hence, by bidding early A was able to avoid searching while giving up no extra surplus.

Now suppose that price-draws are independent, i.e., $o_i \sim U[0,1]$ for $i \in \{A, B\}$, and that bidders use the equilibrium strategy profile from Proposition 1. As before, B initially bids zero and then searches. Bidder A bids $\sqrt{2\varepsilon}$ and passes. Her payoff is be given by (2) where $p_1 = 0$ and $\tau_B = 1$, i.e., the payoff is 1/2. Now suppose that A deviates and searches instead. Then, she draws o_A. Recall, that in the final bidding round bidders bid their respective outside prices. If $o_A \geq o_B$ and A wins the auction, then she pays the final auction price o_B. If $o_A \leq o_B$ and A loses the auction, then we have two cases: 1. $o_B \geq \sqrt{2\varepsilon}$ and 2. $o_B < \sqrt{2\varepsilon}$. In the first case bidder A is outbid in the final round and pays her outside price o_A. In the second case, however, A wins the auction with her initial round bid $\sqrt{2\varepsilon}$ and pays the final auction price o_B. This is the case when she is unable to use her own (low) price-offer o_A. As a result, A's expected payoff from searching is

$$\int_0^{\sqrt{2\varepsilon}} (1-\sqrt{2\varepsilon})do_B + \int_{\sqrt{2\varepsilon}}^1 \left(\int_0^{o_B} (1-o_A)do_A + \int_{o_B}^1 (1-o_B)do_A \right) do_B - \varepsilon$$

or,

$$\frac{2}{3} - \frac{(2\varepsilon)^{3/2}}{6} - \varepsilon.$$

We conclude that all bidder-types that satisfy

$$\frac{2}{3} - \frac{(2c_A)^{3/2}}{6} - c_A \geq \frac{1}{2} \tag{7}$$

will deviate from the proposed EB equilibrium and search even when they are winning the auction after the initial bidding round. Hence, with independent price draws EB equilibrium fails.

This motivates the multiple-bidding equilibrium which is given in the next proposition.

Proposition 4: There is a symmetric *multiple-bidding* (MB) equilibrium in which bidder $i \in \{A, B\}$, with search cost $c_i \geq 1/6$ bids according to increasing and concave bidding function in the initial round and bids her outside price-offer in the final round. If her search cost is low, i.e., $c_i < 1/6$, than she bids zero in the initial round. In the searching round she passes whenever she is the high bidder $W = i$ and her search cost is sufficiently high, i.e., $c_i < 1/6$. Otherwise, she passes. For bidder $i \in \{A, B\}$ and $j \neq i$ we have[7]

$$\beta^1(c_i) = \begin{cases} 0 & \text{if} \quad c_i < 1/6 \\ \sqrt{2c_i} & \text{if} \quad c_i \geq 1/6 \end{cases},$$

in the first round,

$$\tau(c_i; h_1) = \begin{cases} 0 & \text{if} \quad W = i \\ 1 & \text{if} \quad otherwise \end{cases},$$

in the searching round and

$$\beta^2(c_i; h_2) = o_i$$

[7] The strategies profile describes the equilibrium path. The full characterization of the best response strategies that form the equilibrium is given in the Appendix.

in the final round.

Proof: Appendix.

The equilibrium is very similar to the EB equilibrium with the exception that some bidder-types bid zero in the initial bidding round while the remaining types use increasing bidding function. When price-draws are independent each bidder realizes extra private gains from searching which induces some bidder-types (those with low search costs) to pass in the initial bidding round. The equilibrium interprets early bidding as an attempt to avoid searching when it is relatively costly to search, while the late bidding is interpreted as the response to potentially large gains from searching when searching is relatively cheap. Just as it was the case in the EB equilibrium, it is also true here, that at all times at least one of the bidders searches. The main difference between the two equilibria is that in the MB equilibrium it can also happen that both bidders search. These are precisely the bidders who bid zero in the initial bidding round. The power of the equilibrium is in explaining both major empirical bidding regularities within the same incentive structure.

4 Welfare Comparisons

Our model contains two types of equilibria: the *late-bidding* (LBcp/LBip) equilibrium and, the *early-/multiple-bidding* (EB/MB) equilibrium. In this section we compare these two equilibria in terms of how they divide the surplus between buyers and sellers both when price draws are common and when they are independent.

4.1 Common Price Draws

We begin by looking at the simpler, common price draws case, such that $o_A = o_B = o$, and look at the volume of total surplus. Throughout this section we will use Figure 1 to guide us. Figure 1 shows the areas for which the equilibrium behavior in the LBcp equilibrium differs. The behavior in the EB equilibrium is the same for all areas A1-A3. We will focus our attention on the shaded region in which $c_B \leq c_A$. The picture is symmetric around the 45° line.

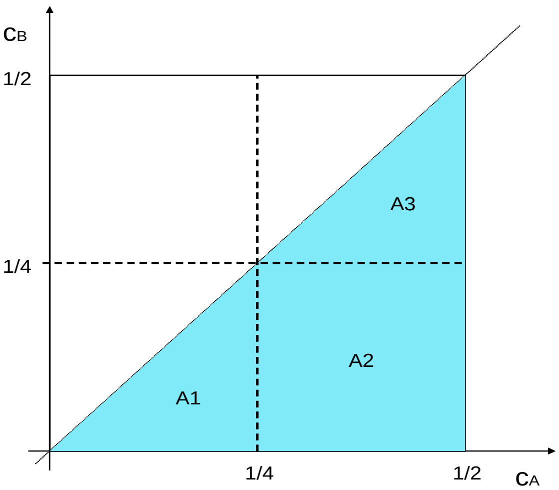

Figure 1.

In each equilibrium two units can be traded – one in the auction and the second for the outside price. The total surplus depends on how many units sell and how many buyers have searched. In the EB equilibrium only the more efficient buyer searches and both units sell with certainty. Therefore the total (ex-post) surplus is

$$2 - c_B.$$

In the LBcp equilibrium both items sell with probability 3/4 (areas A1 and A2) and only a single item is sold with probability 1/4 (area A3). With probability 1/4 both bidders search (area A1) and with probability 1/2 only the more efficient buyer searches (area A2). Finally, with probability 1/4 neither of the two buyers searches (area A3). The total (ex-post) surplus in this equilibrium is

$$\begin{cases} 2 - c_B - c_A & \text{if} \quad (c_A, c_B) \in \text{A1} \\ 2 - c_A & \text{if} \quad (c_A, c_B) \in \text{A2} \\ 1 & \text{if} \quad (c_A, c_B) \in \text{A3} \end{cases} .$$

We compare the two equilibria in terms of their ex-ante surpluses.

Observation 1: The EB equilibrium reaches the total (ex-ante) surplus 11/6 which is greater than the surplus of 13/8 generated in the LBcp equilibrium.

The reason for the loss of surplus in the LBcp equilibrium is that there the implicit coordination of searching is not possible. The LBcp equilibrium suffers from two types of inefficiency. The first is on the area A1 where both buyers search. Here, buyer A searches in vain since her marginal gain from searching is zero. If only the more efficient buyer, B, searched, then buyers would increase their joined surplus. The second inefficiency occurs on area A3 where both buyers pass. Here, one of the buyers does not trade causing a loss of surplus equal to 1. As before, if buyer B searched, then buyers would increase their joined surplus.

Next, we look at the division of surplus between buyers and sellers. The behavior in the EB equilibrium does not vary across buyer-types. In the initial bidding round, both buyers make nontrivial bids which causes the standing price to increase, i.e., $p_1 = \beta^1(c_B)$. Then, in the searching round only the more efficient buyer, B, searches. Buyer B then pays the outside price, o, with certainty while buyer A pays the higher of o and p_1. Below, we break down the total surplus between buyers (the first term) and the sellers (the second term), i.e.,

$$2 - (o + \max[o, p_1]) - c_B, \ o + \max[o, p_1].$$

In the LBcp equilibrium both buyers bid zero in the initial bidding round and search only if $c_i \leq 1/4$, where $i \in \{A, B\}$. In area A1 both buyers search and both pay the outside price, o. In area A2 only the more efficient buyer, B, searches and since the standing price is zero, i.e., $p_1 = 0$, both pay the outside price, o. Finally, in the last area, A3, both buyers pass and bid 1 in the final round. In the expression below the first term gives the surplus of the sellers and the second term gives (ex-post) surplus of the buyers for each respective area, i.e.,

$$\begin{cases} 2(1-o) - c_B - c_A, \ 2o & \text{if} \quad (c_A, c_B) \in \text{A1} \\ 2(1-o) - c_B, \ 2o & \text{if} \quad (c_A, c_B) \in \text{A2} \\ 0, \ 1 & \text{if} \quad (c_A, c_B) \in \text{A3} \end{cases} .$$

We compare the two equilibria in terms of the (ex-ante) surplus they generate for both, buyers and sellers. Table 1 gives the differences in (ex-ante) surplus that buyers and sellers get in the EB and the LBcp equilibrium for each respective area A1, A2 and A3. The results are summarized in the following observation.

Observation 2: By comparing the EB and the LBcp equilibria we find that:

In all areas A1-A3 the sellers are always better off in the EB equilibrium than they are in the LBcp equilibrium. The early bidding in the EB equilibrium deprives buyers of some surplus in the area A2 but this is more than offset by gains from the coordinated searching in the remaining areas A1 and A3.

Table 1 shows distribution of gains and losses between the two equilibria.

Differences in total surplus: (EB - LBcp)	Buyers:	Sellers:	Total:
A_1 :	$1/48$	$1/48$	$1/24$
A_2 :	$-1/16$	$1/16$	0
A_3 :	$1/12$	$1/12$	$1/6$
A_{1+2+3} :	$1/24$	$1/6$	$5/24$

Table 1.

4.2 Independent Price Draws

In this section we compare equilibrium LBip with the MB equilibrium. The division of surplus in the MB and LBip equilibrium differs from the previous section. Figure 2 shows the areas with different equilibrium behavior in both cases of the MB and LBip equilibrium. Areas B1, B2 $\cup$ B4 and B3 $\cup$ B5 $\cup$ B6 describe different equilibrium behavior in the MB equilibrium and areas B1 $\cup$ B2 $\cup$ B3 and B4 $\cup$ B5 $\cup$ B6 do the same for the case of LBip equilibrium. Hence, we have six areas where equilibrium behavior differs between the EB and the LBip equilibrium.

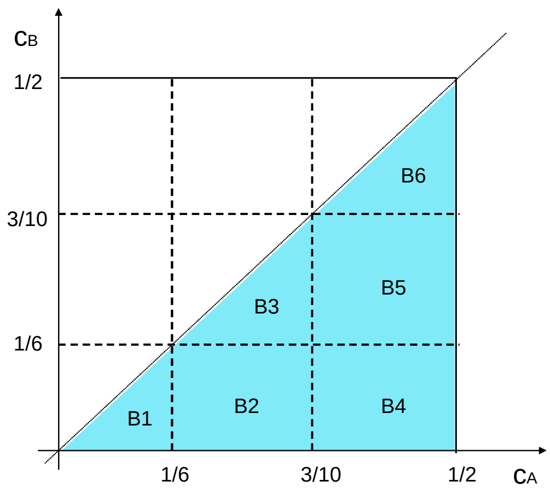

Figure 2.

In the MB equilibrium both units always sell and the more efficient buyer, B, always searches. The less efficient buyer, A, searches only if her search cost is sufficiently low, i.e., $c_A \leq 1/6$. Hence, the the total (ex-post) surplus in the MB equilibrium is given by

$$\begin{cases} 2 - c_B - c_A & \text{if} \quad (c_A, c_B) \in \text{B1} \\ 2 - c_B & \text{if} \quad (c_A, c_B) \in \text{B2-B6} \end{cases} .$$

In the LBip equilibrium both items sell with probability $21/25$ (areas B1-5) and only a single item is sold with probability $4/25$ (area B6). With probability $9/25$ both buyers search (areas B1-3) and with probability $12/25$ only the more efficient buyer, B, searches (areas B4-5). Finally, with probability $4/25$ both buyers pass (area B6). Hence, the total surplus in this equilibrium is given by

$$\begin{cases} 2 - c_B - c_A & \text{if} \quad (c_A, c_B) \in \text{B1-B3} \\ 2 - c_B & \text{if} \quad (c_A, c_B) \in \text{B4-B5} \\ 1 & \text{if} \quad (c_A, c_B) \in \text{B6} \end{cases} .$$

Observation 3: The MB equilibrium reaches greater surplus, 1.82, than the LBip equilibrium, 1.66.

Next we look at the division of surplus between buyers and sellers. We first focus on the MB equilibrium. In the area B1 both buyers bid zero in the initial round and then search. Hence, both of them draw an outside price and bid their price draws it in the auction. The lower draw, i.e., $\min[o_A, o_B]$, becomes the price for both of them. In areas B2 and B4 only the more efficient buyer, B, searches while the other buyer passes. Since buyer B bids zero in the initial round the standing price remains at zero, i.e., $p_1 = 0$, and both buyers pay a final price equal to B's price draw, i.e., o_B. Finally, in areas B3, B5 and B6, both buyers bid positive amounts early raising the standing price above zero. The standing price then becomes a price floor in the auction. Hence, the more efficient buyer, B, pays her price draw, o_B while her opponent, buyer A, wins the auction and pays the higher of o_B and the standing price p_1. The (ex-post) surpluses of buyers (the first term) and sellers (the second term) are summarized below

$$\begin{cases} 2(1-\min[o_A,o_B]) - c_B - c_A,\ 2\min[o_A,o_B] & \text{if} \quad (c_A,c_B) \in \text{B1} \\ 2(1-o_B) - c_B,\ 2o_B & \text{if} \quad (c_A,c_B) \in \text{B2} \cup \text{B4} \\ 2 - o_B - \max[o_B,p_1] - c_B,\ o_B + \max[o_B,p_1] & \text{if} \quad (c_A,c_B) \in \text{B3} \cup \text{B5} \cup \text{B6} \end{cases} .$$

In the LBip equilibrium the division of the surplus is almost the same as it was for the LBcp equilibrium in the preceding section. Here, in areas B1-3 buyers realize extra gains from an additional price draw. Both pay the final price equal to $\min[o_A, o_B]$. The division of surplus is

$$\begin{cases} 2(1-\min[o_A,o_B]) - c_B - c_A,\ 2\min[o_A,o_B] & \text{if} \quad (c_A,c_B) \in \text{B1-B3} \\ 2(1-o_B) - c_B,\ 2o_B & \text{if} \quad (c_A,c_B) \in \text{B4-B5} \\ 0,\ 1 & \text{if} \quad (c_A,c_B) \in \text{B6} \end{cases} .$$

For both, buyers and sellers, we subtract their (ex-ante) surplus in the LBip equilibrium from that in the MB equilibrium and summarize in the following observation.

Observation 4: By comparing MB and LBip equilibria we find that:

(i) In all parts of equilibria (areas B1-B6) sellers capture more surplus in the MB than in the LBip equilibrium.

(ii) Buyers are worse off in the MB equilibrium than in the LBip equilibrium despite their implicit coordination of searching in the MB equilibrium.

Table 2 shows the distribution of gains and losses between the MB and LBip equilibrium. We exclude areas B1 and B4 since in these areas both equilibria render the same behavior and hence the same surplus.

Differences in total surplus: (MB - LBip)	Buyers:	Sellers:	Total:
B_2 :	-0.018	0.06	0.042
B_3 :	-0.021	0.039	0.018
B_5 :	-0.05	0.05	0
B_6 :	0.042	0.059	0.101
$B_{2+3+4+6}$:	-0.047	0.208	0.161

Table 2.

By comparing observations 2 and 4 we notice a little paradox: buyers' gain in the EB relative to the LBcv equilibrium when price draws are common, but lose when price draws become independent. In both cases, the equilibrium (EB/MB) in which early bidding occurs is more efficient than the corresponding late-bidding (LBcp/LBip) equilibrium. However, while the extra surplus in the EB equilibrium (common price draws) is shared by both, buyers and sellers, the extra surplus in the MB equilibrium (independent price draws) is entirely captured by the sellers. Parts of equilibrium where buyers do worse in MB than in LBip equilibrium are in areas B2, B3 and B5. In the reminder of this section we examine what causes this phenomenon.

To resolving this paradox we have to first understand how searching affects buyer's surplus in both cases of common and independent price draws. Suppose that after the initial bidding round the standing price is zero, i.e., $p_1 = 0$. In the case of common price draws buyers can reach the greatest joined surplus if only the more efficient buyer, i.e., buyer B, search. The situation is different in the case of independent price draws. Here, buyers reach their maximum combined surplus if they both search. To illustrate this we focus on areas B1-3 where buyers always search in LBip equilibrium. To make this as simple as possible consider a borderline case where $c_A = 3/10$ and $c_B = 0$. When both buyers search, then their joined surplus is given by

$$2(1 - \frac{1}{3}) - 3/10 = 31/30.$$

If, however, only the more efficient buyer searches, then the expected joined surplus is

$$2(1 - \frac{1}{2}) = 1.$$

This suggests that in the LBip equilibrium buyers have greater incentives to search when outside prices are independent. One direct implication of this is that the equilibrium searching threshold in the LBip equilibrium, 3/10, is higher in the case of independent price draws than it is in the case of common price draws, i.e., 1/4. Consequently, the area B6 is smaller than the area A3 and the loss of trading surplus in the LBip equilibrium is reduced. The second implication is that in areas B2 and B3 buyers maximize their joint surplus in the LBip equilibrium – since they both bid zero in the initial round and then both search. It follows then that the joined surplus of buyers's have to be smaller in the MB than in the LBip equilibrium, since there (in the MB eq.) only the more efficient buyer searches and the other buyer passes. In area B3 buyers' surplus is reduced even more in the MB equilibrium because both buyers make positive bids in the initial bidding round and raise the standing price above zero. Hence, the inefficient searching in the MB equilibrium explains the negative entry in the cell B2 of the buyers' column (Table 2) and a combination of inefficient searching and early bidding explains the negative entry in cell B3.

Finally, the negative entry in cell B5 of buyers column is entirely due to the early bidding. There, in both equilibria only buyer B searches. The early bidding in the MB equilibrium set a price floor in the auction and becomes a pure transfer of surplus from buyer and sellers – a tax for the buyers' actions that implicitly coordinate searching.

5 Conclusion

Empirical studies of bidding behavior in Internet auctions show that large proportion of bidders bid multiple times while others bid only at the end. We designed a model that incorporates two key features found in typical Internet auction: bidding takes place in multiple and discrete rounds and proxy bidding is allowed. There are two types of equilibria. In the first equilibrium some bidders bid multiple times and some only bid late. In the second equilibrium all bidders bid late. The first equilibrium shows that we can see both multiple bidding and late bidding in an auction where all bidders are rational. The key to this finding is that the bidders face different costs of searching for outside prices. Buyers with high search costs bid multiple times, while those with low search costs only bid late. The main distinguishing feature of my model from other models of multiple bidding is that it is symmetric and does not rely on some buyers being uninformed about their private values at the beginning of the auction.

We considered two cases: common outside prices (the outside price is the same for both bidders) and independent outside prices (each bidder draws her own outside price). The late-bidding (LBcp/LBip) equilibrium exists in both cases and remains qualitatively unchanged. The equilibrium in which bidders bid multiple times also exists in both cases but varies substantially from one case to the other. When outside prices are common all bidders bid multiple times – the early-bidding (EB) equilibrium. On the other hand when outside prices are independent only the high-search-cost bidders bid multiple times while the rest bid late – the multiple-bidding (MB) equilibrium. The timing of the bids depends on the statistical structure of the outside prices. The case of independent outside prices is more realistic and the equilibrium in this case generates both the late and multiple bidding behaviors which fits nicely with the empirical patterns found in the bidding data.

Finally, we addressed some issues regarding welfare implications. According to our expectations we found that multiple-bidding (EB/MB) equilibrium dominates the late-bidding (LBcp/LBip) equilibrium in terms of generated total surplus. This is due to the fact that bidders act as if they coordinated searching, and as a result, both bidders always trade in the EB/MB equilibrium as opposed to the LBcp/LBip equilibrium in which some bidders may not always trade. However, a surprising result appears when we look at the division of surplus between buyers and sellers in both equilibria. While sellers are always better off in the equilibrium with early bidding (EB/MB) than in the LBcp/LBip equilibrium, the buyers are better of in the EB relative to LBcp but worse off in the MB relative to LBip equilibrium.

The main contribution of our model is in providing an explanation for the practice of early bidding which is commonly dismissed as a product of irrational or myopic behavior. The next step is to test how our theory fares in the reality. Future research might, for example, use an experiment to address this issue. Our model also suggests important implications for auction design. Since the sellers always receive greater profit in the equilibrium in which bidders coordinate searching, they might be interested in finding ways of implementing that equilibrium. For example, using secret instead of public reserve prices might be one way of accomplishing this.

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6 Appendix

Proposition 1.:

Consider strategy profile given by Proposition 1. Consider two bidders A and B.

Suppoise β is arbitrary function that is strictly increasing. β

Claim 1.: Suppose A is the low bidder, i.e., $W \neq A$. Then, bidder A searches, i.e., $\tau_A = 1$.

Proof: Consider bidders A and B and suppose $W \neq A$. Then, $b_B \geq b_A > 0$ and since β is strictly increasing we have $c_B \sim U[\beta^{-1}(b_B), 1/2]$. Since $W \neq A$, then $\tau_B = 0$ and A's payoff from searching is given by (3), i.e., $\frac{1}{2} - c_A$. When A passes she gets 0, by (4). Since $\frac{1}{2} - c_A \geq 0$, then the profit from searching is no less than zero, i.e. searching is optimal. $\square$

Claim 2.: Suppose $W = A$. Then, bidder A optimally passes, i.e., $\tau_A = 0$.

Proof: Since $W = A$ we have that $b_A \geq (>) b_B > (\geq) 0$ and $\tau_B = 1$. Then, $p_1 = b_b$ and A's payoff from passing is given by (2), i.e.,

$$\frac{1}{2} - \frac{b_A^2}{2}.$$

Her payoff from searching is given by (1), i.e.,

$$\frac{1}{2} - \frac{b_A^2}{2} - c_A.$$

The difference between passing and searching is $c_A \geq 0$, i.e., A optimally passes. $\square$

Claim 3.: In the initial bidding round $\beta^1(c_A) = \sqrt{2c_A}$ is a A's best response.

Proof: Suppose bidder A with $c_A \in [0, 1/2]$ bids $b_A = \sqrt{2c_A}$. When $W = A$ then we have $b_A \geq b_B = p_1 \geq 0$ and $c_B \in [0, c_A)$. Then, in the searching round, A passes, i.e., $\tau_A = 0$ and B searches, i.e., $\tau_B = 1$. The payoff that A gets in this case is given by (2) – the first term of (8). If $W \neq A$, then $b_B \geq b_A = p_1 \geq 0$ and we have $\tau_A = 1$ and $\tau_B = 0$. The payoff in this case is given by (3) – the second term of (8). Hence, A's expected payoff in the initial round is

$$\int_0^{c_A} (\frac{1}{2} - z)2dz + \int_{c_A}^{1/2} (\frac{1}{2} - c_A)2dz, \tag{8}$$

where we used a substitution $b_B = \sqrt{2c_B}$ in the first term.

To establish that $b_A = \sqrt{2c_A}$ is a best response consider a deviation $\hat{b}_A \neq b_A$. Then, with probability $\hat{b}_A^2$ we have $c_B \in [0, \hat{b}_A^2/2]$ and $W = A$. In this case A's payoff is given by (2) where $\tau_B = 1$. With the complementary probability $1 - \hat{b}_A^2$ we have $c_B \in [\hat{b}_A^2/2, 1/2]$ and $W \neq A$. Bidder A's payoff is now given by (2) where $\tau_B = 0$. Hence, the expected payoff from deviation is

$$\int_{1/6}^{\hat{b}_A^2/2} (\frac{1}{2} - z)2dz + \int_{\hat{b}_A^2/2}^{1/2} (\frac{1}{2} - c_A)2dz. \tag{9}$$

Taking the difference between (8) and (9) we get

$$\int_{c_A}^{\hat{b}_A^2/2} (\frac{1}{2} - c_A)2dz - \int_{c_A}^{\hat{b}_A^2/2} (\frac{1}{2} - z)2dz > 0,$$

i.e., bidding b_A gives higher payoff.

Thus, $b_A = \sqrt{2c_A}$ is A's best response since any deviation gives lower payoff. $\square$

Corollary 1:

Assume there are two bidders A and B. Consider a bidding equilibrium $(\beta^1(\cdot), \tau(\cdot; h_1), o)$ in symmetric and pure strategies. Assume bidding function β is strictly monotone, continuous and $\beta(0) = 0$.

Claim 1: For a given history $\{b_B, A\}$ if $\tau(c_B, \{b_B, A\}) = 1$ (0) is part of equilibrium, then $\tau_A = 0$ (1) is also a part of the equilibrium.

Proof: Suppose $h_1 = \{b_B, A\}$ and B searches in the equilibrium, $\tau_B = 1$. Consider bidder A. If A searches as well, $\tau_A = 1$, she gets payoff given by (1),

$$\frac{1}{2} - \frac{b_B^2}{2} - c_A.$$

If she passes, $\tau_A = 0$, then her payoff is given by (2),

$$\frac{1}{2} - \frac{b_B^2}{2}.$$

Since $c_A \geq 0$ passing is a best response for A, i.e., $\tau_A = 0$ is part of equilibrium.

Next, suppose $h_1 = \{b_B, A\}$ and B passes in the equilibrium, $\tau_B = 0$. Then, if A searches, $\tau_A = 1$, her payoff is given by (1), $1/2 - c_A$. And her payoff from passing is given by (2), i.e., 0. Since $c_A \leq 1/2$ searching is a best response for A, i.e., $\tau_A = 1$ is part of equilibrium. □

Claim 2: In any equilibrium when bidder A is the low bidder after the initial bidding round, $W \neq A$, then she searches, i.e., $\tau_A = 1$.

Proof: Assume that in the equilibrium β is strictly monotone, continuous and $\beta(0) = 0$. Suppose that $c_B \geq c_A$. Suppose there was a particular type $\mathring{c}$ such that $\tau(\mathring{c}, \{\beta(\mathring{c}), B\}) = 0$, where τ is the symmetric equilibrium searching function. Then, if $c_A = \mathring{c}$, then in equilibrium we would have $W = B$ ($W \neq A$) and $b_B \geq b_A = p_1 \geq 0$. The history of play after the initial bidding round would be $h_1 = \{b_A, B\}$ and, by Claim 1, B would be searching, $\tau_B = 1$.

Suppose that $c_A = \mathring{c}$ and consider bider B with (the highest) cost $c_B = 1/2$. Then, in equilibrium, her bid is $\beta(c_B) = \bar{b}$ (the maximum bid one would observe in the equilibrium). It follows, that in the searching round B becomes the high bidder, $W = B$, with certainty. Hence, for any given type of her opponent, $c_A \in [0, 1/2]$, if A passes in the equilibrium, $\tau(c_A, \{\beta(c_A), B\}) = 0$, then, by Claim 1, B searches. Her payoff in this case is given by (1), where $\tau_B = 1$ and we use the fact that $1/2 - c_B = 0$. If instead A searches in the equilibrium, $\tau(c_A, \{\beta(c_A), B\}) = 1$, then B passes. Her payoff is given by (2) where $\tau_B = 0$. Thus, her (ex-ante) expected equilibrium payoff is given by

$$\int_0^{1/2} \tau(z, \{\beta(z), B\})(\frac{1}{2} - \frac{\beta(z)^2}{2})2dz. \tag{10}$$

Now, consider following deviation by B. In the initial bidding round she bids $\mathring{b} = \beta(\mathring{c})$ and, in the searching round, she passes if she is the high bidder, i.e., $\tau_B = 0$ if $W \neq B$. In other words, B mimics the equilibrium strategy of a type $\mathring{c}$. Then, for all types of bidder A, $c_A \in [\mathring{c}, 1/2]$, the equilibrium history is $h_1 = \{\mathring{b}, k\}$, i.e., A becomes the high bidder after the initial bidding round and the standing price is $\mathring{b}$. Since in the equilibrium $\tau(\mathring{c}, \{\beta(\mathring{c}), B\}) = 0$, then, by Claim 1, bidder A searches, i.e., $\tau_A = 1$. Bidder B's payoff in this case is given by (4) where $\tau_A = 1$ – don't forget to flip the indexes, now B is A in the formula. For the remaining types, $c_A \in [0, \mathring{c}]$, B's equilibrium play remains unaffected (by the deviation). The expected payoff to bidder B from bidding $\mathring{b}$ is

$$\int_0^{\mathring{c}} \tau(z, \{\beta(z), B\})(\frac{1}{2} - \frac{\beta(z)^2}{2}) + \int_{\mathring{c}}^{1/2} (\frac{1}{2} - \frac{\beta(z)^2}{2})2dz. \tag{11}$$

Since the difference between (11) and (10) is

$$\int_{\mathring{c}}^{1/2} (1 - \tau(z, \{\beta(z), B\}))(\frac{1}{2} - \frac{\beta(z)^2}{2})2dz \geq 0$$

we found a profitable deviation by B, i.e., our supposed equilibrium fails.

Since passing when losing, i.e., $\tau_A = 0$ when $W \neq A$, is not part of any equilibrium, then it must be that in any equilibrium: $\tau_A = 1$ when $W \neq A$. □

Claim 3: In any equilibrium, when $W = A$, then bidder A passes, i.e., $\tau_A = 0$.

Proof: We combine Claims 1 and 2. Suppose $W = A$. Hence, $b_A \geq b_B$ and $W \neq B$. Then, in any equilibrium, by Claim 2, $\tau_B = 1$ and, by Claim 1, we have $\tau_A = 0$. □

Claim 4: Suppose we had an equilibrium such that in the searching stage bidder A passes when she is the high bidder and searches when she is the low bidder, i.e., $\tau_A = 1$ when $W \neq A$ and $\tau_A = 0$ when $W = A$. Then, in any such equilibrium, $\beta^1(c_A) = \sqrt{2c_A}$ is the bidding function in the initial bidding round.

Proof: Suppose that in the equilibrium τ has the property such that for a given type c_A and a history $\{b_A, B\}$, we had $\tau_A = 1$ and for history $\{b_B, A\}$, we had $\tau_A = 0$. Suppose that the equilibrium bidding function, β, is increasing, continuous and $\beta(0) = 0$. Consider bidder A. Suppose she bids $b_A \in [0, 1]$. For $x \in [0, 1]$, define $C(x) = \min[\beta^{-1}(\beta(1/2)), \beta^{-1}(x)]$. Function C returns the β^{-1} for any b_A which is in the range of β and returns 1/2 (the value of the highest type) if b_A is outside of the range of β. For a given bid by bidder A, b_A, she becomes the high bidder, $W = A$, if $c_B \in [0, C(b_A)]$. In that case her equilibrium payoff is given by (2) where $\tau_B = 1$. If $c_B \in [C(b_A), 1/2]$, then she becomes the loosing bidder, i.e., $W \neq A$. Her payoff in that case is given by (3) where $\tau_B = 1$. Her (ex-ante) expected payoff from bidding b_A is

$$\int_0^{C(b_A)} (\frac{1}{2} - \frac{\beta(z)^2}{2})2dz + \int_{C(b_A)}^{1/2} (\frac{1}{2} - c_A)2dz. \tag{12}$$

To find her optimal bid we maximize (12). The first order condition is

$$C'(b_A)((1 - \beta(C(b_A))^2) - (1 - 2c_A)) = 0.$$

Notice that C is a continuous function which is differentiable everywhere but at a single point $\beta(1/2)$. Hence, for all $b_A < \beta(1/2)$ we have

$$\begin{aligned} 1 - {b_A}^2 &= 1 - 2c_A \\ b_A &= \sqrt{2c_A}, \end{aligned}$$

The optimal bid b_A^* satisfies optimality condition

$$b_A^* \begin{cases} = \sqrt{2c_A} & \text{if} \quad \sqrt{2c_A} < \beta(1/2) \\ \in [\beta(1/2), 1] & \text{if} \quad \sqrt{2c_A} \geq \beta(1/2) \end{cases}. \tag{13}$$

Since, in any equilibrium, bidders behave optimally at each information set the equilibrium bidding function β has to satisfy the optimality condition (13). There is only a single such function $\beta(c_A) = \sqrt{2c_A}$ which satisfies (13). Hence, in any equilibrium the bidding function is $\beta^1(c_A) = \sqrt{2c_A}$. $\square$

Proposition 2:
Consider strategy profile given by Proposition 2.

Claim 1: Bidder A best responds by searching, $\tau_A = 1$, when $c_A < 1/4$ and by passing, $\tau_A = 0$, when $c_A > 1/4$.

Proof: Consider bidder A with search cost c_A. Then, $b_A = b_B = p_1 = 0$. Furthermore, $\tau_B = 1$ for $c_B < 1/4$ and $\tau_B = 0$ for $c_B \geq 1/4$. Since $b_A = b_B = 0$, then expressions (1) and (3) are equal and (2) and (4) are also equal which implies that W has no effect on bidder A's payoff.

Bidder A's payoff from searching is

$$\int_0^{1/2} (\frac{1}{2} - c_A)2dz = 1/2 - c_A, \tag{14}$$

where we used (1) inside the integral. On the other hand when A passes her payoff is

$$\int_0^{1/4} \frac{1}{2}2dz = 1/4, \tag{15}$$

by using (3). Hence, bidder A optimally searches when (14) is greater than (15), i.e.

$$\begin{aligned} 1/2 - c_A &> 1/4 \\ c_A &< 1/4, \end{aligned}$$

and she optimally passes when when

$$c_A > 1/4.$$

Thus, A best responds by playing $\tau_A = 1$ when $c_A < 1/4$ and $\tau_A = 0$ when $c_A > 1/4$. $\square$

Claim 2.: Bidding zero, i.e., $b_A = 0$, is A's best response for all $c_A \in [0, 1/2]$.

Proof: Consider bidder A with $c_A < 1/4$. Since $b_A = b_B = p_1 = 0$ we have $W = \oslash$. Furthermore, when $c_B < 1/4$, then $\tau_B = 1$ and when $c_B \geq 1/4$, then $\tau_B = 0$. Bidder A searches, i.e., $\tau_A = 1$. The payoff from $b_A = 0$, is

$$\int_0^{1/2} (\frac{1}{2} - c_A)2dz = \frac{1}{2} - c_A, \tag{16}$$

where we used (3). Next, suppose that A makes a different bid, i.e., $\tilde{b}_A > 0$. Now, $\tilde{b}_A > b_B = p_1 = 0$. Hence, $W = A$ and $\tau_A = 0$. Then, the expected payoff is

$$\int_0^{1/2} (\frac{1}{2} - c_A) 2dz = \frac{1}{2} - c_A, \tag{17}$$

where we used (1). Since (16) and (17) are equal all bids are optimal, i.e., $b_A = 0$ is optimal.

Next, suppose that $c_A > 1/4$. All remains the same from the previous case except that now $\tau_A = 0$, i.e., A passes after the first stage. The payoff from bidding zero is

$$\int_0^{1/4} \frac{1}{2} 2dz = \frac{1}{4},$$

by using (4). When A bids more than zero, i.e., $\tilde{b}_A > 0$, then $W = A$ but the payoff stays unaffected since (3) and (4) are the same. Thus, bidder A is indifferent and hence bidding zero is optimal.

In both cases, when $c_A < 1/4$ and when $c_A > 1/4$, $b_A = 0$ is a best response by bidder A. $\Box$

* * *

Before we give proofs of Propositions 3 and 4 we first generalize payoff functions (1)-(4) to accomodate the independent price draws. In the searching stage, suppose b_A and b_B are bids submitted in the initial round. First we take up (1). This is the case when A is the high bidder, i.e., $b_A \geq b_B$ and the history is $h_1 = \{b_B, A\}$. When A searches, i.e., $\tau_A = 1$, then her payoff is

$$\begin{gathered} \tau_B (\int_0^{b_B} (1 - b_B) dy + \\ \int_{b_B}^{b_A} (1 - o_B) do_B + \int_{b_A}^{1} (\int_0^{o_B} (1 - o_A) do_A + \int_{o_B}^{1} (1 - o_B) do_A) do_B) + \\ (1 - \tau_B) \int_0^1 (1 - o_A) do_A - c_A \\ = \frac{1 - \tau_B}{2} + \tau_B (\frac{2}{3} - \frac{b_B^2}{2} - \frac{b_A^3}{6}) - c_A. \end{gathered} \tag{18}$$

Notice that the difference from (1) is only in the middle term (the nested integral). This is the case when B searches, i.e., $\tau_B = 1$, and $o_B \geq b_A$. In the final bidding round B rises her bid to o_B. Now A is outbid and hes a chance to use her price-offer o_A. When $o_A \geq o_B$ and A wins the auction she pays o_B. When $o_A \leq o_B$ and A loses the auction she gets the object for her price offer o_A.

When A passes, i.e., $\tau_A = 0$, her payoff is equivalent to that under common price draws, i.e., (2),

$$\tau_B(\int_0^{b_B}(1-b_B)do_B + \int_{b_B}^1(1-o_B)do_B) = \tau_B(\frac{1}{2} - \frac{b_B^2}{2}). \tag{19}$$

The other case is when A is the loosing bidder, i.e. $b_A \leq b_B$ and the history is $h_1 = \{b_A, B\}$.The expression (3) becomes

$$\begin{gathered}\tau_B(\int_0^{b_B}(1-o_A)do_A+ \\ \int_{b_B}^1(\int_0^{b_B}(1-b_B)do_B + \int_{b_B}^{o_A}(1-o_B)do_B + \int_{o_A}^1(1-o_A)do_B)do_A)+ \\ (1-\tau_B)\int_0^1(1-o_A)do_A - c_A \\ = \frac{1-\tau_B}{2} + \tau_B(\frac{2}{3} - \frac{b_B^2}{2} + \frac{b_B^3}{3}) - c_A.\end{gathered} \tag{20}$$

The difference from (3) is again in the middle term. This is the case when B searches, i.e., $\tau_B = 1$, and $o_A \geq b_B$. In the final bidding round A rises her bid to o_A and auction price rises to b_B. If $o_B \leq b_B$ then B does not rise her bid and A wins the auction at price b_B. If $b_B \leq o_B < o_A$, then B bids o_B but A still wins the auction at price o_B. Finally, when $o_A \leq o_B$, then A pays o_A no matter if she winns or loses the auction.

When A passes, i.e., $\tau_A = 0$, her payoff is equivalent to (19).

Proposition 3:

Consider strategy profile given by Proposition 3.

*The proof is eqvivalent to the proof of Proposition 2 only here we use expressions (18)-(20) in place of expressions (1)-(4).

Proposition 4.:

In equilibrium bidders use

$$\beta^1(c_i) = \begin{cases} 0 & \text{if } \; c_i < 1/6 \\ \sqrt{2c_i} & \text{if } \; c_i \geq 1/6 \end{cases}$$

iin the first round and

$$\tau(c_i; h_1) = \begin{cases} 0 \;\; \text{if} & \begin{array}{c} W = i \text{ and } c_i > 1/6(1 - b_A^3) \\ \text{or} \\ W \neq i \text{ and} \\ c_A > \frac{1}{6(1-2k)}(3 - 2(2q + \beta^{-1}(b_A)) + (q - \beta^{-1}(b_A))4b_B^3) \end{array} \quad \text{when } p_1 > 0 \\ 1 \;\; \text{if} & \qquad \qquad \textit{otherwise} \\ \quad 0 \;\; \text{if} & \begin{array}{c} W = i \text{ and } c_A > \frac{1}{20y}(1 - b_A^3 - 10y) \\ \text{or} \\ W \neq i \text{ and } c_i > 3/10 \end{array} \quad \text{when } p_1 = 0 \\ \quad 1 \;\; \text{if} & \qquad \qquad \textit{otherwise} \end{cases}$$

in the searching round, where $\beta(c) = 0$ for $c < x$ and $\beta(c)$ is strictly increasing for $c \geq x$; $y = \max[3/10, x]$ and $q = 1/6(1 - \beta(q)^3)$. In the final round

$$\beta^2(c_i; h_2) = o_i.$$

For all histories $\{h_1 \in H_1 \mid p_1 \in h_1 \text{ and } p_1 \in (0, 1/3)\}$ bidder i believes that $c_j \sim U[0, 1/6]$.

Suppose that $\beta(c) = 0$ for $c < x$ and $\beta(c)$ is strictly increasing for $c \geq x$.

Claim 1.: Suppose $p_1 = 0$. Then, when $W \neq A$, i.e., bidder A is the low bidder, she optimally searches if $c_A < 3/10$ and passes when $c_A \geq 3/10$. For $W = A$, bidder A optimally searches when $c_A \leq \frac{1}{20y}(1 - b_A^3 - 10y)$ and optimally passes when $c_A \leq \frac{1}{20y}(1 - b_A^3 - 10y)$, where $y = \max[3/10, x]$.

Proof: Further suppose that $x \leq 3/10$. Consider bidder A. Since $p_1 = 0$ then we have either $b_B \geq b_A = 0$ or $b_A > b_B = 0$. In the first case, $W = \oslash$ and $b_B \geq b_A = 0$. Beliefs remain unchanged, i.e., $c_B \sim U[0, 1/2]$. Since $b_B = 0$ and $\tau_B = 1$ for $c_B < x$ and $b_B > 0$ and $\tau_B = 0$ for $c_B \geq x$ the payoff to A from searching is

$$\int_0^x \frac{2}{3} 2dz + \int_x^{1/2} \frac{1}{2} 2dz - c_A = \frac{1}{2} + \frac{x}{3} - c_A, \tag{21}$$

by using (20) and the payoff from passing is

$$\int_0^x \frac{1}{2} 2dz = x, \tag{22}$$

by using (19). Notice that (21) is strictly greater than (22) for all $c_A < 3/10$ and hence bidder A optimally searches. When $c_A \geq 3/10$ she optimally passes.

If $x > 3/10$, then since $\tau_B = 1$ for $c_B < 3/10$ and $\tau_B = 0$ for $c_B \geq 3/10$ the payoff to A from searching, i.e., expression (21) is $\frac{6}{10} - c_A$. The payoff from passing, i.e., expression (22) is $\frac{3}{10}$. Hence, A will search when $c_A < 3/10$ and pass otherwise.

The second case is when $b_A > b_B = 0$. Here, $W = A$ and the beliefs are $c_B \sim U[0, \beta^{-1}(x)]$. Suppose that $x > 3/10$ first. Then, bidder B searches, i.e., $\tau_B = 1$, if $c_B \leq 3/10$ and passes if $c_B > 3/10$. Then, if A searches she gets

$$\int_0^{3/10} (\frac{2}{3} - \frac{b_A^3}{6})\frac{1}{x} dz + \int_{3/10}^x \frac{1}{2}\frac{1}{x} dz - c_A \tag{23}$$

and if she passes she gets $1/2$. Then she will search if (23) is greater than $1/2$, i.e., when

$$c_A \leq \frac{1}{20x}(1 - b_A^3 - 10x).$$

For $x \leq 3/10$ we have that bidder B searches, i.e., $\tau_B = 1$, and hence A's payoff from searching is

$$\frac{2}{3} - \frac{b_A^3}{6} - c_A. \tag{24}$$

The payoff from passing remains at $1/2$ and so A searches when (24) exceeds $1/2$, i.e., when $c_A \leq 1/6(1 - b_A^3)$, and pass otherwise. $\square$

Suppose that $\beta(c) = 0$ for $c < x$ and $\beta(c)$ is strictly increasing for $c \geq x$

Claim 2.: Suppose $p_1 > 0$. Then, when $W \neq A$, i.e., bidder A is the low bidder, she optimally searches if $c_A \leq \frac{1}{6(1-2k)}(3 - 2(k + 2q) + (q - k)4b_B^3)$ and optimally passes when $c_A > \frac{1}{6(1-2k)}(3 - 2(k + 2q) + (q - k)4b_B^3)$. For $W = A$, bidder A optimally searches when $c_A < 1/6(1 - b_A^3)$ and optimally passes when $c_A \geq 1/6(1 - b_A^3)$.

Proof: Consider bidder A and suppose that $p_1 > 0$ and $W \neq A$. Then, $b_B \geq b_A > 0$ and since β is strictly increasing we have $c_B \sim U[k, 1/2]$, where $k = \beta^{-1}(b_A)$. Since $c_B \geq c_A$ and $W = B$, we have that $\tau_B = 0$ when $c_B \geq 1/6(1-b_B^3)$ and $\tau_B = 1$ when $c_B \geq 1/6(1-b_B^3)$. Define q as a solution to $q = 1/6(1-\beta(q)^3)$ and suppose that $q > x$. Then, A's payoff from searching is given by (20), i.e.,

$$\int_k^q (\frac{2}{3} - \frac{b_B^2}{2} + \frac{b_B^3}{3}) \frac{1}{1/2 - k} dz + \int_q^{1/2} \frac{1}{2} \frac{1}{1/2 - k} dz - c_A. \tag{25}$$

Her payoff from passing is given by

$$\int_k^q (\frac{1}{2} - \frac{b_B^2}{2}) \frac{1}{1/2 - k} dz. \tag{26}$$

Bidder A optimnally searches when (25) is greater than (26), i.e., when

$$c_A \leq \frac{1}{6(1-2k)}(3 - 2(k + 2q) + (q - k)4b_B^3).$$

Otherwise she passes. When $q \leq x$, then the payoff from searching is

$$\int_{c_A}^{1/2} \frac{1}{2} \frac{2}{c_A} dz - c_A \tag{27}$$

and payoff from passing is 0. As before, (27) is always greater than 0, i.e., searching is optimal.

Next suppose that $W = A$, i.e., A is the high bidder. Than $b_A \geq b_B > 0$ and the standing price equals to B's bid, i.e., $p_1 = b_B$. Then, since β is increasing and invertible on $[x, 1/2]$, we update beliefs to $c_B = (\beta)^{-1}(b_B)$. Since $W \neq B$ we have that $\tau_B = 1$, and when A searches she gets

$$\frac{2}{3} - \frac{b_B^2}{2} - \frac{b_A^3}{6} - c_A, \tag{28}$$

by using (18). Notice that when $c_B = x$ we could be dealing with off-equilibrium path history, i.e., $0 < b_B < \beta(x)$, and in that case beliefs that support this equilibrium are that $c_B \sim U[0, c_A)$, i.e. B searches, $\tau_B = 1$. The payoff that A gets is given by (28).

The payoff from passing is

$$\frac{1}{2} - \frac{b_B^2}{2}, \tag{29}$$

by using (19). Hence, when $c_A \geq 1/6(1 - b_A^3)$ passing is optimal and when $c_A < 1/6(1 - b_A^3)$ searching is optimal. $\square$

Claim 3.: When $c_A < 1/6$, then $\beta(c_A) = 0$ is A's best response.

Proof: Consider bidder A with $c_A < 1/6$ and suppose that $b_A = 0$. Then $b_A = p_1 = 0$. When $c_B \in [0, 1/6)$ we have $b_B = b_A$ and hence both bidders search, i.e., $\tau_A = \tau_B = 1$. The payoff that A gets in this case is given by (18) or (20). When $c_B \in [1/6, 1/2]$ we have $b_B > b_A$ and hence bidder B passes, i.e., $\tau_B = 0$, and bidder A searches, i.e., $\tau_A = 1$. In this case A's payoff is given by (20). Thus, the total payoff that A gets is given by (21), i.e. $\frac{5}{9} - c_A$.

Now suppose A bids $\tilde{b}_A > 0$ instead. We have to differentiate between four cases:

1. $\tilde{b}_A < \sqrt{1/3}$ and $c_A < (1 - \tilde{b}_A^3)/6$,
2. $\tilde{b}_A < \sqrt{1/3}$and $c_A > (1 - \tilde{b}_A^3)/6$,
3. $\tilde{b}_A \geq \sqrt{1/3}$ and $c_A < (1 - \tilde{b}_A^3)/6$ and
4. $\tilde{b}_A \geq \sqrt{1/3}$ and $c_A > (1 - \tilde{b}_A^3)/6$.

Case 1.: Suppose $0 < \tilde{b}_A < \sqrt{1/3}$ and $c_A < (1 - \tilde{b}_A^3)/6$. When $c_B \in [0, 1/6)$ we have $\tilde{b}_A > b_B = p_1 = 0$ and hence $W = A$. Thus, both bidders search, i.e., $\tau_A = \tau_B = 1$. The payoff A gets in this case is given by (18) where $\tau_B = 1$ and $b_B = 0$. When $c_B \in [1/6, 1/2]$ we have $b_B > \tilde{b}_A = p_1 > 0$ and hence $W \neq A$. In this case bidder A searches, i.e., $\tau_A = 1$ and bidder B passes, i.e., $\tau_B = 0$. The payoff that A gets is given by (20) where $\tau_B = 0$. Thus A's total expected payoff is

$$\int_0^{1/6} (\frac{2}{3} - \frac{\tilde{b}_A^3}{6})2dz + \int_{1/6}^{1/2} \frac{1}{2} 2dz - c_A = \frac{5}{9} - \frac{\tilde{b}_A^3}{18} - c_A. \tag{30}$$

Notice that since $\tilde{b}_A > 0$, (30) is less than $5/9 - c_A$, i.e., bidding zero gives a higher payoff.

Case 2.: Suppose $0 < \tilde{b}_A < \sqrt{1/3}$ and $c_A > (1 - \tilde{b}_A^3)/6$. All remains the same as in Case 1 only with one exception. When $c_B \in [0, 1/6)$, i.e., we have $W = A$, then the condition $c_A > (1 - \tilde{b}_A^3)/6$ implies that A passes, i.e.,

$\tau_A = 0$. Hence in this case A's payoff changes and is now given by (19), where $\tau_B = 1$. Thus, the total expected payoff is

$$\int_0^{1/6} \frac{1}{2} 2dz + \int_{1/6}^{1/2} (\frac{1}{2} - c_A) 2dz = \frac{1}{3} - \frac{c_A}{3}. \tag{31}$$

As before, (31) is less than $5/9 - c_A$ when $c_A < 1/6$, i.e., bidding zero is gives greater payoff.

Case 3.: Here, suppose that $\tilde{b}_A \geq \sqrt{1/3}$ and $c_A < (1 - \tilde{b}_A^3)/6$. When $c_A \in [0, 1/6)$ we have $\tilde{b}_A > b_B = p_1 = 0$ and hence $W = A$. In this case both bidders search, i.e., $\tau_B = 1$ and $\tau_A = 1$. Bidder A gets payoff given by (18) in which $\tau_B = 1$ and $b_B = 0$. When $c_B \in [1/6, \tilde{b}_A^2/2]$ then $\tilde{b}_A > b_B = p_1 > 0$ and hence $W = A$. Thus, A's payoff is again given by (18) in which now we have $\tau_B = 1$ and $b_B = \sqrt{2c_B}$. Finally, when $c_B \in (\tilde{b}_A^2/2, 1.2]$ we have $b_B > \tilde{b}_A = p_1 > 0$ and hence $W \neq A$. In this case A searches, i.e., $\tau_A = 1$, and B passes, i.e., $\tau_B = 0$. Thus, A's payoff is given by (20) where $\tau_B = 0$. Hence A's total expected payoff is

$$\int_0^{1/6} (\frac{2}{3} - \frac{\tilde{b}_A^3}{6}) 2dz + \int_{1/6}^{\tilde{b}_A^2/2} (\frac{2}{3} - z - \frac{\tilde{b}_A^3}{6}) 2dz + \int_{\tilde{b}_A^2/2}^{1/2} \frac{1}{2} 2dz - c_A. \tag{32}$$

It helps to rewrite (21) - this is the payoff A gets from bidding zero – as

$$\int_0^{1/6} \frac{2}{3} 2dz + \int_{1/6}^{\tilde{b}_A^2/2} \frac{1}{2} 2dz + \int_{\tilde{b}_A^2/2}^{1/2} \frac{1}{2} 2dz - c_A. \tag{33}$$

Notice that the first two terms in (33) are both greater than their counterparts in (32) since $\tilde{b}_A > 0$. And, with the last terms in both expressions being the same we have (33) greater than (32), i.e. A prefers to bid zero.

Case 4.: Suppose $\tilde{b}_A \geq \sqrt{1/3}$ and $c_A > (1 - b_A^3)/6$. This case is identical to Case 3. but with the difference that now A passes, i.e., $\tau_A = 0$, when $W = A$, i.e., when $c_B \in [0, \tilde{b}_A^2/2]$. Thus, when $c_B \in (0, 1/6)$, then A's payoff is given by (19) with $\tau_B = 1$ and $b_B = 0$; and, when $c_B \in [1/6, \tilde{b}_A^2/2]$, then A's payoff is given by (19) with $\tau_B = 1$ and $b_B = \sqrt{2c_B}$. Hence, the total expected payoff changes to

$$\int_0^{1/6} \frac{1}{2} 2dz + \int_{1/6}^{\tilde{b}_A^2/2} (\frac{1}{2} - z) 2dz + \int_{\tilde{b}_A^2/2}^{1/2} (\frac{1}{2} - c_A) 2dz. \tag{34}$$

As before, all terms in (33) are at least as big as those in (34), i.e., bidding zero gives greater payoff.

Thus, $b_A = 0$ is A's best response $\square$

Claim 4.: When $c_A \geq 1/6$, then $\beta^1(c_A) = \sqrt{2c_A}$ is a A's best response.

Proof: Suppose $c_A \geq 1/6$ and $b_A = \sqrt{2c_A}$. When $c_B \in [0, 1/6)$ we have $b_B = 0$ and hence $W = A$. In the middle stage A passes, i.e., $\tau_A = 0$ and B searches, i.e., $\tau_B = 1$. The payoff that A gets in this case is given by (19) in which $b_B = 0$ and $\tau_B = 1$. When $c_B \in [1/6, c_A)$ then $b_A \geq b_B = \sqrt{2c_B}$ and $W = A$.. In the middle stage A passes, i.e., $\tau_A = 0$ and B searches, i.e., $\tau_B = 1$. To get A's payoff, we plug $b_B = \sqrt{2c_B}$ and $\tau_B = 1$ in expression (19). The last case occurs when $c_B \in [c_A, 1/2]$. Then, $b_A \leq b_B = \sqrt{2c_B}$ and we have $W \neq A$. Hence, in the middle stage, A searches, i.e., $\tau_A = 1$ and B passes, i.e., $\tau_B = 0$. Now A's payoff equals to (20) in which $\tau_B = 0$. The total expected payoff that A gets form bidding $b_A = \sqrt{2c_A}$ is

$$\int_0^{1/6} \frac{1}{2} 2dz + \int_{1/6}^{c_A} (\frac{1}{2} - z) 2dz + \int_{c_A}^{1/2} (\frac{1}{2} - c_A) 2dz. \tag{35}$$

To establish that $b_A = \sqrt{2c_A}$ is a best response suppose that $\tilde{b}_A \neq b_A$ and consider three cases:

1. $\tilde{b}_A \geq \sqrt{1/3}$,
2. $0 < \tilde{b}_A < \sqrt{1/3}$, and
3. $\tilde{b}_A = 0$.

Case 1. $\tilde{b}_a \geq \sqrt{1/3}$.

In this case $\tilde{b}_A \geq b_B$ occurs when $c_B \in [0, \tilde{b}_A^2/2]$ in which case we have $W = A$. The behavior of both bidders corresponds to that which we described above. Further notice that $\tilde{b}_A^2/2 > 1/6$. Hence, when $c_B \in [0, 1/6]$, then the payoff that bidder A gets is given by (19) with $b_B = 0$ and $\tau_B = 1$. When $c_B \in [1/6, \tilde{b}_A^2/2]$, then A's payoff is given by (19) with $b_B = \sqrt{2c_B}$ and $\tau_B = 1$. The last case is when $c_B \in (\tilde{b}_A^2/2, 1/2]$. Then, $W \neq A$ and bidder A gets payoff given by (20) with $\tau_B = 0$. Thus, A's total expected payoff is

$$\int_0^{1/6} \frac{1}{2} 2dz + \int_{1/6}^{\tilde{b}_A^2/2} (\frac{1}{2} - z) 2dz + \int_{\tilde{b}_A^2/2}^{1/2} (\frac{1}{2} - c_A) 2dz. \tag{36}$$

Taking the difference between (35) and (36) we get

$$\int_{c_i}^{\tilde{b}_A^2/2} (\frac{1}{2} - c_A) 2dz - \int_{c_i}^{\tilde{b}_A^2/2} (\frac{1}{2} - z) 2dz > 0,$$

i.e., bidding b_A gives higher payoff.

Case 2.: $0 < \tilde{b}_A < \sqrt{1/3}$.

In this case $\tilde{b}_A > b_B$ when $c_B \in [0, 1/6)$ and $\tilde{b}_A < b_B$ when $c_B \in [1/6, 1/2]$. Thus, when $c_B \in [0, 1/6)$ we have $W = A$. Then bidder A's payoff is given by (19) in which $b_B = 0$ and $\tau_B = 1$. When $c_B \in [1/6, 1/2]$ we have $W \neq A$ and A's payoff is given by (19) with $\tau_B = 0$. Thus, the total expected payoff from bidding $\tilde{b}_A$ is

$$\int_0^{1/6} \frac{1}{2} 2dz + \int_{1/6}^{1/2} (\frac{1}{2} - c_A) 2dz, \tag{37}$$

which is clearly less than (35).

Case 3.: $\tilde{b}_A = 0$.

In this case we have $\tilde{b}_A = p_1 = 0$. Thus, when $c_B \in [0, 1/6)$, then $b_B = \tilde{b}_A$ and hence bidder B searches, i.e., $\tau_B = 1$. Bidder A searches only when $W \neq A$ and $c_A \in [1/6, 3/10)$. Otherwise, she passes. Therefore, when $c_A \geq 3/10$ then A always passes, i.e., $\tau_B = 0$. The payoff that A gets in that case is

$$\int_0^{1/6} \frac{1}{2} 2dz$$

which is less than (35). On the other hand, when $c_A \in [1/6, 3/10)$, then A searches, i.e., $\tau_A = 1$. Notice that when $c_B \in [0, 1/6]$, then bids are tied and hence $W = \oslash$. Then, A's total expected payoff is

$$\int_0^{1/6} (\frac{2}{3}) 2dz + \int_{1/6}^{1/2} (\frac{1}{2}) 2dz - c_A,$$

which is less than (35) since $c_A \geq 1/6$.

Thus, $\beta^1(c_A) = \sqrt{2c_A}$ is A's best response since any deviation gives lower payoff. $\square$