Proportional Pie-Cutting

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Abstract

Cutting a pie into wedge-shaped pieces with radial cuts is, surprisingly, quite different from cutting a cake with parallel cuts. If two players have unequal entitlements to the pie, a minimal number of cuts can always be used to divide the pie into proportional pieces that reflect these entitlements, whereas this is not always possible for a cake. Two procedures are given that induce the players truthfully to reveal their preferences for different portions of the pie such that they receive pieces that are at least equal to their entitlements and, consequently, do not envy the other player for getting a disproportionally valuable piece. Under the more information-demanding procedure, the allocation is also efficient. If there are three of more players, it is not always possible to make proportional, envy-free allocations.

1 Introduction

It would seem that dividing a pie is not much different from dividing a cake. If we represent a pie by a circle, cutting it at any point and "straightening out" the circle gives a line segment, which can be used to represent a cake. So isn't a cake just a pie that has been cut?

In fact, they are quite different objects, and the division of each may give very different results. We may think of a cake as a rectangle valued along its horizontal axis, and a pie as a disc valued along its circumference. We will use vertical, parallel cuts to divide a cake into pieces, and radial cuts from the center to divide a pie into wedge-shaped pieces. A pie-cutting procedure can be used to divide a shoreline on a lake into three connected pieces, whereas a cake-cutting procedure produces disconnected pieces of shoreline (Figure 1). Note that the minimal number of cuts necessary to divide a cake among n people is $n - 1$, one less than the minimal number n necessary to divide a pie.

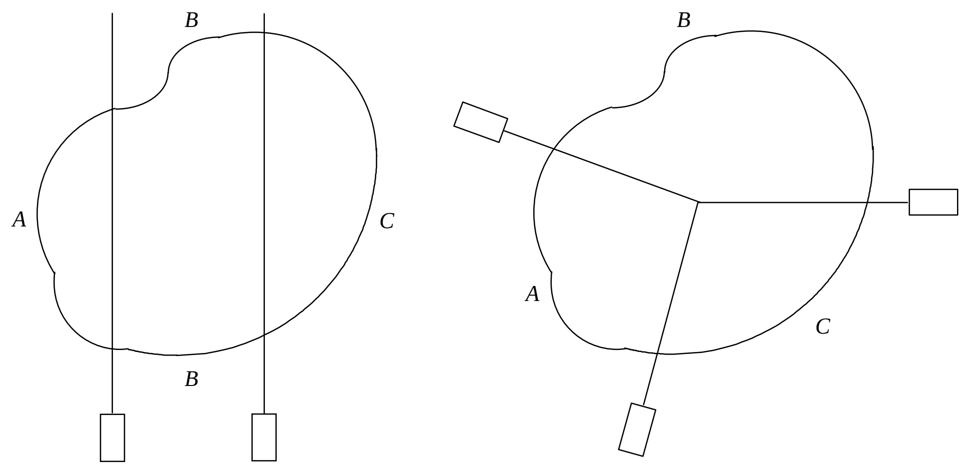


Figure 1: Using a cake-cutting procedure to divide lake-front property among players A, B, and C results in a disconnected shoreline for B (left), whereas each player gets connected pieces using a pie-cutting procedure (right).

More than a decade ago, Gale [5] posed the following question: If there are n players, does there always exist an *envy-free* allocation of a pie (no player desires another player's wedge-shaped piece) that is *undominated* (no other allocation is better for at least one player and not worse for the others) using the minimal number of n cuts? When there are just two players that have equal claims or entitlements to the pie, not only is the answer to Gale's question "yes," but there is also a procedure that produces an envy-free, undominated allocation [2].

But what if they don't have equal entitlements – say, one player is entitled to twice as much as the other player, and the pie must therefore be divided into unequal proportions. We show that the answer is still "yes" for two players if we extend the notion of envy-freeness to the case of unequal entitlements. However, the answer is "no" if there are three or more players, which we demonstrate with two 3-player examples. Thus, proportional pie-cutting by three or more players is not always possible.

On the other hand, there are envy-free procedures for cutting a cake among three players, using the minimal number of 2 cuts, that yield an undominated (also called *efficient*) allocation [1, 8]. Envy-free cake-cutting procedures for four players exist, but they do not use the minimal number of 3 cuts. For overviews of the cake-cutting literature, see [2, 4, 7, 9].

Because most disputes are between two parties that have equal claims to a disputed item or items (*e.g.*, spouses in a divorce), dividing objects when the parties have unequal claims has received relatively little attention in the cake-cutting literature (an exception is [7, pp. 35-48]). But there is a more important reason for this neglect: It is not always possible to use the minimal number of cuts to divide a cake between players if their entitlements are unequal, which we will demonstrate with a 2-player example. By contrast, we will prove that there always exists a division of a pie using the minimal number of 2 cuts that reflects the unequal entitlements of the two players. More precisely, if the two players are entitled to p and $1-p$ proportions of a pie, such that $0 < p < 1$, then we show that there exists an envy-free and efficient allocation in which the players' pieces are in the ratio $p : 1-p$, as each values the pieces (players' valuations are assumed to be subjective).

Furthermore, we provide two procedures that induce risk-averse players to reveal their preferences truthfully. The first procedure requires players to divide the pie into equally valued pieces and gives the players pieces that are exactly equal to their entitled shares, according to their preferences, but it may leave a surplus. The second procedure, which leaves no surplus, requires the players to submit their valuations as probability measures. This additional information improves the outcome by giving an envy-free and efficient allocation of the entire pie in the ratio $p : 1-p$ of the players' entitlements, according to their respective measures. If the players' preferences are not absolutely continuous, however, then this allocation may be efficient only with respect to the ratio. That is, there may be a different allocation that violates the ratio but which gives one player more, as we will illustrate later.

2 Unequal Entitlements between Two Players: Existence Results

Assume that players' preferences for pie are defined by finitely additive, nonatomic probability measures over the unit disk. Finite additivity ensures that the value of a finite number of disjoint pieces is equal to the value of their union. It follows that no subpieces have greater value than the larger pieces that contain them. The measure is nonatomic, so a single radial cut, which defines one border of a piece, has no area and so contains no value. We also require the measures of the players to be absolutely continuous, so that there is no portion of the pie with measure zero for one player and positive measure for the other.

We assume that the value placed on an entire pie by each player is 1. For n players, entitlements to the pie are defined by $(p_1, \ldots, p_n)$ such that $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$. An allocation is *proportional* if player i receives a piece valued at v_i according to its measure, and the proportions reflecting these values $v_1 : v_2 : \cdots : v_n$ are equal to the proportions $p_1 : p_2 : \cdots : p_n$ reflecting the entitlements.

An allocation is *envy-free* if player i, who is entitled to p_i and values its piece at v_i, does not think that another player, entitled to p_j, received more than v_j in player i's measure. For equal entitlements in which $p_i = \frac{1}{n}$ for $i = 1, \ldots, n$, our definition of envy-freeness can be interpreted to mean that no player desires another player's piece. For unequal entitlements, it means that no player thinks another player got a disproportionally large piece, based on the latter player's entitlement. That is, no player would envy another player if it had its entitlement.

For cake, these probability measures satisfy the same properties as they do for pie, but we define them over the unit square instead of the unit disk. The following example shows that proportional allocations that guarantee players their unequal shares of cake may not be possible using a minimal number of cuts.

Example 1. (Dividing cake between two players with unequal entitlements) We assume a cake given by the unit square is cut perpendicularly to the x-axis, and players' measures are over $[0, 1]$ along this axis. Let player A's measure be given by the uniform distribution (dotted line in Figure 2), and player B's measure by

$$f_B(x) = \begin{cases} 4x & \text{for } x \leq \frac{1}{2} \\ 4 - 4x & \text{for } x > \frac{1}{2}, \end{cases}$$

which is a triangular distribution (dashed lines in Figure 2).

Assume that players A and B are entitled to unequal portions of the cake given by p and $1-p$, respectively, where $\frac{1}{2} < p < 1$. Due to the symmetry of the probability measures, it is sufficient to

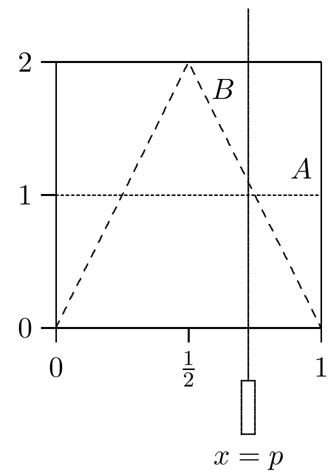


Figure 2: Cutting the cake at $x = p$ for the measures of players A and B in Example 1.

consider the case where player A cuts the cake at $x = p$, giving it $[0, p]$ that it values at p. Player B gets the remainder $(p, 1]$, which it values at $\int_p^1 (4 - 4x)dx = 2(1-p)^2$. Because $2(1-p)^2 < 1 - p$ when $p > \frac{1}{2}$, player B receives less than its entitled share, $1 - p$.

A proportional allocation is possible by solving for the cut-point x in $x : 2(1-x)^2 = p : 1 - p$. However, in this case neither player receives a piece that it values as much as its entitlement. Thus, the players cannot both get their entitled shares. As we will show, player B must get a piece in the middle of the cake, but one cut precludes this division.

Using radial cuts, it is possible to divide a pie between two players wherein each receives a wedge-shaped piece that it values as equal to at least its entitled share, according to its measure. Our first theorem assumes the players have rational entitlements.

Theorem 1. *For integers k and $n > 0$ such that $0 \leq k \leq n$, there exist two wedge-shaped pieces such that A values one piece at $\frac{k}{n}$, and B values the other piece at $\frac{n-k}{n}$ or more, according to their respective measures.*

Proof. Divide the pie into n sectors by marking n angles $\alpha_0 = 0, \alpha_1, \ldots, \alpha_{n-1}$ in $[0, 2\pi)$ with $\alpha_i < \alpha_{i+1}$ such that the value of each sector (the pie between two consecutive angles) is $\frac{1}{n}$ according to player A's measure. Let sector i be the wedge-shaped piece between angles α_i and $\alpha_{(i+1) \bmod n}$. Define piece i to be the k *consecutive* counterclockwise sectors, beginning with and including sector

i. Equivalently, for $i = 0$ to $n-1$, piece i is the wedge-shaped piece defined by the counterclockwise inclusion of pie between the radii α_i and $\alpha_{(i+k) \mod n}$. As defined, A values piece i at $\frac{k}{n}$ for all i.

Let player B's value of the complement of piece i be denoted by v_i. Because the complements of all the pieces i cover the pie $n-k$ times, then $\sum_{i=0}^{n-1} v_i = n-k$. Hence, the *average* value of a complementary piece is $\frac{n-k}{n}$. This implies that there exists at least one i such that $v_i \geq \frac{n-k}{n}$. $\square$

Although rational entitlements could be used to approximate irrational entitlements, we show the existence of proportional allocations in which the players receive their irrational entitlements by using the density of the rational numbers in the real numbers. For cake-cutting, Robertson and Webb [7] consider unequal and irrational entitlements.

Theorem 2. *For any $p \in [0,1]$, there exist two wedge-shaped pieces such that A values one piece at p and B values the other at $1-p$, according to their respective measures.*

Proof. Because existence for rational numbers p holds from Theorem 1, assume that p is irrational. If there exists a wedge-shaped piece that player A values at p and player B values the complementary piece at $1-p$ or more, then the complementary piece can be trimmed so that B receives exactly $1-p$. Hence, for every wedge-shaped piece that player A values at p, assume that player B values the complement at less than $1-p$. For every angle $\theta \in [0, 2\pi)$, let piece θ begin at θ and counterclockwise sweep out a wedge-shaped piece valued at p according to player A's measure. Let v_θ be B's value of the complementary piece. Absolute continuity of the measures ensures that v_θ is a continuous function over $[0, 2\pi]$. Hence, v_θ achieves a maximum v^* that, by assumption, is strictly less than $1-p$.

Because the rational numbers are dense in the real numbers, there exists a rational number q such that $v^* < 1-q < 1-p$. By Theorem 1, there exist two wedge-shaped pieces in which A receives q and B receives $1-q$ according to their measures. Because $p < q$, A's piece can be trimmed to be of value p. But B receives $1-q$, which is strictly greater than v^*, yielding a contradiction. Hence, B cannot receive a piece that is strictly less than $1-p$. $\square$

In general, allocations that give the players pieces valued at exactly their entitlements will generate surpluses (white space in Figure 3). If A receives a piece valued at p according to its measure but B values the complement at more than $1 - p$, then it can be trimmed to give B a piece valued at $1 - p$. This can be done as in Figure 3 so that the surplus is a wedge-shaped piece. Then this surplus can be divided by cutting it at some angle c_θ (between α and β in Figure 3) that mirrors the entitlements of the players; adding these portions of the surplus to the players' pieces gives them larger pieces, but still in the proportion of their entitlements. Thereby the ratio of the entitlements is preserved, but the players value their pieces at more than their entitlements.

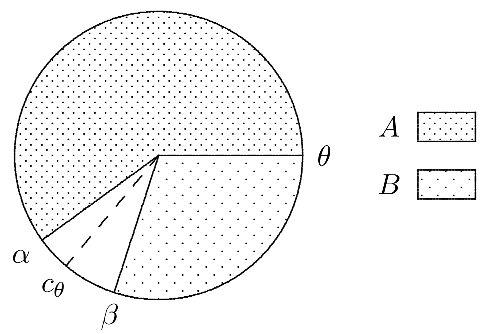


Figure 3: According to their respective measures, player A receives a piece valued at p while player B receives a piece valued at $1 - p$. The surplus (white space) can be divided in proportion to the players' entitlements by determining the angle c_θ between α and β.

For every angle θ, there exists a c_θ such that if player A receives the piece counterclockwise between θ and c_θ and player B receives the complement, then the pieces to A and B are in the ratio $p : 1 - p$. For some θ, however, the ratio $p : 1 - p$ does not guarantee that the players receive pieces that they value as much as their entitlements. But from Theorem 2, there exists a θ that ensures that the players can receive pieces equal to their entitlements; moreover, if there exists a surplus as illustrated in Figure 3, there will be pieces in the ratio $p : 1 - p$ that are valued at more than the players' entitlements. By considering all possible angles θ so that the pieces to A and B are in the ratio of the entitlements $p : 1 - p$, we can find the allocation that maximizes the value of the pieces to the players. This results in an envy-free and efficient allocation whereby each player receives at least its entitled share according to its measure.

To guarantee an efficient allocation such that the value of the players' pieces is in the ratio $p : 1 - p$ requires that the players' measures be absolutely continuous. Measures on a pie are *absolutely continuous* with respect to each other if, whenever a piece of pie has positive measure to one player, it has positive measure to all players. Without absolute continuity, an initial allocation could be modified by giving more pie to one player, thereby increasing the value that that player receives, without decreasing the value of the piece that the other player receives. The following example demonstrates the necessity of absolute continuity.

Example 2. (If preferences are not absolutely continuous, there may not exist a proportional, efficient allocation.)

Assume that players A and B are equally entitled to a pie. Let player A's measure be uniformly distributed over the unit disk. Assume that player B's measure in polar coordinates is given by

$$f_B(r,\theta) = \begin{cases} \frac{4}{\pi} & \text{for } \theta \in \left[\frac{3\pi}{8}, \frac{5\pi}{8}\right] \text{ and } \theta \in \left[\frac{11\pi}{8}, \frac{13\pi}{8}\right] \\ 0 & \text{otherwise,} \end{cases}$$

so player B places equal value on each of two sectors, centered around 12 o'clock and 6 o'clock, and this value is uniformly distributed over each sector (shaded wedges in Figure 4).

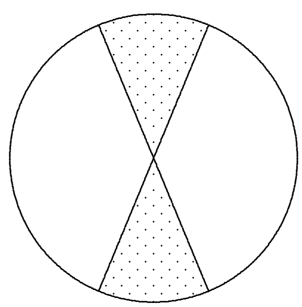


Figure 4: Player B's value is uniformly distributed over the sectors between angles $\frac{3\pi}{8}$ and $\frac{5\pi}{8}$ and between angles $\frac{11\pi}{8}$ and $\frac{13\pi}{8}$.

Any diameter that falls within the two sectors equally divides the pie for both players. However, such an allocation, in which the players receive values in the ratio $\frac{1}{2} : \frac{1}{2}$, is inefficient. This follows from the fact that there is an allocation in which player B receives $\frac{1}{2}$ and player A receives more than $\frac{1}{2}$: Give player B the top sector (the pie between angles $\frac{3\pi}{8}$ and $\frac{5\pi}{8}$, centered around 12 o'clock) that it values at $\frac{1}{2}$, and give Player A the remainder, which it values at $\frac{7}{8}$. Although this efficient allocation is envy-free, it violates the players' equal entitlements.

Theorem 3. *For any $p \in [0, 1]$, there exists an envy-free and efficient allocation using 2 cuts such that each player receives at least its entitled share, according to its measure, and the proportion of the players' pieces is $p : 1 - p$, according to their measures, if these measures are absolutely continuous.*

Proof. For every $\theta \in [0, 2\pi]$, consider the angle $c_\theta \in [0, 2\pi]$ (see Figure 3) such that if the pie were cut at θ and c_θ, then the ratio of A's value of the piece defined counterclockwise between θ and c_θ to B's value of the complement is $p : 1 - p$. Because the players' measures are nonatomic and absolutely continuous, player A's value of the piece between θ and c_θ is a continuous function that achieves a maximum for some $\theta^* \in [0, 2\pi]$, which defines c_{θ^*}. This also maximizes player B's value for the piece it receives. Cut the cake at θ^* and c_{θ^*}, giving A the piece counterclockwise between θ^* and c_{θ^*} and B the complement.

Because Theorems 1 and 2 show the existence of an allocation whereby each player receives at least its entitlement, it follows that the above maximization gives A a piece that it values at least as much as p, and B a piece that it values at least as much as $1 - p$. The allocation is envy-free, because each player believes the other player has received at most its entitled piece. The allocation is efficient with respect to other 2-cut allocations, because the continuous value function was maximized at θ^*. □

When a pie is cut with two radial cuts, the first cut can be viewed as selecting the endpoints of the interval defining a cake, whereas the second cut divides the interval into two pieces. We return to Example 1 to illustrate how cutting a pie with 2 cuts can yield an envy-free and efficient allocation, whereas cutting a cake with 1 cut may not.

Example 3. (The cake-cutting example with two cuts: A continuation of Example 1) Because player B values the cake near $\frac{1}{2}$ more than near 0 and 1, player B needs to receive the middle piece of cake for its division to be envy-free and efficient. Due to symmetry, this division occurs by cutting the cake at x and $1 - x$. Player A values the sum of the left and right pieces at $2x$. Player B values the middle piece at $1 - 4x^2$. Proportionality requires cutting the cake at x and

$1-x$ such that x satisfies $2x : 2(1-4x^2) = p : 1-p$. For players to receive pieces in ratio $p : 1-p$ for $p > \frac{1}{2}$, then $x = \frac{(p-1)+\sqrt{5p^2-2p+1}}{4p}$. Player A gets two pieces that it values more than p because

$$2x = \frac{p-1+\sqrt{5p^2-2p+1}}{2p} > p \iff \sqrt{5p^2-2p+1} > 2p^2 - p + 1$$

is equivalent to

$$5p^2 - 2p + 1 > (2p^2 - p + 1)^2 = 4p^4 - 4p^3 + 5p^2 - 2p + 1 \text{ or } 0 > 4p^4 - 4p^3 = 4p^3(p-1).$$

Because the values of the pieces received by the players, according to their respective measures, are proportional to and greater than their entitlements, it follows that the allocation is envy-free. Figure 5 demonstrates how the cake can be viewed as a pie: Connecting the endpoints of the line segment (cake), 0 and 1, to form a circle (pie), we can cut the pie at x and $1-x$ to obtain an envy-free and efficient allocation.

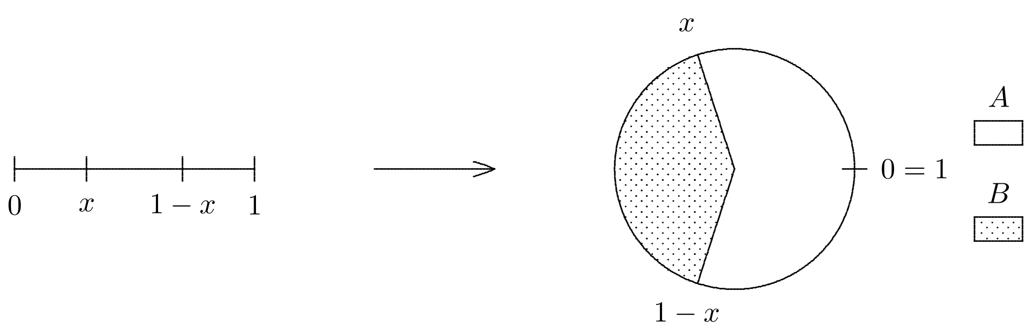


Figure 5: By identifying the endpoints of the unit interval, the cake in Example 3 can be viewed as a pie. Player B receives the cake between x and $1-x$, which forms a wedge-shaped piece of pie.

3 Proportional Pie-Cutting Procedures for Two Players

The existence of an allocation with specific properties does not imply a procedure whereby the players would truthfully reveal their preferences to arrive at the allocation. For example, Jones [6] shows the existence of a one-cut allocation of cake for two players such that the allocation is envy-free, equitable (both players receive pieces that they value the same according to their measures), and efficient (with respect to single-cut allocations). But Brams *et al.* [3] prove that there does not exist a procedure to induce the truthful revelation of preferences that would lead to this allocation, although there is a procedure that can approximate this allocation.

When a procedure is used, we assume that players try to maximize the minimum-value pieces (*maximin pieces*) that they can guarantee for themselves, regardless of what the other players do. In this sense, the players are risk-averse and never strategically announce false measures if it does not guarantee them more-valued pieces. Further, we assume that the players do not know each other's preferences. This uncertainty, coupled with risk-aversion, induces the players to reveal their preferences, or measures, truthfully under the procedures.

We next present a moving-knife procedure that guarantees an allocation wherein each player receives a piece exactly equal to its rational-number entitlement. We show that risk-averse players will truthfully divide the pie into n equally valued sectors to maximize the minimum value of the pieces they receive. The main drawback of the procedure is that the allocation may be inefficient, because there is almost always a surplus that goes unallocated to the players. Even though the procedure may leave no surplus, it does not guarantee an efficient outcome, as the proof of Theorem 3 demonstrates.

Proportional Pie-Cutting Procedure for Rational-Number Entitlements

Suppose that player A is entitled to $\frac{k}{n}$ of the pie and player B is entitled to $\frac{n-k}{n}$ of the pie.

1. Select a point on the circumference of the pie at random. Denote the radius from the center of the pie to this point as 0 radians. Let this angle be angle 0.

2. Player A, unobserved by player B, marks $n-1$ additional angles that, together with angle 0, divide the pie into n sectors (dashed lines in Figure 6*a*).

3. Player B places one knife along the radius at angle 0 and places $n-1$ knives from the center of the pie to the circumference at $n-1$ angles that, together with angle 0, divide the pie into n sectors (solid lines in Figure 6*a*).

4. Player B rotates the n knives counterclockwise in such a way that the knives continue to define n sectors.

5. Player A stops the rotation when one of player B's knives is coincident with one of player A's n angles (k_4 and 4 in Figure 6*b*), and there are k consecutive sectors in the counterclockwise direction from this knife, according to A's angles, that do not intersect $n-k$ consecutive sectors

in the clockwise direction from this knife, according to B's knives.

6. Player A reveals its angles. The pie is cut in three places: the two radii defining the boundary of A's k consecutive sectors; and at the knife that, together with the knife coincident with A's angle, forms the boundary of the $n-k$ consecutive sectors, according to B's knives (see Figure 6*b*).

7. If A does not call stop before player B's knives traverse one sector – that is, when the knife at angle 0 reaches the position of the first knife counterclockwise from angle 0 (knife k_1 in Figure 6) – then neither player receives any of the pie.

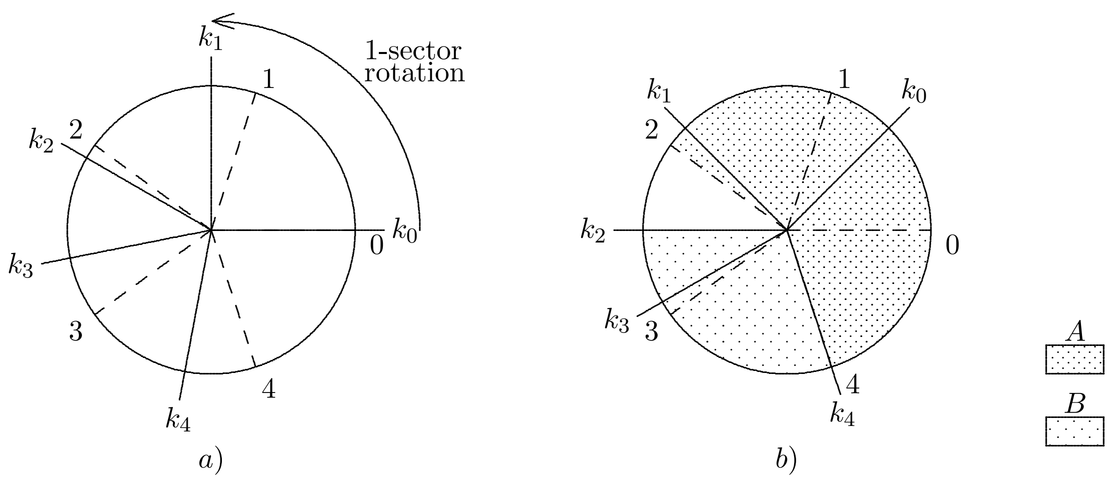


Figure 6: Players A and B use the proportional pie-cutting procedure for rational-number entitlements to divide the pie in a ratio of 3 : 2. *a*) Player A's five marks divide the pie into five sectors, as do player B's five knives. *b*) Player B rotates its knives until player A stops the rotation when knife k_4 coincides with A's angle 4. Player A receives the 3 consecutive sectors counterclockwise from knife k_4, and player B receives the pie between knives k_2 and k_4.

Theorem 4. *Under the proportional pie-cutting procedure for rational entitlements, a risk-averse player A will truthfully submit $n-1$ angles that, with angle 0, divide the pie into n equally valued sectors. A risk-averse player B will rotate its knives so as to keep the value between them equal. The players will receive pieces exactly equal to their entitlements, according to their respective measures.*

Proof. Because neither player A nor player B is aware of which consecutive sectors it will receive, for A and B to ensure that they receive k and $n-k$ sectors, respectively, A will submit $n-1$ angles to create, with angle 0, n equally valued sectors, according to its measure, and B will rotate its n knives to keep the value of pie between the knives equal.

It remains to show that the procedure terminates with the players receiving pieces valued at their entitlements when they follow their risk-averse strategies. Beginning with angle 0, number player A's n angles 0 to $n-1$ in a counterclockwise direction. Beginning with the knife k_0 at angle 0, if player B's knives are numbered 0 through $n-1$ in a counterclockwise direction, then a one-sector counterclockwise rotation moves knife k_i to the initial position of knive k_{i+1} (except knife k_{n-1} ends at angle 0), as shown in Figure 6a. One knife traverses each sector, so the entire pie is covered. Therefore, as player B rotates its n knives one sector, each of player A's angles will coincide at some point with one of player B's knives. We will show that at some angle j, there will be k consecutive sectors counterclockwise from angle j, according to player A's angles, that do not intersect with the $n-k$ consecutive sectors clockwise from angle j, according to player B's knives.

Define piece i to be the k consecutive sectors counterclockwise from angle i for $i = 0$ to $n-1$, according to player A's angles. Let v_i be the value of the complementary piece, according to player B's measure. As in the proof of Theorem 1, the complementary pieces for $i = 0$ to $n-1$ cover the pie $n-k$ times, and $\sum_{i=0}^{n-1} v_i = n-k$. Because the average value of a complementary piece is $\frac{n-k}{n}$, then there exists an angle j such that the complement of piece j is valued at least as much as $\frac{n-k}{n}$. It follows that when a knife is coincident with angle j, then the complement to piece j consists of $n-k$ consecutive sectors, according to player B's knives, that do not intersect piece j, as in Figure 6b). □

We next present, in four steps, a procedure that guarantees an envy-free and efficient allocation of the entire pie when the ratio of player A's entitlement to player B's entitlement is $p : 1-p$ and p may be an irrational number. Because angle 0 is chosen at random, the first procedure requires the players to know their measures in order to divide the pie into n equally valued pieces. In contrast, the second procedure requires the players to submit their measure functions; then the players or a referee must solve an optimization problem. In return for its increased informational demands and its increased computational complexity, the procedure gives an efficient allocation for any p.

Efficient Proportional Pie-Cutting Procedure

1. Each player submits its (possibly false) measure over the unit disk.

2. A referee places a first mark at $\theta = 0$ and a second at c_0 (see Figure 7; the position c_0 is chosen so as to make the ratio of A's value $v_A(0)$ of the piece, counterclockwise between 0 and c_0, to B's value $v_B(0)$ of the remainder equal to $p : 1-p$, according to the players' submitted measures.

3. A referee rotates the first mark θ counterclockwise between 0 and 2π, simultaneously moving c_θ to keep the proportion between the players' values $v_A(\theta) : v_B(\theta)$ at $p : 1-p$, according to their submitted measures.

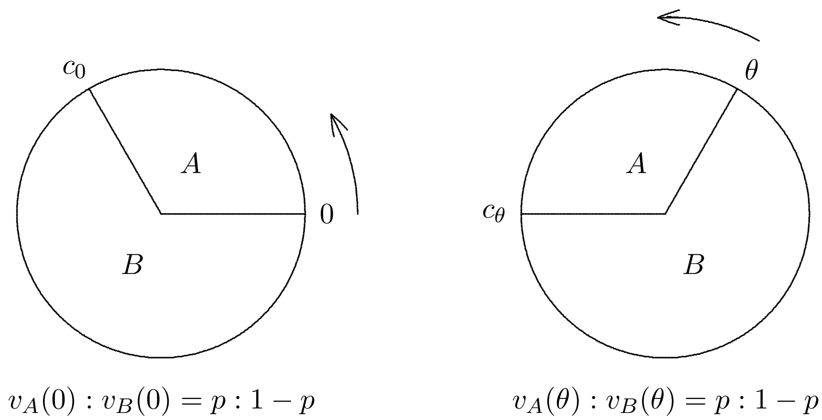


Figure 7: A referee rotates θ counterclockwise from 0 to 2π, keeping $v_A(\theta) : v_B(\theta) = p : 1-p$.

4. After a complete rotation, the referee selects an angle θ^* that maximizes value of player A's piece counterclockwise between θ^* and c_{θ^*}. This maximizes the value of B's piece, according to B's submitted measure, too.

This and the previous procedure are similar to the moving-knife procedures found in the cake-cutting literature. But the outcome under the second procedure can be determined without knives and without a referee. For example, a computer program could determine the angles θ^* and c_{θ^*} on the basis of the players' submitted measures. Robertson and Webb [7] discuss the information requirements for moving-knife and other procedures in the cake-cutting literature.

Theorem 5. *Under the efficient proportional pie-cutting procedure, risk-averse players will truthfully reveal their measures. The procedure gives an envy-free and efficient allocation such that the ratio of the values of the wedge-shaped pieces that A and B receive is $p : 1-p$, according to their submitted measures.*

Proof. By Theorem 3, there exists an allocation that gives an envy-free and efficient allocation according to the submitted measures.

Suppose that player A submits the false measure g_A. Because A does not know B's measure, it is possible that player B's measure is g_A, too. Then player A will receive (at random) one of the infinitely many pieces valued at p according to B's measure. Not all such pieces can be valued at more than p according to A's measure, so A may receive a piece valued at less than p. However, A can guarantee p by being truthful. Hence, risk-averse players will truthfully reveal their measures. □

4 Three or More Players

Surprisingly, increasing the number of players by just one – from two to three – may rule out proportional allocations for players with unequal entitlements. The following theorem is supported by two three-player examples that can be extended to more than three players. The first example is simple and intuitive, whereas the second example is more general; the latter demonstrates that proportional allocations may not exist when player entitlements (rational or irrational) are not equal.

Theorem 6. *For three or more players with unequal entitlements, there may be no proportional allocation of pie, using the minimal number of cuts, such that each player receives a piece at least equal to its entitlement.*

Example 4. (No proportional solution in which one player is entitled to more than $\frac{1}{2}$ the pie may exist.)

Assume that players A, B, and C are entitled to pie in the proportions $3:1:1$. The sectors representing the players' measures appear in Figure 8. Player A distributes $\frac{1}{10}$ uniformly on each of the ten sectors in Figure 8; players B and C distribute $\frac{1}{5}$ uniformly over each of their corresponding five sectors.

Number player A's sectors clockwise from 1 to 10, as shown in Figure 8 (we omit numbers 2-5 and 7-10 inclusive). To give A the $\frac{3}{5}$ of the pie to which it is entitled, A must receive a piece that

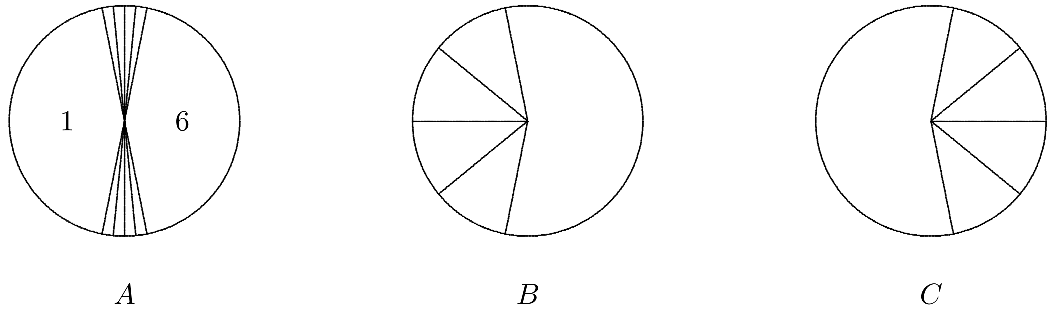


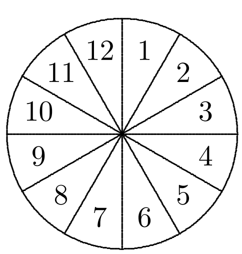
Figure 8: An allocation of pie in the proportion 3 : 1 : 1 does not exist for players A, B, and C.

contains five consecutive sectors and a portion of at least one other sector. Any five consecutive sectors must contain sector 1 or sector 6. Because player B values A's sector 1 at $\frac{4}{5}$ and player C values A's sector 6 at $\frac{4}{5}$, at least one of players B or C cannot receive a piece that it values at $\frac{1}{5}$.

Lest one think that one player must be entitled to more than $\frac{1}{2}$ to preclude a proportional solution in the three-player case, consider the following example.

Example 5. (No proportional solution in which no player is entitled to more than $\frac{1}{2}$ the pie may exist.)

Players A and B are each entitled to $\frac{1}{3}+x$, where $0 < x < \frac{1}{24}$, and player C is entitled to $\frac{1}{3}-2x$. Assume that the pie consists of twelve sectors, each player has a uniform distribution over each sector, and the sectors are valued as shown in Figure 9. Player C has a uniform distribution over the whole pie.



	1	2	3	4	5	6	7	8	9	10	11	12
A	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$
B	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{x}{3}$	$\frac{1}{3}-x$	$\frac{x}{3}$
C	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

Figure 9: An allocation of pie in the proportion $\left(\frac{1}{3}+x\right) : \left(\frac{1}{3}+x\right) : \left(\frac{1}{3}-2x\right)$ does not exist for players A, B, and C.

For any division of the pie into equal thirds, not necessarily restricted to consist of the whole sectors shown in Figure 9, players A and B value each one-third at $\frac{1}{3}$. Hence, to receive pieces valued at their entitlements, players A and B must receive pieces that represent more than two-thirds of the pie. Player C must receive more than one-quarter of the pie because $\frac{1}{3} - 2x > \frac{1}{4}$ when $x < \frac{1}{24}$.

Assume that A and B have each received $\frac{1}{3}$ of the pie and C has received $\frac{1}{4}$, leaving $\frac{1}{12}$ to be shared among A, B, and C. Now A and B each must get x of the pie, and C must get $\frac{1}{12} - 2x$, according to their respective measures. Because sectors valued at $\frac{1}{3} - x$ by player A and B are always between sectors valued at $\frac{x}{3}$ by both players, there is no one-twelfth of the pie that is valued at more than x by both players A and B; at least one of these players must receive a piece valued at no more than $\frac{x}{3}$. This guarantees that no proportional allocation exists whereby each player receives a piece at least equal to its entitlement.

Just as we showed that proportional *cake* division breaks down if there are two or more players (Example 1), proportional *pie* division breaks down if there are three or more players. Still, it is pleasing that for two players, not only can a pie always be divided between them so that each, by getting at least a proportional share, is envy-free, but there is also a procedure that induces risk-averse players to be truthful, allowing such a fair division actually to be implemented.

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