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Essays on Macroeconomic Models with Imperfect Information

Volha Audzei

Dissertation

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Abstract

This dissertation analyzes how relaxing the assumption of rational expectations modifies the output of macroeconomic models. In particular we show how imperfect information among the financial agents modifies their risk-taking decisions, the effect of monetary policy on bank lending or equilibrium selection.

In the first paper we incorporate a model of the interbank market into a standard DSGE model, with the interbank market rate and the volume of lending depending on market confidence and the perception of counterparty risk. As a result, a credit crunch occurs if the perception of counterparty risk increases. Changes in market confidence then can generate credit crunches and contribute to the depth of recessions. We conduct an exercise to mimic certain central bank policies: targeted and untargeted liquidity provision, and reduction of the reserve rate. Our results indicate that policy actions have a limited effect on the supply of credit if they fail to influence agents' expectations. A policy of a low reserve rate worsens recessions due to its negative impact on banks' revenues. Liquidity provision stimulates credit slightly, but its efficiency is undermined by liquidity hoarding.

The second paper is devoted to the problem of excessive risk-taking by financial agents. Recent central bank policies have stimulated a debate as to whether these policies contribute to the building up of another credit boom. In this paper we build a theoretical model which captures excessive risk-taking in the form of an increased risk appetite and decreased incentives to acquire information. As a result, with market risk being reduced, agents tend to acquire more risk in their portfolios than they would with the higher market risk. The same forces increase portfolio risk when the safe interest rate is falling.

In the third paper, together with Sergey Slobodyan, we study if initially mis-specified equilibrium (the Restricted Perceptions Equilibrium, or RPE) is compatible with the equilibrium choice of sparse weights, developed recently by Gabaix, 2014. We find that the agents stick to their initial mis-specified AR(1) forecasting model choice if the feedback from expectation in the model is strong or an included variable becomes more persistent. We also identify a region in the parameter space where the agents find it advantageous to pay attention to no variable at all.

Abstrakt

V této disertaci je zkoumáno, jak uputní od pedpokladu racionálních oekávání ovlivní výstupy makroekonomických model. Konkrétn ukazujeme, jak nedokonalé informace ekonomických agent na finanních trzích psobí na jejich rozhodovací proces za pítomnosti rizika, dále pak na efekt monetární politiky na banky, pjky a rovnováný výbr.

V svém prvním paperu zaleujeme mezibankovní trh do standardního DSGE modelu, piem mezibankovní úroková míra a objem pjek závisí na dve na trhu a na riziku protistrany. Toto ve výsledku znamená, e pokud na trhu panuje pedstava, e dolo ke zvýení míry rizika protistrany, pak dochází k zamrznutí úvrových trh. Zmny v dve v trhy pak mohou zapíinit zamrznutí úvrových trh, a tudí prohloubit recesi. Ve své práci teoreticky napodobujeme nkteré politiky centrálních bank. Konkrétn napodobujeme cílené a necílené poskytnutí likvidity a snížení úrokové míry z rezerv. Nae výsledky ukazují, e pokud centrální banka svými zákroky nijak nezmní oekávání ekonomických agent, pak mají opatení centrální banky pouze omezený vliv na nabídku úvr. Sníení úrokové míry z rezerv prohlubuje recesi, protoe toto opatení sniuje výnosy bank. Poskytnutí dodatené likvidity vede k mírnému zvýení úvrové aktivity bank, ale zárove toto opatení sniuje efektivitu, protoe vede k hromadní likvidity.

Ve druhém paperu se zamujeme na problematiku nadmrného riskování ekonomických agent na finanních trzích. Nedávná opatení centrálních bank rozpoutala debatu o tom, zda tato opatení nepispívají k dalímu rozmachu úvr. Ve svém druhém paperu konstruujeme teoretický model, který zachycuje nadmrné riskování ve form zvýené touhy riskovat a ve form nií poptávky po informacích. Výsledky ukazují, e pi sníení trního rizika jsou ekonomití agenti více náchylní k tomu mít ve svých portfoliích více rizikových aktiv ne v pípad, kdy je trní riziko vysoké. Stejné síly zvyují riziko portfolia pi prudkých sníeních úrokových mr.

Ve tetím lánku zkoumáme spolu se Sergeyem Slobodyanem, zda je poátení chybn specifikovaná rovnováha (zvaná Restricted Perception Equilibrium neboli RPE) kompatibilní s rovnovánou volbou rozptýlených vah, které byly poprvé zavedeny Gabaixem, 2014. Zjiujeme, e pokud je zptná vazba oekávání silná nebo pokud jsou promnné zahrnuté v modelu perzistentní, pak se agenti drí svých pvodních modelových rozhodnutí, která uinili na základ pedpovdí zakládajících se na AR(1) procesu. Dále také nacházíme v parametrovém prostoru oblast, v ní agenti povaují za výhodné nevyhledávat informace o ádné promnné.

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All errors remaining in the text are my responsibility.

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Introduction

This dissertation studies how imperfect information affects the predictions of macroeconomic models. The expectations of financial agents affect the functioning of financial markets and can propagate shocks to the real economy or become a source of shocks themselves. These expectations are not necessarily perfect. Agents can have limited information or a limited ability to process it. Studies[1] have shown that the expectations of professional forecasters demonstrate inertia, and it takes time for them to learn when changes occur. Therefore, after crisis episodes, agents can have pessimistic forecasts. Imperfect information and/or overly pessimistic expectations influence the efficiency of policy actions aimed at mitigating a recession and can undermine their effect or lead to unintended consequences.

The first paper contributes to the literature by addressing how imperfect information among financial agents – banks – influences the functioning of the interbank market and the supply of credit to the real economy. We start with a simple model in which the supply of interbank market credit depends on banks' expectations about economic activity. Banks have heterogeneous expectations about risky asset returns and are endogenously divided into lenders and borrowers. The lenders assess counterparty risk as borrowers' ability to meet their obligations given their portfolio returns. After periods with low returns, lenders anticipate higher risk on the interbank market and demand a higher interbank rate. Given the banks' beliefs distribution, lower market beliefs result in less lending and can even lead to an interbank market crunch. As banks are creditors to the

[1] Examples being (Coibion and Gorodnichenko 2015a) and (Andrade and Le Bihan 2013).

real sector, the functioning of the interbank market then affects the real economy, generating or amplifying business cycle fluctuations. Within this framework we study possible central bank policy actions for stimulating credit or restoring the interbank market. We show that when lending is impaired because of low market sentiment, policy actions have a rather limited effect.

We consider the following central bank policy responses: (i) liquidity provision, targeted and untargeted, (ii) a policy rate cut, and (iii) relaxation of the collateral constraint on the interbank market. Our findings suggest that liquidity provision can help restore credit to the real sector, but its effect is limited by banks' pessimism, with a significant share of the central bank's funds ending in reserves. Interestingly, reducing the policy rate results in a worse outcome than the scenario with no policy response. In our model, the policy rate is the reserves rate, with reserves being the only safe asset. The low return on reserves erodes banks' revenues, resulting in a smaller supply of credit and a subsequent fall in capital accumulation.

The second paper is motivated by the debate about whether a low policy rate contributed to the recent financial crisis and if the ongoing policy of low interest rates is contributing to the building up of a new financial bubble. The question asked is if endogenous information acquisition can drive overaccumulation of risk when safe interest rates or market volatility is reduced. It is common that in portfolio choice models with rational expectations, investment into a risky asset is linear in excess return. In our model, when the policy rate or market volatility falls, risk accumulation in the economy increases in a nontrivial way, because agents acquire less information about the risky asset.

We capture the excessive risk accumulation by modeling information decisions. Financial agents invest in information to reduce the variance of their forecasts. We show that when market volatility declines, agents invest into information less and acquire more of a risky asset. This results in a portfolio risk comparable to that in an economy with higher market volatility. With interest rates being lowered, our model not only captures the standard "search-for-yield " effect, where financial intermediaries invest more into risky assets. We also show an increase in agents' ignorance about the asset quality. With low information investment and large risky asset holdings this implies a larger portfolio risk accumulation.

The main contribution of our model to the current debate is that it mimics excessive risk-taking of financial agents [2]. We show that average risk monitoring declines with lower interest rates despite the growth in excess return on a risky asset. Another result is overaccumulation of risky assets in a low risk environment. That is to say, with low variance of risky asset return, agents take as much or more risk in their portfolio than they would have with a high risky asset variance. This effect is explained in our model with just one deviation from rational expectations: agents do not know the future return, but only its distribution, i.e. there is no assumption of agent irrationality. In our model, this result is driven by a decline in risk monitoring in a low risk environment. Combined with an increase in risky asset acquisition, this results in higher portfolio variance compared to a high variance environment.

In the third paper we study under what conditions a mis-specified forecasting rule survives in equilibrium. We study an economy n which adaptively learning agents choose a strict subset of variables for their forecasting functions, thus inducing a RPE. We then allow these agents, inhabiting the RPE, to reconsider their forecast rules, subject to the informational cost constraint modeled as in Gabaix (2014). We consider if the model parameters which make the RPE stable are sufficient to ensure that the same subset of variables is selected by informationally constrained agents.

We find, in line with the literature, that when the feedback parameter from expectations is large enough, the mis-specified forecasting rule prevails in equilibrium. For a business cycle model studied, consistent with empirical evidence, our results show that when inflation becomes very persistent, agents switch to using a simple AR(1) rule. We also find a region in the parameter space where agents choose not to pay attention to any variables.

[2] By excessive risk-taking we understand an increase in the portfolio risk caused by information decisions.

Chapter 1

Confidence Cycles and Liquidity Hoarding [1]

Market confidence has proved to be an important factor during past crises. However, many existing general equilibrium models do not account for imperfect expectations or overly pessimistic investor forecasts. In this paper, we incorporate a model of the interbank market into a standard DSGE model, with the interbank market rate and the volume of lending depending on market confidence and the perception of counterparty risk. As a result, a credit crunch occurs if the perception of counterparty risk increases. Changes in market confidence then can generate credit crunches and contribute to the depth of recessions. We conduct an exercise to mimic some central bank policies: targeted and untargeted liquidity provision, and reduction of the reserve rate. Our results indicate that policy actions have a limited effect on the supply of credit if they fail to influence agents' expectations. A policy of a low reserve rate worsens recessions due to its negative impact on banks' revenues. Liquidity provision stimulates credit slightly, but its efficiency is undermined by liquidity hoarding.

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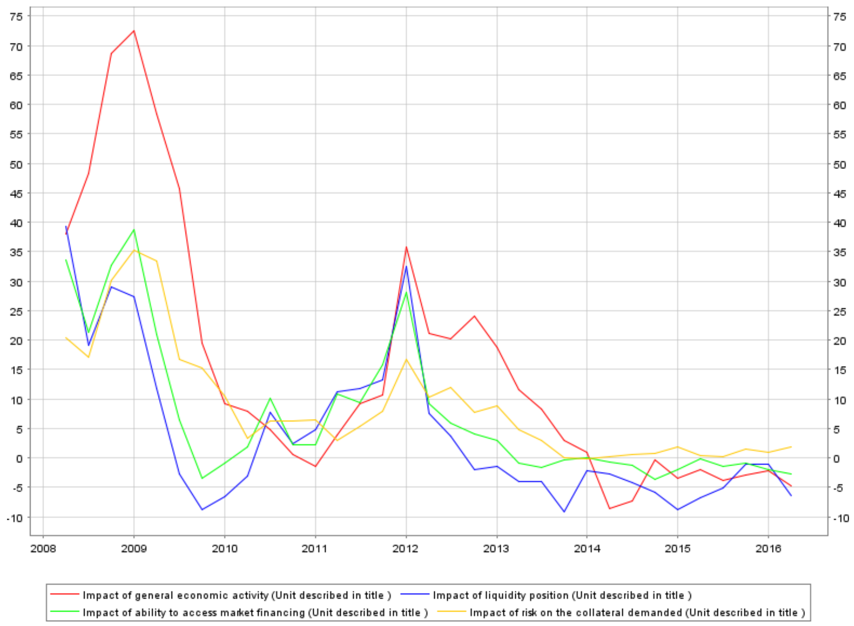
1.1 Introduction

The recent financial crisis was one of the deepest and longest in modern history. Having started in the financial sector, it then spread into the real economy, causing a recession the length of which has yet to be determined. Not surprisingly, it drew the attention of academics and policy makers to the interconnections between the financial and real sectors. A possible explanation of the origin and development of the crisis, or at least of its depth, lies in market imperfections and the limited rationality of economic agents. Figure

Figure 1.1: Net Tightening of Banks' Lending Standards



Source: ECB Bank Lending Survey

1.1 illustrates some change in European bankers' expectations occurring at the start of the crisis.[2] The figure shows the tightening of banks' lending standards in response to different factors. With the start of the crisis in 2008, there is a huge spike in the percentage of banks who tightened their credit standards due to the impact of general economic activity. This can be interpreted as a rise in banks' concerns about the economy. Other factors also contributed significantly to the tightening of lending standards, with banks' liquidity position being the least important. One can see another, somewhat smaller, spike around 2012, when most of the factors had the same impact. This spike can be

[2]In the Bank Lending Survey conducted by the ECB, one of the questions was about how selected factors contributed to the lending standards of the bank: either to tightening or to easing. Net tightening is defined as the difference between the percentage of respondents who tightened their lending standards and those who eased them. Here, we show the lending standards applied to enterprises.

attributed to the recent euro crisis. As banks' assessment of the economy becomes more optimistic and monetary policy actions mitigate their liquidity or collateral risk concerns, their lending standards become slightly eased (negative values on the graph) or factors become irrelevant for lending standards (values around zero). The question could be if the banks were overly pessimistic or just rationally predicting the downturn? There is no obvious way to find hard proof of overly pessimistic or overly optimistic expectations. One possible proxy is the survey of professional forecasters. A number of papers studying the survey of professional forecasters, examples being Coibion and Gorodnichenko (2015a) and Andrade and Le Bihan (2013), have found sluggishness in forecasters' expectations. We interpret this finding as meaning that agents form their forecasts based on backward-looking data. It is not surprising, then, that after an episode of low returns or high risk, forecasters underestimate returns or overestimate risks in the next period. Therefore, it is not unrealistic to consider that after the crisis, when central banks started to implement unconventional measures, banks had overly pessimistic expectations.

This paper contributes to the literature by addressing banks' imperfect information about general economic activity and counterparty risk in a DSGE model. Unlike a number of papers which regard interbank market collapse as being due to a liquidity shortage, we focus on the role of counterparty risk. In our model, banks lend to the real economy depending on their heterogeneous return expectations. A decline in return expectations increases their evaluation of counterparty risk on the interbank market. When lenders expect a low return on a risky asset, they assign a high probability to the scenario of their borrowers not being able to honor the debt. These expectations can drive the interbank market rate to a level where no bank is willing to borrow. Without access to the interbank market, even the most optimistic banks reduce their lending to the real economy and pessimistic banks just hoard their funds, i.e., keep them in reserves. Such an enriched banking sector is incorporated into a workhorse DSGE model, with only the moments of the beliefs distribution entering the equilibrium solution. With the number of expert surveys and market volatility indices at hand, our developed framework becomes a tractable version of a DSGE model for analyzing the role of expectational shocks and their propagation to the real economy. We also consider the question of the efficiency of the policy measures applied during the economic downturn. Our model allows us to account for the hoarding behavior by banks observed during the crisis, which is often missing from DSGE models analyzing unconventional central bank policy. Hoarding was observed in the form of banks being reluctant to lend while keeping funds in excessive reserves or investing in short-term assets.[3] We consider several types of central bank policy actions that resemble those taken during the crisis and the subsequent recession, including liquidity provision to all banks at a fixed rate and targeted liquidity provision to support lending to the real sector. We also consider the policies of reducing the rate on reserves and relaxing collateral constraints on the interbank market.[4] Our findings suggest that investors' expectations and their uncertainty instigate large swings in the real economy, where manufacturers are dependent on credit. In our model, when banks are concerned about economic prospects the liquidity provision policy dampens the magnitude of the crisis but neither stops nor shortens it. Moreover, a significant share of funds received from the central bank is invested in safe assets instead of flowing into the real economy. This result is in line with the banks' observed behavior. This also suggests that making policy evaluations without accounting for investors' sentiment and market volatility may overstate policy efficiency. Lowering policy rate makes hoarding less attractive, but reduces the banks' revenues, resulting in even worse outcomes than in the case of no central bank action. This paper is related to several strands of literature. The first concentrates on the role of the financial sector and credit in the economy. Studies have incorporated the banking sector into general equilibrium models. Examples include Gertler and Karadi (2011), Curdia and Woodford (2011), Del Negro et al. (2011), and Gertler and Kiyotaki (2010). Having introduced the financial sector, these papers address central banks' crisis-mitigation policies. While the first two papers consider the effect of policies on the transfer of credit between households and financial intermediaries, the latter two analyze credit supply to entrepreneurs subject to a liquidity constraint of the Kiyotaki and Moore (2008) type. Our study also addresses the efficiency of central bank policy, but accounts for the role of investor sentiment. The closest to our paper is Gertler and Karadi (2011). We use their framework as a backbone, allowing banks to have imperfect expectations and to lend to each other. We also use a similar approach to simulate the crisis and consider policy efficiency.

There are papers on interbank market structure related to our model. They include Gale and Yorulmazer (2013), Heider, Hoerova, and Holthausen (2009), and Allen, Carletti, and Gale (2009), who consider liquidity hoarding through the interbank market

[3]For evidence on hoarding see Gale and Yorulmazer (2013) and Heider, Hoerova, and Holthausen (2009) and references therein.

[4]The policy of relaxing collateral constraints actually involved widening the set of assets accepted as collateral by the central bank. In our model, it takes the form of banks being willing to lend up to a larger fraction of borrowers assets.

structure. In these models the reason for banks to hoard liquidity is anticipation of a liquidity shock. Our interbank market structure can allow for a liquidity shock, but we consider the role of counterparty risk in breaking the interbank market. Another related paper is by Bianchi and Bigio (2014). They focus on the liquidity management problem and assume no default risk on the interbank market and analyze the possible causes of crises and policy responses. Here, we focus more on counterparty risk on the interbank market and its implications for monetary policy. Bank heterogeneity in a DSGE model is introduced by Hilberg and Hollmayr (2011), who study liquidity provision and relaxation of collateral constraints. In Hilberg and Hollmayr (2011), bank heterogeneity is caused by exogenous separation into investment and commercial banks; only investment banks are allowed to borrow from the central bank. We consider a different interbank market structure consisting of a number of ex-ante identical banks who differ ex post depending on their subjective interpretation of public information.

There is body of literature suggesting that market expectations and uncertainty about the future can be important factors in generating economic fluctuations.[5]. The importance of sentiment shocks is empirically supported by Fuhrer (2011) and Beaudry, Nam, and Wang (2011), Beber, Brandt, and Luisi (2013) and Bloom (2009). The informational structure of the model is motivated by a branch of empirical literature analyzing survey data on economic forecasts.[6]

This paper proceeds as follows. We first analyze a simple model of the interbank market to illustrate the role of market expectations in causing a credit crunch and consider policy actions by a central bank. Next, the general equilibrium model is completed. Within a DSGE model, we show the implications of market mood swings for the propagation of crises and policy efficiency when there is feedback from household decisions and market prices.

[5] Among the seminal contributions to this branch of literature are Woodford (2001) and Mankiw and Reis (2007a).

[6] Among the papers analyzing diversity and systematic mistakes in professional and central bank forecasts are Franses, Kranendonk, and Lanser (2011) and Fildes et al. (2009), who study the bias in expert adjustment of forecasts

1.2 A Simple Model of Credit and the Interbank Market

In this section we describe the main mechanism of the model in a simplified setting. Later in the paper, the described sector is incorporated into a DSGE model to compare our results with the literature and to consider the general equilibrium effects of policy actions.

There are two time periods and two types of investment opportunities for banks: a storage asset pays R^{res} and a risky asset R^k in the next period. Decisions are made in period 1 and payoffs are realized in period 2. At $t = 1$ banks attract deposits d from the household. We simplify the problem by assuming that deposits are distributed equally among all banks and set $d = 1$. Banks pay R to depositors in the next period. The time subscripts are dropped. There is a continuum of banks normalized to 1 and indexed by i, each with different expectations about the risky asset return, $E^i\hat{R}^k$. In the general equilibrium context the risky asset is credit to the real sector, so in the simple model we sometimes refer to the risky asset position as credit. Banks can participate in the interbank market. If a bank chooses to borrow on the interbank market, we limit its borrowing to its share of liabilities. We call this share λ_b. The interbank market rate, R^{ib}, is determined endogenously by clearing the market. Clearly, portfolio decisions depend on the bank's expectations about the risky asset return, the safe asset return, the state of the interbank market (functioning or not), and the interbank market rate. We show below that, given the safe asset rate, the distribution of banks' beliefs defines their decisions and interbank market conditions. Then we consider possible policy actions and show how they affect liquidity hoarding and credit. In this framework we consider the state of the interbank market and the amount of credit given the moments of the beliefs distribution – the average market belief and its standard deviation.

Lending on the interbank market is risky: there is a probability that due to low portfolio returns borrowers will not repay their loans. We assume that not repaying part of a loan and not paying the full amount are equally costly for borrowers and that the cost is exclusion from the interbank market. That is, the lender only considers the probability that the borrower's return is smaller than his liabilities and disregards the set of possible partial loan repayments, which simplifies the math significantly. We also abstract from the agency problem here, assuming that banks will honor their debt unless their returns do not allow it.

Suppose for simplicity that beliefs are distributed uniformly among banks with mean m and variance σ^2.[7] We think of each banker as being a statistician making her best forecast conditional on the available information. Each bank's individual estimate, $E^i\hat{R}^k$, is then assumed to be distributed uniformly with the same variance σ^2. That is, each banker has her own prediction of the risky asset return, $E^i\hat{R}^k$, with variance σ^2, and these predictions are distributed uniformly among banks with mean $\bar{E}\hat{R}^k = m$ and the same variance σ^2. These assumptions are made for the sake of simplicity and more intuitive presentation of the results. Later in the paper we relax these simplifying assumptions and let banks have a model-consistent beliefs distribution.

Every bank is risk neutral and optimizes its next-period return by maximizing the following function:

$$\max_{\alpha^i,h^i,L^i} \alpha^i E^i\hat{R}^k + h^i * R^{res} + \left(1-\alpha^i-h^i\right) p^i R^{ib} + (E^i\hat{R}^k - R^{ib}) * \Lambda^i \tag{1.1}$$

subject to a collateral constraint on the interbank market: $\Lambda^i = \lambda_b$ or 0. Bank i chooses the portfolio shares α^i (the share of the risky asset) and h^i (the share of hoarded or reserve assets), and the rest of the assets $(1-\alpha^i-h^i)$ are then lent on the interbank market. The return then consists of the expected return on the risky asset $\alpha^i E^i\hat{R}^k$, the return on the safe asset $h^i * R^{res}$, and the expected return on interbank lending $p^i\,(1-\alpha^i-h^i)\,R^{ib}$. The lenders are uncertain whether the borrowers will be able to repay their debt, hence they assign a loan repayment probability p^i. Those banks which are willing to borrow on the interbank market and invest in the risky asset get the expected return on borrowed funds: $(E^i\hat{R}^k - R^{ib}) * \Lambda^i$,[8] where Λ^i is the amount borrowed on the interbank market. Because every bank is risk neutral, the problem results in a corner solution.

Let us now consider the subjective loan repayment probability. The probability of a loan being repaid, p^i, is lender i's subjective probability that the borrower will repay the loan, in other words, that the borrower's return on the risky asset will be higher than her payments on the loan and other liabilities. Because of risk neutrality, all borrowers invest everything in the risky asset. By construction, all banks have an equal amount of deposits and also borrow the same amount on the interbank market: λ_b. That is, from

[7]The bounds of the uniform distribution a and b are then: $a = m - \sigma\sqrt{3}$ and $b = m + \sigma\sqrt{3}$. In this simplest model, a can be negative.

[8]Banks only borrow on the interbank market to invest in the risky asset. Consider the case where a banker borrows and invests in the safe asset. This would mean that $R^{res} > R^{ib}$. In this case, no one would lend on the interbank market. Therefore, the interbank market only functions when $R^{res} < R^{ib}$.

the lender's perspective, each borrower has the same amount of assets and liabilities. And p^i is determined by the lender's belief about the risky asset return, $E^i\hat{R}^k$:

$$p^i = Prob\left\{(1+\lambda_b)\,E^i\hat{R}^k \geq R + \lambda_b R^{ib}\right\} \tag{1.2}$$

In (1.2) the borrower's expected return is $(1+\lambda_b)\,E^i\hat{R}^k$, where 1 is the borrower's own funds and λ_b is the share borrowed on the interbank market. The liabilities are then $R*1$ to households and $\lambda_b R^{ib}$ to the interbank market lender. If the return is higher than the liabilities, a bank pays interbank market debt. As we have assumed for this section that the bank's estimate of the risky asset return is uniformly distributed with variance σ^2, we can write p^i as a cumulative density function of uniform distribution:

$$p^i = 1 - U_{E^i\hat{R}^k,\sigma_v^2}\left(\frac{R+\lambda_b R^{ib}}{1+\lambda_b}\right) = \frac{1}{2} - \frac{\left(R+\lambda_b R^{ib}\right)}{2\sigma\sqrt{3}(1+\lambda_b)} + \frac{E^i\hat{R}^k}{2\sigma\sqrt{3}} \tag{1.3}$$

Equation (1.3) shows the connection between the individual probabilities and the interbank interest rate. Aggregating the interbank market, we show how these individual probabilities translate into the interbank market rate.

For the interbank market to clear, lending must be equal to borrowing. Lenders are risk neutral and lend everything they have, and all have the same funds (equal to 1), so the amount of lending is given by:

$$\underbrace{1}_{\text{lenders' funds}} * \underbrace{\int_{E^l\hat{R}^k}^{E^m\hat{R}} f(x)d(x)}_{\text{share of lenders}} * \underbrace{1}_{\text{number of banks}}$$

where $f(x)$ is a probability density function of uniform distribution. To calculate the share of lenders, we use the frequency of banks between the marginal lender and the marginal investor. Multiplying the share by the total number of banks and lenders' funds we get the total supply of interbank lending. Similarly, the amount of borrowing on the interbank is:

$$\underbrace{1}_{borrower's\ funds} * \underbrace{\lambda_b}_{collateral\ constraint} * \underbrace{\int_{R^{ib}}^{\bar{R}} f(x)\,dx}_{share\ of\ \ borrowers} * \underbrace{1}_{number\ of\ banks}$$

where the upper bound on return expectations is given by $\bar{R}$. Multiplying the share by the total number of banks we get the number of borrowers. As every borrower faces the collateral constraint $\lambda_b * borrower's\ funds$, the demand for interbank funds is given by the number of borrowers times this collateral constraint.

Knowing the distribution of beliefs across banks, in the case of uniform distribution we can write the interbank market clearing condition as follows (a detailed derivation is given in Appendix A):

$$E^m \hat{R}^k - E^l \hat{R}^k = \lambda_b \left(\sigma\sqrt{3} + m - R^{ib} \right) = \lambda_b \left(\bar{R} - R^{ib} \right) \tag{1.4}$$

The interbank market rate, R^{ib}, clears the market, and m and σ are the mean and the standard deviation of banks' beliefs distribution. The supply of loans is given simply by the difference between the belief of the marginal investor and that of the marginal lender: $E^m \hat{R}^k - E^l \hat{R}^k$. The demand for loans is the difference between the largest belief in the market, $\bar{R}$, and the belief of the marginal borrower. Note that the marginal borrower expects the risky asset return to be equal to the interbank market rate. When the distribution is uniform, the largest belief can be written as the sum of the mean and the standard deviation $\sigma\sqrt{3} + m$. Each individual probability is also a function of the standard deviation.

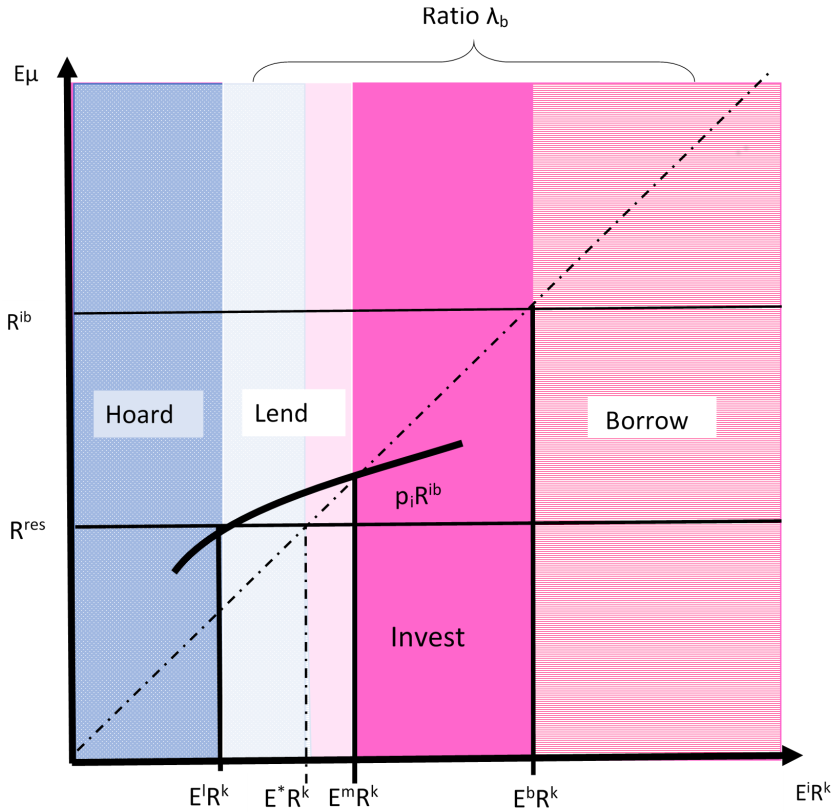
The marginal lender's belief determines the lower bound on the interbank interest rate. It must be such that there exists at least one banker whose expected interbank market return, $p^l R^{ib}$, is greater than or equal to the reserve rate, R^{res}. And at the same time, her belief about the risky asset return must be lower than the interbank market return:

$$E^l \hat{R}^k \leq p^l R^{ib} \geq R^{res} \tag{1.5}$$

Equation (1.5) also shows where the non-linearity in the model comes from. If the market beliefs are such that there is no banker expecting the interbank market return to be higher than the reserve rate or the risky asset return, there is no lending. The reason for this may be that market beliefs about the risky asset are very low. Then, a lender would expect interbank lending to be very risky (p^i, the probability of loan repayment, is very low) and demand a high interbank rate, as $R^{ib} \geq \frac{E^i \hat{R}^k}{p^i}$ and $R^{ib} \geq \frac{R^{res}}{p^i}$. At the same time, no banker would believe that the risky asset pays more than this high interbank rate, and none would be willing to borrow. The marginal borrower determines the upper bound on the interbank market rate. It should not exceed the largest belief in the economy: $R^{ib} \leq \bar{R}$. This upper bound, however, disappears in the full model with normal distribution of banks' beliefs.

Depending on the beliefs and their mean and dispersion, the interbank market can have three different states: a functioning market, or a market where no one lends, or a market where no one borrows. Every bank has different risky asset return expectations, which results in different portfolio decisions. Banks' choices are illustrated in Figure 1.2.

Figure 1.2: Banks' Expectations and Investment Decisions



Dotted blue area – hoarders, white transparent area – lenders, solid pink area – direct investors, red stripes ares – borrowers

In Figure 1.2 banks are distributed according to their risky asset return expectations. On the vertical axis there are beliefs of individual banks. When the beliefs are too low for an interbank market to exist, banker are divided into investors (pink and red stripes area) and hoarders (dotted blue area). Crossing of the 45 degree line and the safe asset rate determins the marginal investor when interbank market is not functioning, E^*R^k. Bankers to the right of E^*R^k invest, to the left - hoard. When the beliefs increase, increasing probability of the repayment on the interbank market, some of the hoarders and investors become lenders (transparent white area). Let us start with the most pessimistic bankers. Their estimates of the risky asset return are so low that: 1) they are lower than the rate on reserves: $E^i\hat{R}^k < R^{res}$, 2) the subjective probability of loan repayment on the interbank market is so low that the expected interbank market return is lower than the rate on reserves: $p^iR^{ib} < R^{res}$. These bankers hoard (invest in reserves) (blue dotted are) The less pessimistic bankers (transparent wide area) assign a higher loan repayment probability; for them $p^iR^{ib} \geq R^{res}$. At the same time, they do not expect the risky asset to pay more than the interbank market: $E^i\hat{R}^k < p^iR^{ib}$. We denote a banker who is indifferent between lending and hoarding as a marginal lender with beliefs $E^l\hat{R}^k$ such that $p^lR^{ib} = R^{res}$. The optimistic banks – those investing in the risky asset – believe that the risky asset pays more than lending on the interbank market $E^i\hat{R}^k > p^iR^{ib}$. The marginal investor is then a banker indifferent between investing and lending - the intersection of 45 degree line and p^iR^{ib} curve; we denote her beliefs as $E^m\hat{R}^k = p^mR^{ib}$. Among the optimists there are those who believe that the risky asset pays even more than the interbank market rate: $E^i\hat{R}^k > R^{ib}$. They borrow on the interbank market and invest everything in the risky asset. The marginal borrower is then defined simply as $E^b\hat{R}^k = R^{ib}$ and is given by an intersection of interbank rate and 45 degree line.

The solution to the model is then given by (1.4), (1.3), and the definitions of the marginal lender and investor:

$$\begin{aligned} p^lR^{ib} &= R^{res} \\ E^m\hat{R}^k &= p^mR^{ib} \end{aligned}$$

The expression for marginal investors in terms of the interbank market rate are:

$$\begin{aligned} E^l\hat{R}^k &= \frac{R}{\lambda_b+1} + \frac{\lambda_b R^{ib}}{\lambda_b+1} - \frac{\sqrt{3}\sigma(R^{ib}-2R^{res})}{R^{ib}} \\ E^m\hat{R}^k &= \frac{R^{ib}(\sqrt{3}\lambda_b R^{ib} - 3\lambda_b\sigma + \sqrt{3}R - 3\sigma)}{(\lambda_b+1)(\sqrt{3}R^{ib}-6\sigma)} \end{aligned}$$

Combining these 4 equations gives an equation for the interbank market rate, summarized in the proposition 1.

Proposition 1. *The necessary condition for the interbank market to exist is that there is a unique R^{ib}, solving*

$$a*(R^{ib})^3 + b*(R^{ib})^2 + c*(R^{ib}) + d = 0, \tag{1.6}$$

that is real and non-negative.[9] *With $a > 0$, $b < 0$, and $d > 0$, if a positive root exists, it is unique. This positive root exists only if:*

$$R^{res} > A + \frac{R}{\lambda_b + 1} < 0, \tag{1.7}$$

where [10] $A < 0$.

The sufficient conditions for the equilbrium with the interbank market is:

$$\begin{aligned} R^{res} < R^{ib} &< \frac{1+\lambda_b}{\lambda_b}\sigma\sqrt{3} \\ R^{ib} &< \bar{R} = m + \sigma * sqrt3 \end{aligned}$$

for $\sigma > \frac{\lambda_b R^{res}}{(1+\lambda_b)\sqrt{3}}$.

The mathematical proof is given in Appendix A. (1.7) states, that R^{res} should not be too much smaller than payments on deposits and not too "negative". This is a mathematical restriction, if R^{res} drops too much, than for $E^l \hat{R}^k > 0$, R^{ib} becomes negative. [11] That is, there is always a unique positive R^{ib} for positive (or not "too negative" R^{Res}).

Proposition 1 states that for an interbank market to exist, beliefs must be sufficiently diverse. It also determines the upper bound on the interbank rate proportionally to banks' diversity. If the diversity of beliefs is small, the beliefs of borrowers and lenders do not differ much, borrowers' beliefs are not much more optimistic than those of lenders, and borrowers are not willing to pay a high interbank rate. In the opposite case, with large diversity, borrowers expect much higher returns from the risky asset than lenders do, and the interbank rate is higher. The illustration of the region for sufficient conditions for a given R^{res}, R, and λ_b is given on Figure 1.10 in Appendix 1.A.

Proposition 2. *Low market beliefs result in a lower interbank rate and lower lending.*

The mathematical proof is given in Appendix A. The proposition is illustrated in panel *a* of Figure 1.3. If the dispersion is fixed, a decrease in market beliefs about the risky asset return means a shift in the bounds of the beliefs distribution: the most pessimistic banker becomes even more pessimistic and the most optimistic banker becomes less optimistic.

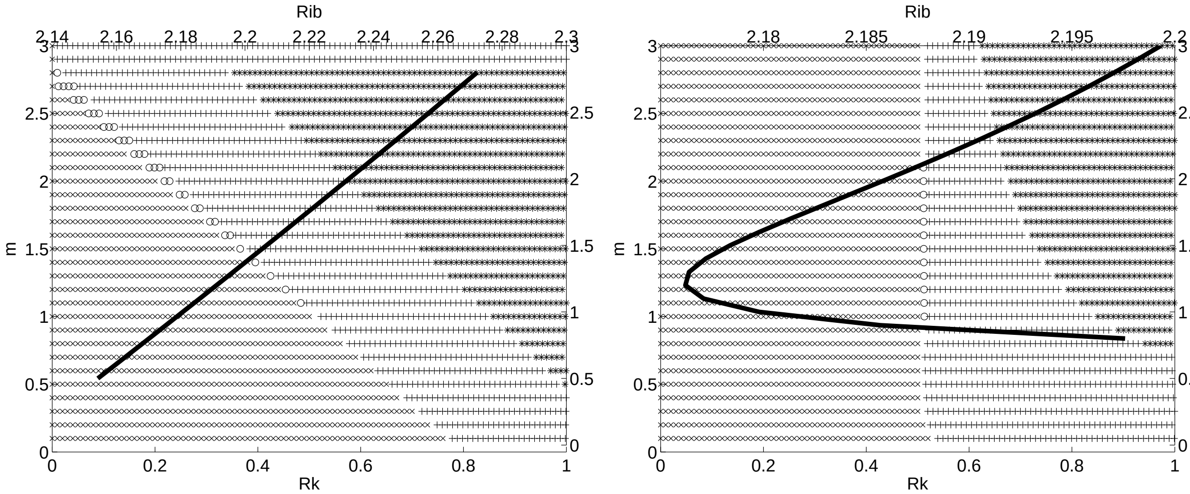
---

[9] $a = \sqrt{3}\lambda_b(1 + \lambda_b)$, $b = -\lambda_b\left(\sqrt{3}(\lambda_b+1)m + 9\lambda_b\sigma + 3\sigma\right)$, $c = 6\sigma\left(\lambda_b\left(\lambda_b m + \sqrt{3}\lambda_b\sigma + m - R^{res}\right) + R - R^{res} - \sqrt{3}\sigma\right)$, *and* $d = 12\sqrt{3}(\lambda_b+1)R^{res}\sigma^2$

[10] $A = \frac{-3\left(3\lambda_b^3+7\lambda_b+6\right)\sigma^2 - \lambda_b(\lambda_b+1)^2 m^2 + 4\sqrt{3}\lambda_b(\lambda_b+1)m\sigma}{6\sqrt{3}(\lambda_b+1)^2\sigma}$

[11] If pessimistic market expectations about risky asset return, $E^l \hat{R}^k$, can be negative, this would violate the structure of the ξ shock which is on $[0,1]$ interval.

Figure 1.3: Banks' Beliefs Distribution



(a) **(b)**

Numerical simulations. From left to right: x area – hoarders, o area – lenders, + area – investors who do not borrow, * – investors who borrow, solid line – interbank rate (upper horizontal axis). The numerical values for the parameters $R^{res} = R = 1.0101$,$\lambda_b = 01$, for the panel a $\sigma = 0.98$, for the panel $m = 1$
.

Borrowers (the red area) use interbank loans to invest in the risky asset. Intuitively, when borrowers expect a lower return on the risky asset they are willing to pay less for the interbank loan. With all bankers being less optimistic about the risky asset return, there is a larger share of those who do not invest themselves and expect a lower loan repayment probability: there is more hoarding (the blue area). Those bankers who are considering whether to lend on the interbank market or to invest in the risky asset evaluate these two options at a lower interbank rate. This makes interbank lending less attractive and the share of lenders (the green area) also shrinks.

Corollary 1. *With very low beliefs diversity, there is no lending on the interbank market. With very high beliefs diversity, lending is possible but small.*

From proposition 1 there is a lower bound on beliefs diversity $\sigma > \frac{\lambda_b R^{res}}{(1+\lambda_b)\sqrt{3}}$. Because a lender compares the expected interbank market return with the return on reserves, the rate on reserves defines this upper bound. The role of the standard deviation is two-fold in the model, as shown in panel b of Figure 1.3. First, it measures the dispersion of beliefs among banks. With very low dispersion there is little difference in beliefs across bankers and there is little or no lending. Second, it reflects how each bank is uncertain about its own estimate of the future return. If a lender is almost certain about her low return expectation, she will assign a low loan repayment probability and ask for a high interbank market rate. At the same time, the borrowers are more convinced about their high return expectations and are willing to pay that high rate. Consequently, with a low standard deviation, there is either no lending, or very low lending at a high interbank rate. As the standard deviation increases, so does the uncertainty among the borrowers. They are willing to pay a lower interbank rate. For the same reason, the lenders are less certain about their pessimistic returns. They expect a higher loan repayment probability and agree to a lower interbank rate. When the uncertainty and dispersion are very large, there is still lending, but its volume is negligible (see Figure 1.3). If the diversity of beliefs is large, the bounds of the distribution widen and there are some very optimistic borrowers. This pushes the interbank market rate up. Figure 1.3 summarizes possible scenarios of interbank market functioning, where panel *a* shows the impact of mean market beliefs on banks' equilibrium allocations and panel *b* shows the impact of the beliefs dispersion and the standard deviation of banks' forecasts. On the horizontal axis are the shares of bankers: hoarders, lenders, investors that do not borrow, and investors that borrow. The black line is the interbank market rate, measured on the upper horizontal axis. When the interbank market rate is not defined, the interbank market collapses: when the average beliefs and diversity are too small.

Policy effects

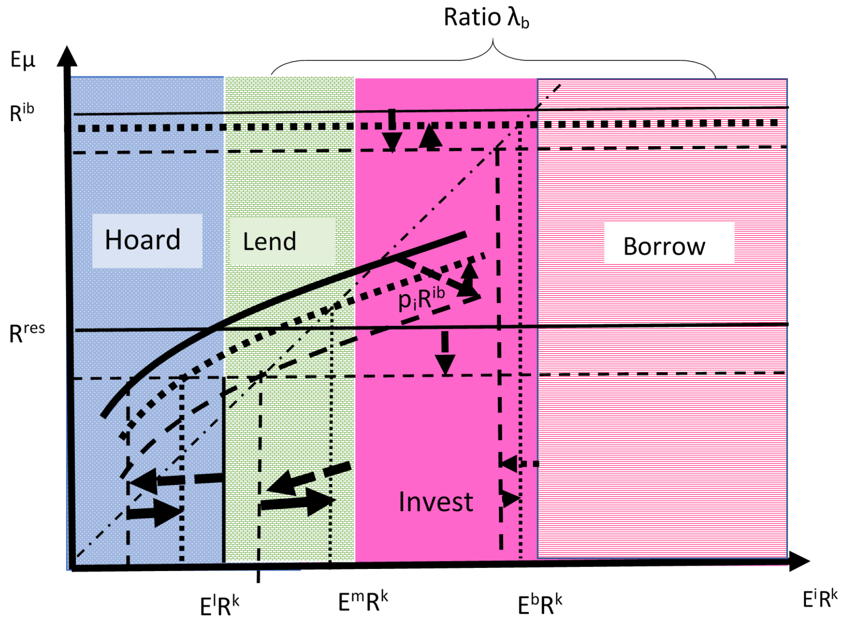
A functioning interbank market

Let us now consider the impact of policy actions in the context of the simple model we have developed. First, suppose that the interbank market is functioning, but some superior agent, which we call the central bank, would like to increase interbank lending and/or stimulate credit to the real economy.

Proposition 3. *A low policy rate increases lending and lowers the interbank market rate, and increases the supply of credit to the real economy.*

The mathematical proof is given in the appendix. The mechanism is illustrated on Figure 1.4. Suppose there is a decline in the reserve rate, R^{res} - a downward shift to a dashed line. As the safe asset becomes less attractive, some of the hoarders start to lend - $E^l\hat{R}^k$ shifts to the left - dashed line. The larger supply of lending results in a lower interbank market rate, increasing the set of borrowers, $E^b\hat{R}^k$ shifts to the left and lowering p^iR^{ib} curve (to the dashed curve). From the point of view of the marginal

Figure 1.4: Banks' Beliefs Distribution and Interest Rate Decline



Dotted blue area – hoarders, grean squares area – lenders, solid pink area – direct investors, red stripes ares – borrowers

investor, the risky asset becomes more attractive, as the interbank market return falls. A less optimistic bank becomes a marginal investor, as $E^m\hat{R}^k$ falls too. The lower expected return on the interbank market, in turn, will decrease the set of lenders - $E^l\hat{R}^k$ shifts to the right (dotted line). This pushes the interbank rate up (to a dotted line), declines the set of investors and borrows. Yet, it only partially offsets the initial increase in lending and investing, with lending and investing increasing after a safe rate decline. Figure 1.4 also shows that the effect of a safe rate decline is limited, as a decline in the interbank market rate pushes some lenders out of the market. In the case, interbank market does not function, shown in Figure 1.3, the effect of a safe rate decline is trivial - it simply affects the margin at which banks start to credit the real economy.

Proposition 4. *Relaxing the collateral constraint increases lending and the interbank market rate, but lowers credit to the real economy.*

The mathematical proof is given in the appendix. The collateral constraint in our model is more a mathematical restriction and is not meant to represent the real-life interbank market or the central bank's policy. However, we find it instructive to analyze what impact relaxing the constraint could have on the interbank market. When borrowers are less restricted, one would expect there to be more interbank market lending and more credit supply to the real economy. In our model, allowing borrowers to borrow more does indeed increase lending and the interbank market rate. With relaxed restrictions and a higher interbank rate, lenders expect a lower loan repayment probability, with the higher interbank rate partially compensating for it. However, the most pessimistic lenders leave the market. That is, the marginal lender must now have higher return expectations, and all bankers that have lower beliefs (and there are now more of them, as the marginal lender shifts to the right) hoard. That is, hoarding increases.

Liquidity provision

In this simple framework without liquidity concerns, the provision of liquidity to banks, whether targeted or untargeted, does not affect the functioning of the interbank market. All the bankers would allocate all their available funds according to the decision rules discussed above. The provision of funds to optimists increases credit to the real economy, given that optimists exist. If the liquidity provision is untargeted and the funds are distributed equally among the banks, the pessimistic banks hoard it, as their main concern is counterparty risk and a low risky asset return. In this regard, targeting only optimistic banks can increase credit. Again, in the general equilibrium context, the feedback from prices and banks' balance sheets reverses the predictions: the untargeted policy results in better general outcomes than the targeted one.

To sum up, in the light of our model, if a central bank wants to increase lending on a market where banks are concerned about counterparty risks, a policy that does not address those concerns can do very little. Moreover, it could even have opposite-than-desired effects.

Interbank market collapse

Let us now consider the case where the interbank market collapses due to low market expectations about the risky asset return.

Suppose that expectations are such that no lenders are willing to lend. Obviously, providing them with additional funds will not revive interbank lending, but, if provided to optimists, such funds could increase credit to the real economy. The size of this effect is conditional on the share of investors: the more investors there are in the economy, the more efficient the policy will be. On the other hand, if the funds are provided to pessimistic banks, the policy increases hoarding and has only a small impact on credit to the real sector.

Proposition 5. *The effect of a policy rate reduction is limited by the mean market belief.*

A formal proof is provided in the appendix. The only tool that might have a potential effect is **a reduction in the policy rate**. However, this policy has a very limited or zero effect if market beliefs are very low, which also means a very low interbank rate. In this case, even with a low reserve rate, hoarding is still more attractive than interbank lending.

Proposition 6. *Relaxing the collateral constraint does not restore the functioning of the interbank market or credit to the real economy*

The mathematical proof is given in the appendix. Intuitively, as was discussed in proposition 4, when borrowers are more leveraged lenders expect a lower loan repayment probability, implying less lending.

To sum up, if banks are concerned about a low risky asset return and expect a low loan repayment probability, policy actions have a very limited effect. Liquidity provision policies enhance credit through optimistic bankers only, with the rest of the funds ending up in reserves. A low interest rate policy restores the market only if market beliefs are not very low, and stimulates credit to the real economy among banks expecting the risky asset to pay more than the storage asset.

1.3 Closing the General Equilibrium Model

In this section we drop our simplifying assumptions about banks' beliefs. In particular, banks now form expectations based on past data on risky asset returns and private signals about future returns. A bank's belief is then the result of the Kalman filter and follows a normal distribution. We then input the banking sector developed above into a linearized DSGE model as in Gertler and Karadi (2011). In their model, agents have perfect expectations about future risky asset returns. We modify it so that risky asset returns are harder to predict. Besides publicly observable risky returns, our banks have private signals about future returns. These signals are heterogeneous, though correlated among banks, and are subject to mood swings, that is, mood shocks, which we model as a decline in banks' average private expectations about asset returns. Another difference is that, in Gertler and Karadi (2011), banks frictionlessly transfer their liabilities to credit to the real sector. In our model, we allow the banks to keep (hoard) liquidity if they choose to. Thus, it is possible to address the question of whether the liquidity provided by the central bank is transmitted to the real economy, or ends up in bank reserves. Last but not least, heterogeneous expectations give rise to an interbank market. In our model, the interbank market serves as a propagation mechanism, increasing or decreasing the credit supply as interbank market conditions change.

We start with a description of the main building blocks of the model: the financial sector with heterogeneous beliefs and the interbank market. Then we proceed to complete the general equilibrium model and consider crisis and policy effects when there is feedback from the rest of the economy. The rest of the sectors are standard as in Smets and Wouters (2007) and Gertler and Karadi (2011), so we outline them only briefly. For a more rigorous discussion the reader is referred to these papers.

1.3.1 The Financial Sector

There is a continuum of banks normalized to one. Every period, a fraction $(1-\theta)$ of banks exit the sector and join the households. At the same time, the same number of household members become bankers and receive starting capital from the households. This starting capital equals a share of total banking sector assets. Banks receive deposits from the households, paying a gross real rate R_t. We model deposits as distributed equally among the banks regardless of their portfolio holdings. Banks allocate their funds between a safe asset paying the same gross real rate R_t, a risky asset with an uncertain gross real return, R^k_{t+1}, and interbank market lending with a gross real return R^{ib}_t. Banks are aware of the risk that some borrowers may not repay their debt. The debt repayment probability is reflected in the interbank market rate.[12] In order not to track the distribution of each banker's worth, we treat the bankers as members of one family, where each member maximizes his own return. At the beginning of a new period, before making investment decisions, they all average their net worth.

The risky asset in the model is credit to the real sector. Banks buy the claims of non-financial firms, S_t, at price Q_t and return R^k_{t+1}. The non-financial firms are intermediate goods manufacturers that need funding to buy capital. They transfer the return on the capital as a payment on S_t. The uncertainty comes from a so-called capital quality shock, ξ_{t+1}, as in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). As opposed to physical depreciation, δ_t, it is intended to capture unexplained fluctuations in the value

[12] Note the timing of the interest. Although it is paid in period $t+1$, the rate on the safe asset and the interbank market rate are set in period t. For the rest of the model description we use the same convention to refer to the timing of the variables when they are decided upon.

of capital: it influences not only the capital stock $\delta_t \xi_t K_{t-1}$, but also its value $Q_t \xi_t K_{t-1}$. The value of undepreciated capital is then defined as the difference between the value of new capital and the value of depreciated capital.

$$(Q_{t+1} - \delta_{t+1})\, \xi_{t+1} K_t \tag{1.8}$$

The return on capital consists of the value of the marginal product of capital, $\alpha \frac{P_{m,t+1} Y_{t+1}}{\xi_{t+1} K_t}$, plus the value of new capital, Q_{t+1}, minus depreciated capital, δ_{t+1}. The quality shock, ξ_{t+1}, then influences expectations of the return on capital:

$$R^k_{t+1} = \frac{\left(\alpha \frac{P_{m,t+1} Y_{t+1}}{\xi_{t+1} K_t} + Q_{t+1} - \delta_{t+1}\right) \xi_{t+1}}{Q_t} \tag{1.9}$$

Equations (3.13) and (3.14) are identical to those in Gertler and Karadi (2011). Now, though, we adjust the process for the quality shock. It is observable by all the sectors, but the composition of the shock is unobservable. With this process we intend to capture developments in capital value which are not predictable by the market. We assume that capital quality is subject to two types of shocks – persistent and transitory. The combination of these two shocks creates uncertainty in predicting future values of capital quality.

$$\xi_t = \xi_{t-1}^{\rho_\xi} \mu_t e^{\varepsilon_{\xi,t}} \tag{1.10}$$

μ_t is a persistent shock

$$\mu_t = \mu_{t-1}^{\rho_\mu} e^{v_t} \tag{1.11}$$

where ρ_μ and ρ_ξ are persistence parameters, v_t and $\varepsilon_{\xi,t}$ are transitory Gaussian shocks, serially uncorrelated with zero contemporaneous correlation and variances σ_v^2 and σ_ε^2. Neither the intermediate goods producers nor the banks observe either μ_t or $\varepsilon_{\xi,t}$. Next we explain how the banks set their expectations about ξ_{t+1}.

Expectations Formation

Banks do not observe whether the change in ξ in (1.10) is due to a transitory or a persistent shock. They have access to past data on returns and they use it to form a homogeneous economic forecast. There are, however, private signals - expert adjustments - about the value of μ_t. The inclusion of expert forecast adjustments is motivated by an extensive literature that provides evidence of the widespread use of expert factors in forecasting practice.[13]We model expert adjustment as an additional signal about the value of μ_t:

$$\theta_t^i = \rho_\theta \theta_{t-1}^i + \eta_t^i \tag{1.12}$$

where η_t^i is the noise in the opinion of bank i's expert, with η_t^i being correlated draws from $N(\mu_t, \sigma_\eta)$.

The noise in expert opinions is correlated with correlation coefficient ρ^c.[14] This correlation can be interpreted in two ways. First, experts tend to react to similar news in a similar fashion – being overly optimistic or overly pessimistic. Second, even though formally they do not share their forecasts with each other, we retain the possibility of convergence of their opinions or coordination on an additional public signal. That is, when the correlation coefficient is one, experts' opinions are fully converged and are the same. Conversely, when ρ^c is zero, they are fully diverged. We assume that the correlation coefficient lies between zero and unity. Appendix B shows that such a correlation among expert errors shifts the average of the draws away from the distribution mean, so that the error in expert opinions is not averaged away. Banks have two sources of information – the past and current realizations of ξ_t and the expert opinion about μ_t. Banks use the Kalman filter to combine the two signals, with the weights of the signals in the final forecasts depending on their relative variance. A description of the Kalman filter setup is given in Appendix C.

The Interbank Market and Banks' Problem

The banks' problem is very similar to the one in the simplified model (1.1). At time t, banks choose their portfolio allocation: invest a share in the risky asset, α_t^i, leave a share in the reserves (or hoard), h_t, lend on the interbank market, $(1 - \alpha_t^i - h_t^i)$, or borrow on the interbank market, Λ_t^i:

$$\max_{\alpha_t^i, h_t^i, L_t^i} E_t^i \hat{R}_{t+1}^k \alpha_t^i + h_t^i * R_t^{res} + p_t^i \left(1 - \alpha_t^i - h_t^i\right) R_t^{ib} + (E_t^i \hat{R}_{t+1}^k - R_t^{ib}) * \Lambda_t^i \tag{1.13}$$

subject to

$$\Lambda_t^i = 0 \text{ or } \lambda_b$$

[13]For an example of this literature and a survey see Franses, Kranendonk, and Lanser (2011) or Fildes et al. (2009)

[14]ρ^c is the Pearson correlation coefficient for each pair of experts.

where R_t^{ib} is the gross real interbank market rate to be paid at $t+1$, and p_t^i is the subjective probability that the loan will be repaid. $E^i \hat{R}_{t+1}^k$ is bank i's subjective expectation about the risky asset return, and R_t^{res} is the gross real safe asset return. If a bank is a borrower on the interbank market, borrowing is restricted to a fraction λ_b of its net worth. Because we assume that net worth is averaged up at the beginning of the period, all borrowers borrow the same amount – λ_b. For a lender $\Lambda_t^i = 0$. The interbank market rate, R_t^{ib}, is determined by the market clearing condition. The main modification from the simple case is a different beliefs distribution. The distribution affects the subjective loan repayment probability and the share of bankers hoarding, etc. Recall that each bank's subjective probability of borrowers being able to meet their obligations is:

$$p_t^i = 1 - F_{E_t^i \hat{R}_{t+1}^k, \sigma_R^2} \left(\frac{\left(R_t d_t + \lambda_b R_t^{ib}\right)}{(1+\lambda_b)} \right) \tag{1.14}$$

where $(1+\lambda_b)\, E_t^i \hat{R}_{t+i}^k$ is the bank's expected risky asset return on its own funds plus those borrowed on the interbank market, λ_b. For a bank to be able to honor its interbank market loan (assuming that debt to the household has priority) the return should be higher than payments to the household, R_t, times the amount of deposits per bank, d_t, and the interbank loan repayment, $\lambda_b R_t^{ib}$, because each bank's belief is distributed normally with variance σ_R^2. Now, p_t^i is the cumulative density function of the normal distribution.

However, to consider what fraction of banks actually invest in credit to the real sector or borrow in the interbank market, we need the distribution across banks. The distribution is Gaussian with some mean, m, and the same variance σ_R^2. The variance of the forecasts is the same for all the banks because they use the same observable and we assume the same variance in their expert adjustments. Both the mean and the variance enter the rest of the model as state variables. Appendix B shows how the correlation of banks' expert adjustment affects the mean and variance of the forecast distribution across banks. When simulating the general equilibrium model, we use only the mean and the variance of the forecast across banks and evaluate the cumulative distribution function for the beliefs of the marginal bankers – the investor, the lender, and the borrower. To sum up, we have a continuum of banks with beliefs distributed across banks $E_t^i \hat{R}_{t+i}^k \sim N(m, \sigma_R)$. Thus, the share of banks investing in the real economy is simply the share of banks with beliefs equal to or higher than the marginal investor's, whose belief is $E_t^m \hat{R}_{t+1}^k = p_t^i R_t^{ib}$:

$$s_t^{inv} = \int_{E_t^m \hat{R}_{t+1}^k}^{\infty} f(x)\, dx = 1 - F_{m,\sigma_R^2}\left(E_t^m \hat{R}_{t+1}^k\right) \tag{1.15}$$

where $F_{m,\sigma_R^2}\left(E_t^m \hat{R}_{t+1}^k\right)$ is normal cdf with mean m and variance σ_R^2. With a functioning interbank market, the marginal investor is indifferent between lending to the real sector and lending on the interbank market. [15]

The share of banks borrowing on the interbank market can be defined as the probability that their belief is higher than the interbank interest rate:

$$s_t^b = \int_{R_t^{ib}}^{\infty} f(x)\, dx = 1 - F_{m,\sigma_v^2}\left(R_t^{ib}\right) \tag{1.16}$$

The share of banks lending is then defined as the probability that the belief is higher than the belief of a marginal lender, $E_t^l \hat{R}_{t+1}^k$ with $p_t^l R_t^{ib} = R_t^{res}$:

$$\begin{aligned} s_t^l &= \int_{E_t^l \hat{R}_{t+1}^k}^{E_t^m \hat{R}_{t+1}^k} f(x)\, dx = \\ &= F_{m,\sigma_R^2}\left(E_t^m \hat{R}_{t+1}^k\right) - F_{m,\sigma_R^2}\left(E_t^l \hat{R}_{t+1}^k\right) \end{aligned}$$

The share of those keeping money in reserves (hoarding) is then defined as those neither investing nor lending $\left(1 - s_t^b - s_t^l\right)$. Multiplying these shares by the total funds of the banking family, we get the respective amounts of credit, borrowing, lending, and hoarding.

Interbank Market Clearing

For the interbank market to clear, demand should be equal to supply. Each borrower demands $\Lambda_t^i = \lambda_b$, and each lender supplies 1 if she finds the interbank market rate more attractive than alternative investments (hoarding or risky asset investment). So, market clearing is:

$$s_t^l = \lambda_b s_t^b$$

[15] switching to normal distribution does not allow us to have nice analytical solution, but makes shock structure more intuitive and simplifies filtering problem.

Supply on the interbank market is the share of lenders, as they supply all their available funds. Demand is then the share of borrowers multiplied by the funds demanded – λ_t^b. Plugging in the expressions for the shares, one can re-write the market clearing condition as:

$$\int_{E_t^l \hat{R}_{t+1}^k}^{E_t^m \hat{R}_{t+1}^k} f(x)\,dx = \lambda_b \int_{R_t^{ib}}^{+\infty} f(x)\,dx \tag{1.17}$$

where $f(x)$ is a normal density function. Alternatively:

$$F_{m,\sigma_R^2}\left(E_t^m \hat{R}_{t+1}^k\right) - F_{m,\sigma_R^2}\left(E_t^l \hat{R}_{t+1}^k\right) = \lambda_b \left(1 - F_{m,\sigma_R^2}\left(R_t^{ib}\right)\right)$$

The banks' mean beliefs and their variance enter (1.17) as the moments for the cumulative density function. Then the variance enters the definition of the marginal investor and the marginal lender: $E_t^m \hat{R}_{t+1}^k = p_t^m R_t^{ib}$ and $E_t^l \hat{R}_{t+1}^k$ such that $p_t^l R_t^{ib} = R_t^{res}$. Combining these two definitions and (1.17) one gets a solution for the interbank market rate and the corresponding amount of lending.

Bank's Net Worth and the Financial Accumulator

Similarly to Gertler and Karadi (2011) we define the bank's net worth as N_t^i, and B_t^i are the deposits from households. In our model, however, banks can invest in three types of assets: risky, safe (hoarding), and interbank loans. So, the bank's balance sheet in our model is given by:

$$Q_t S_t^i + Res_t^i + Lend_t^i = N_t^i + B_t^i, \tag{1.18}$$

where $\frac{Q_t S_t}{N_t^i + B_t^i} = \alpha_t^i$, $\frac{Res_t^i}{N_t^i + B_t^i} = h_t^i$, and $\frac{Lend_t^i}{N_t^i + B_t^i} = (1 - \alpha_t^i - h_t^i)$ is the realized return from lending on the interbank market for the lender or $\frac{Lend_t^i}{N_t^i + B_t^i} = -\lambda_b$ for the borrower in (1.13). Given each asset return, the evolution of a bank's net worth over time can be formulated as

$$N_{t+1}^i = R_{t+1}^k Q_t S_t + R_t^{ib} Lend_t^i + R_t^{res} Res_t^i - R_t B_t^i. \tag{1.19}$$

Note, that for a borrower term $R_t^{ib} Lend_t^i$ is negative and is equal to $R_t^{ib}(-\lambda_b N_t^i)$.

As the agency problem is a slight modification of that in Gertler and Karadi (2011), we put the solution in Appendix D and here present the resulting constraint:

$$(Q_t S_t + Res_t) = \frac{\eta_t}{\lambda - \upsilon_t \left(1 - s_t^h\right)} N_t = \varphi_t N_t, \tag{1.20}$$

where φ_t is the banking sector leverage ratio, and ν_t and η_t are described in Appendix D.

To finalize the law of motion for banks' net worth, recall that in each period a fraction $(1-\theta)$ of the bankers exit and take a $(1-\theta)$ share of the banking family's assets. At the same time, households transfer a fraction $\frac{\omega}{1-\theta}$ of the exit value to the new bankers. That is, the law of motion of banks' net worth is given by:

$$N_{t+1} = \theta \left\{ \left[\left(1 - s_t^h\right) \left(R_{t+1}^k - R_t\right) + s_t^h \left(R_t^{res} - R_t\right) \right] \varphi_t + R_t \right\} N_t + \omega \left(Q_t S_{t-1} + Res_{t-1}\right). \tag{1.21}$$

Credit Support Policies

We consider several credit support policies. Under the first two, the central bank funds asset purchases through intermediaries. The untargeted liquidity provision is modeled as the funding of a share ψ_t of banks' asset purchases:

$$Q_t S_t + Res_t = \varphi_t N_t + \psi_t \left(Q_t S_t + Res_t\right).$$

For targeted credit support, the central bank limits the set of assets to be purchased to risky claims on firms. Let ψ_t^{tar} denote the fraction of risky assets funded by the central bank. Then

$$Q_t S_t + Res_t = \varphi_t N_t + \psi_t^{tar} Q_t S_t.$$

A bank pays R_t for central bank support. One can think of this support as being financed through selling government debt to households paying R_t, so that it does not appear in the government budget constraint. There are, however, operational costs of conducting the policy, $\tau\psi_t \left(Q_t S_t + Res_t\right)$ or $\tau\psi_t^{tar} Q_t S_t$. We assume that both policies are equally costly. As in Gertler and Karadi (2011) the central bank selects ψ_t and ψ_t^{tar} as a proportion of the rise in the risk premium. When there are disturbances in the economy, the risk premium rises above the steady-state level.

$$\psi_t = \kappa \left(R_{t+1}^k - R_t - (\overline{Rk - R})\right), \tag{1.22}$$

where κ is a reaction parameter.

We further consider relaxing the collateral constraint on the interbank market and lowering the real gross return on the safe asset, R_t^{res}, both of which policies involve no operational costs. Relaxing the collateral constraint takes the form of increasing the fraction of borrowers' liabilities up to which borrowing is restricted – $\lambda_{b.}$ An increase in this fraction, denoted as ∇_t^{λ}, and a reduction in R_t^{res}, ∇_t^R, follow the same decision rule as the two previous policies considered:

$$\nabla_t^i = \kappa^i \left(R_{t+1}^k - R_t - (\overline{Rk - R}) \right),$$

where i stands either for λ or for R^{res}. We allow for a different feedback parameter κ^i in the rules.

1.3.2 The Household

There is a representative risk-averse household in the economy which has utility from consumption and disutility from labor. The household solves the following problem subject to a budget constraint:

$$\max_{C_t, L_t, D_t} E_t \sum_{i=0}^{\infty} \beta^i \left[\ln(C_{t+i} - hC_{t+i-1}) - \frac{\chi}{1+\varphi} L_{t+i-1}^{1+\phi} \right] \tag{1.23}$$

$$\text{s.t. } C_t + B_t = W_t L_t + R_{t-1} B_{t-1} + \Pi_t + T_t, \tag{1.24}$$

where C, L, B, and T stand for consumption, labor supply, deposits in banks, and tax, respectively. W and R are the real wage and the real gross return on bank deposits. Π_t is net transfers from financial and non-financial firms to the household. β, ϕ ,$\chi > 0$, and $\beta < 1$.

Bank deposits are guaranteed by the government, which, in the case of bank insolvency, pays the deposits and interest to the household.

The first-order conditions (see Appendix F) state that the marginal disutility of labor is equal to the marginal utility of consumption and that the nominal return on bank deposits should, at the margin, compensate the consumer for postponing consumption to the next period.

1.3.3 Intermediate Goods Producers

The sector is perfectly competitive. Producers combine labor and capital using the Cobb-Douglas production function:

$$Y_t = A_t \left(U_t \xi_t K_{t-1} \right)^{\alpha} L_t^{1-\alpha}, \tag{1.25}$$

where K_{t-1} stands for capital, L_t stands for labor, and A_t is total factor productivity. U_t is the utilization rate of capital. That is, shock ξ_t influences effective capital.

Investment in capital should be made one period in advance. In other words, to produce in period $t+1$ the investment should be made in period t. To invest in the next period's capital, K_t, intermediate goods producers issue claims S_t at price Q_t^S. The value of the capital they can buy at price Q_t^K is then $Q_t^K K_t = Q_t^S S_t$. In the next period, intermediate goods producers sell the depreciated capital to capital producers at the market price Q_{t+1}^K. Because of the perfect competition among intermediate goods producers, the price of capital equals the price of producers' claims: $Q_t^K = Q_t^S \equiv Q_t$. The amount of depreciated capital is equal to $\delta_t\left(U_t\right)\xi_t K_{t-1}$, where δ_t is the physical depreciation rate and ξ_t reflects the capital quality shock discussed above. At $t+1$, the firm pays a gross return $R_{k,t+1}$ to the bankers per each unit of investment. As firms are identical, investment in capital pays the same return to all banks.

In each period, an intermediate goods producer chooses labor demand and demand for capital to maximize its current and next-period profits. Profit consists of the revenues from production and the resale value of the depreciated capital net of payments on claims S_t and labor costs. The price of a unit of the intermediate good is $P_{m,t}$, the cost of replacing used capital is unity, and the cost of buying new capital is Q_t. The producer then chooses the utilization rate and labor demand as:

$$\begin{aligned}(1-\alpha)\frac{P_{m,t}Y_t}{L_t} &= W_t \\ (\alpha)\frac{P_{m,t}Y_t}{U_t} &= \delta'\left(U_t\right)\xi_t K_{t-1}.\end{aligned}$$

As firms make zero profit, they distribute the return on capital to holders of their claims:

$$R_t^k = \frac{\left(\alpha\frac{Pm_tY_t}{K_{t-1}} + Q_t - \delta\left(U_t\right)\right)\xi_t}{Q_{t-1}}. \tag{1.26}$$

The first-order conditions determine the expected return on the firm's claim as the expected value of the marginal product of capital plus the expected resale value of the capital divided by the price of the claim. The wage is then determined by the marginal product of labor. The price of the intermediate good equals the marginal costs.

1.3.4 Capital-Producing Firms

Capital goods producers are competitive firms. They buy depreciated capital from intermediate goods producers and renovate it at the unit costs and sell it at the unit price. They also produce new capital and sell it at price Q. There are no adjustment costs for renovating worn-out capital, but there are flow adjustment costs when producing new capital. Capital producers are risk neutral and maximize the following utility (first order conditions are in Appendix F):

$$\max_{In_t} E_t \sum_{k=t}^{\infty} \beta^{T-k} \Omega_{t,k} \left((Q_k - 1) I_{nk} - f\left(\frac{In_k + I_{ss}}{In_{k-1} + I_{ss}} \right) (In_k + Iss) \right), \tag{1.27}$$

where In is net investment, defined as $In_t \equiv I_t - \delta(U_t)\xi_t K_{t-1}$, where $\delta(U_t)\xi_t K_{t-1}$ is the quantity of renovated capital. Iss is steady-state investment and Q_t is the price of capital. Function f is an investment adjustment cost function satisfying the following properties $f(1) = f'(1) = 0$ and $f"(1) > 0$.

1.3.5 Final Goods Producers (Retailers)

Retailers combine output from intermediate goods producers using the production function:

$$Y_t = \left[\int_0^1 Y_{ft}^{\frac{\varepsilon-1}{\varepsilon}} df \right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{1.28}$$

where Y_{ft} is composite goods output from retailer f and ε is the elasticity of substitution. We follow the Calvo-pricing convention and each period allow only a fraction γ of firms to optimize their prices. The solution is in the Appendix F.

1.3.6 The Government and the Central Bank

The government collects lump-sum taxes from households, T_t, and accepts reserves (the safe asset), Res_t. It also bears some costs of conducting policy, Po_t. The government's budget constraint is satisfied when the following holds:

$$G_t + Po_t = T_t + Res_t - R_t Res_{t-1} \tag{1.29}$$

The resources in the economy are then distributed between consumption, investment, and government expenditure on policy:

$$Y_t = C_t + I_t + f\left(\frac{In_t + I_{ss}}{In_{t-1} + I_{ss}}\right)(In_t + Iss) + G_t + Po_t \tag{1.30}$$

The central bank conducts monetary policy according to the simple rule:

$$i_t = (1 - \rho_i)\left(i + \kappa_\pi \pi_t + \kappa_y(\log Y_t - \log Y_t^*)\right) + \rho_i i_{t-1} + \epsilon_t \tag{1.31}$$

where Y^* is flexible output, ε_t is an exogenous monetary policy shock, and i is the steady-state nominal rate. ρ_i is a smoothing parameter lying between zero and one. The real and nominal interest rates are linked via the Fisher equation: $1 + i_t = R_t E_t\left(1 + \pi_{t+1}\right)$

1.4 Calibration and Simulations

To compare our results with the literature, where possible we follow the calibration choices of (Gertler and Karadi 2011); we list their parameter choices in Table 1.2 in Appendix 1.F. There are, however, some parameters specific to our model: σ_R^2, λ_b, ω, and $\bar{E}\hat{\xi}$. We set average expectations of the capital quality shock, $\bar{E}\hat{\xi}$, to be equal to the steady-state value of $\bar{\xi} = 1$. The variance of banks' forecasts and the dispersion between them, σ_R^2, the collateral constraint, λ_b, and the transfer to new entering bankers, ω, are meant to be suggestive. We set their values to match the following pre-crisis data: the interbank market rate, the share of interbank loans in banks' portfolios, and the share of loans in banks' portfolios. In our model, banks exchange loans for one period on the interbank market, with the period being a quarter. Therefore, the 3-month Euribor is a natural choice for the empirical counterpart for the model interbank rate. The share of interbank loans resembles the share of lenders in our model and is calculated as the ratio of euro area banks' loans to monetary and financial institutions to total assets. Similarly, the share of loans is the ratio of total loans to the total assets of European banks. Total loans include loans to enterprises and to other banks. In our model, what is not lent either to banks or to firms is hoarded. That is, the share of loans is useful for calculating the share of hoarded assets as (1 - share of loans).[16] We choose the parameters to roughly match

[16] In our model, reserves represent safe assets, but in reality there are a number of assets that can be considered "safe." Consequently, we cannot use the amount of reserves as an empirical counterpart for hoarding.

Table 1.1: Calibrated Parameters Specific to our Model

ω	0.0059	proportional transfer to entering bankers
λ_b	0.24	collateral constraint on interbank market
σ_R^2	0.1	variance of return expectations
σ_v^2	0.001	variance of persistent shock to capital quality
σ_η^2	0.5	variance of expert opinion shock
σ_e^2	0.03	variance of capital quality transitory shock
$\sigma_{\varepsilon\eta}$	0.1	covariance of errors in econometric and expert forecasts
ρ_θ	0.66	persistence of expert opinion shock
ρ_ξ	0.66	persistence of capital quality shock
ρ_μ	0.66	persistence of persistent shock to capital quality
ρ_c	0.62	correlation of experts' opinion
κ^λ	2.5	policy reaction for collateral constraint
κ^R	0.1	policy reaction for reserve rate

an interbank rate of 1.31%, a share of hoarded assets of 40%, and a share of assets lent on the interbank market of 20%, and choose σ_R^2, λ_b, and ω to be 0.1, 0.24, and 0.0059, respectively. The parameters for Kalman filter updating, σ_v^2, σ_η^2, σ_e^2, and $\sigma_{\varepsilon\eta}$, are set to match the steady-state variance of the bank's forecast, σ_R^2. Recall that in our model, ξ is subject to two shocks: a persistent one and transitory one: $\xi_t = \rho_\xi \xi_{t-1} + \mu_t + \varepsilon_t$, so the variance of ξ, $\sigma_\xi^2 = \left(\sigma_\nu^2 + \sigma_\varepsilon^2\right)/(1-\rho_\xi^2)$. At the same time, σ_R^2 is a function of σ_ξ^2, as shown by (3.14). That is, the steady-state value of σ_R^2 defines the sum of the variances of the persistent and transitory shocks, which are set at 0.001 and 0.03, respectively. We choose the variance of the expert opinion shock, σ_η^2, so that the share of expert adjustments in the final forecast lies within the bounds defined by the literature on forecasting.[17] We set σ_η^2 to be equal to 0.5, and the resulting share of expert adjustments is then 0.37. The persistence of the capital quality shock is $\rho_\xi = 0.66$, as in (Gertler and Karadi 2011). We set the persistence of all other shocks to be the same, at 0.66 for both the persistent shock and the expert opinions shock.[18] The policy reaction parameter for the reserve rate is set to match the decline in the real policy rate during 2008–2009 relative to the pre-crisis 10-year average: the resulting deviation from the steady state during the crisis is a fall of 0.23 percentage points. The collateral constraint in our model does not have an intuitive empirical counterpart. The value is set for illustrative purposes and alternative values are

[17]For example, (Fildes et al. 2009) analyze a data set containing 70,000 business organizations and their forecasts. They find the mean expert adjustment for monthly forecasts to vary between 18% and 46% depending on the type of business.

[18]In previous research – (Audzei 2012) – we calculated the persistence of the expert opinion shock using the SPF GDP forecast. The resulting value was very close to the current calibration – 0.61.

discussed. The parameters are listed in Table 1.1. With the parameters described above, we then proceed with an analysis of the linearized model and its performance relative to Gertler and Karadi (2011). The model is simulated using Dynare 4.

1.4.1 Defining a Crisis

We consider a crisis to be a transitory shock to capital quality, ξ_t. To make the dynamics of our model comparable to the literature, we consider a crisis to be a 5% decline, as in Gertler and Karadi (2011). They set this value to match a 10% decline in the effective capital stock over a two-year period. We consider several types of crises: a drop in ξ_t of 5%, a drop in ξ_t of 5% combined with banks believing that this was a permanent shock,[19] and an unchanged ξ_t with banks believing in a 5% drop in the persistent component of ξ_t. In other words, we consider a crisis without an expectational shock, a crisis with an expectational shock, and a pure expectational shock, respectively. In the simulations without an expectational shock, banks observe a change in ξ_t, but they do not have perfect information on how persistent this change is. They use past and current observations via the Kalman filter to predict ξ_{t+1}. In the simulations with an expectational shock, in addition to a change in ξ_t, banks get a "pessimistic shock": experts start to believe that a persistent shock has occurred. These expert opinions are combined with past observations again via the Kalman filter.

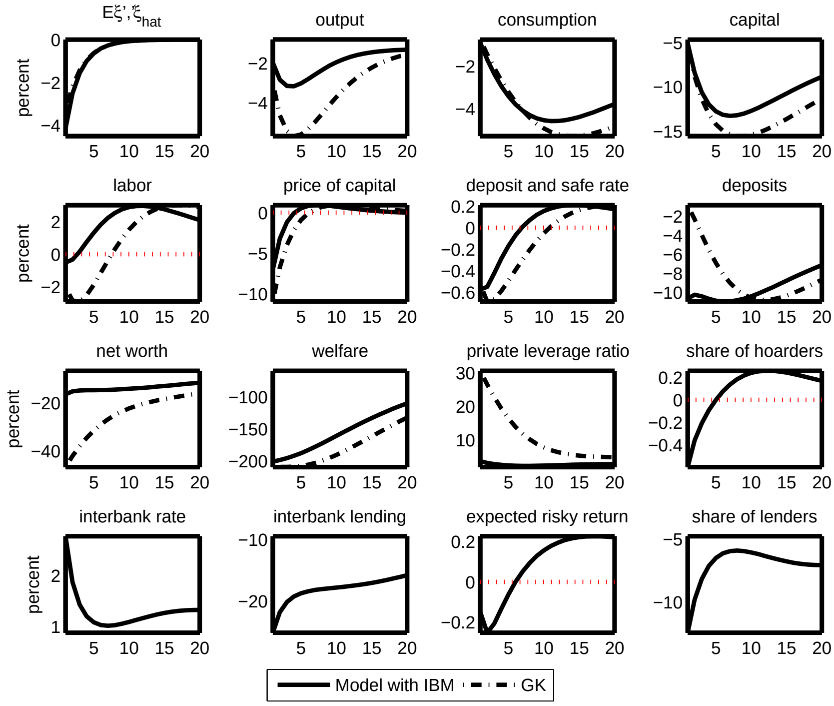
1.4.2 The Role of Expectations and the Interbank Market

In our economy, expectations determine credit to the real sector. They also affect the functioning of the interbank market: the numbers of borrowers and lenders and the equilibrium interbank market rate. As was shown in the simple model case, when market expectations become too low, this results in an interbank market crunch, with no lending occurring. In the general equilibrium model, we consider model responses linearized around the steady state with a functioning interbank market. A decrease in market expectations then results in lower credit supply and lower lending between banks.

If banks have "rational expectations" as in Gertler and Karadi (2011), then there is no interbank market and our model would have identical responses. For this reason, we treat Gertler and Karadi (2011) as a baseline to study the role of expectations and the

[19] This is modeled as a shock to banks' average belief about the drop in the persistent shock. Recall that in the model this is the average belief that matters for the simulations.

Figure 1.5: Crisis Simulations



interbank market. Let us start our analysis with a comparison of the model behavior and the baseline when there is no policy response, and crisis shock is only a shock to ξ_t. In this scenario, the policy rate, R_t^{res}, is set to be equal to the deposit rate so that banks earn nothing on a safe asset. Figure 1.5 shows the responses of our model and the Gertler and Karadi (2011) model [20] where applicable.[21] In period 1 there is a 5% temporary shock to ξ_t. Because ξ_t is itself a persistent process, it remains below the steady-state value for about ten periods. The first subplot shows the expectations about ξ_{t+1}. In a model with perfect expectations, this will be $\overline{E}_t\xi_{t+1} = \rho_\xi * \xi_t$, resulting in a decline of 3.3% in the first period. It coincides with the decline in the ξ_{t+1} in the Gertler and Karadi (2011) model. However, in our model it is not observable whether this was a transitory or a persistent shock. Recall that agents combine two observables to form their forecast: historical values of ξ and expert opinions about the value of the persistent shock to ξ_t. Even when the experts consider the persistent shock to be zero, analysis of historical observations leaves a possibility for it to exist. Consequently, even in a model where expert opinions are not disturbed, the future values of ξ_t are underestimated.

[20] Welfare is calculated as in Gertler and Karadi (2011) as a second order approximation of household utility.

[21] Obviously, such variables as the interbank market rate or safe asset holdings do not exist in their paper.

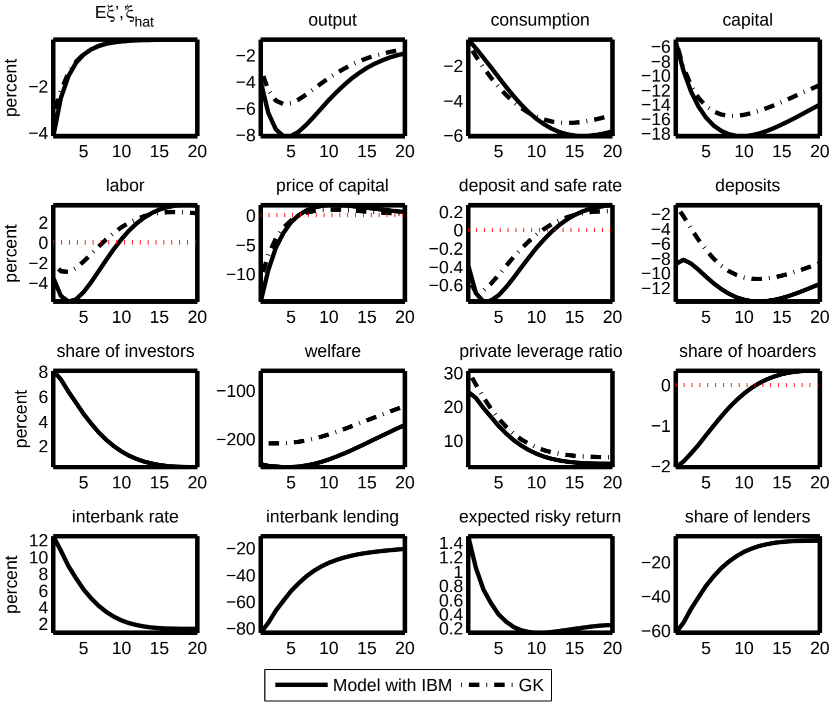
When ξ_t is hit by a shock, there is an immediate decline in the contemporaneous return on banks' investment – R_t^k, as in (1.26). This lowers banks' returns and has a negative effect on their net worth. With a lower net worth of banks, current net investment falls and the price of capital decreases. Because capital and, accordingly, net investment fall, there is less demand for capital and capital producers sell it at a lower price. Price of capital also reflects the resale value of capital. Consequently, its fall contributes to a further decline in banks' worth. A fall in the net worth leads to a decline in both safe and risky asset holdings. With smaller net worth, banks are able to attract fewer deposits from households, as their deposits are limited to a fraction of their net worth through the agency problem. Hence, net investment falls even more, as banks simply have less funds for it.

With investment falling, the return on it rises. The return on the banks' investment is the sum of the marginal product of capital and the resale value of capital. Although the resale value is low in a crisis, the marginal product is large. This leads to a higher expected return on the risky asset (albeit smaller than the actual future return) and a larger share of investors. Banks would like to invest more with such returns, but their investment is already reduced by the fall in net worth.

Note that in the simple model, a fall in expectations results in lower lending and a lower interbank market rate. The crisis in a general equilibrium results in a fall in lending, but the interbank rate rises. This is because of the feedback from banks' investment to the risky asset and the risky asset return. The share of lenders shrinks because more banks would like to invest themselves. This puts upward pressure on the interbank rate, which therefore rises.

Now compare Gertler and Karadi (2011) and our model with no expectational shock. In our model, banks do not invest all their funds in the risky asset, but leave some share in the safe one. Consequently, only a proportion of banks' net worth is affected by the fall in the risky asset return. Our law of motion of net worth (1.21) is similar to the dynamics of N_t in Gertler and Karadi (2011), but the term $\left(R_t^k - R\right)$ is also multiplied by the share of banks investing in the risky asset. As a result, our model demonstrates almost half as large a fall in investment and net worth compared to Gertler and Karadi (2011). In table 1.3 in Appendix 1.G we compare the standard deviations resulting from both models to the standard deviations observed in the data. When it comes to the net worth of the financial intermediaries, our model captures the deviation rather close and better than the benchmark. This is not surprising as banks in reality have a

Figure 1.6: Crisis Simulations Comparable Net Worth



more diversified portfolio than those in the benchmark model holding the risky asset only. The fall in capital and output is smaller in our model. Our banks, though, experience a smaller drop in net worth.

Because the baseline model features only a capital asset and demonstrates a larger fall in the net worth, to isolate the role of the interbank market, we simulate our model controlling for the difference in the net worth.[22] Such a comparison is shown in Figure 1.6. With the net worth in our model decreased as much as in the baseline, the recession is larger in our model, where the interbank market serves as a propagation mechanism. The primary effect is borrowers having less net worth, so they can borrow less. This lowers the demand for capital and the price of capital. The increasing (due to a fall in K) expected risky return, which is return on capital, attracts some lenders to become direct investors themselves, pushing the interbank rate up. This drives even more borrowers out of the market. The higher interbank rate attracts some of the hoarders to become lenders, but it does not offset the fall in lending. An increase in investors' set is responsible for

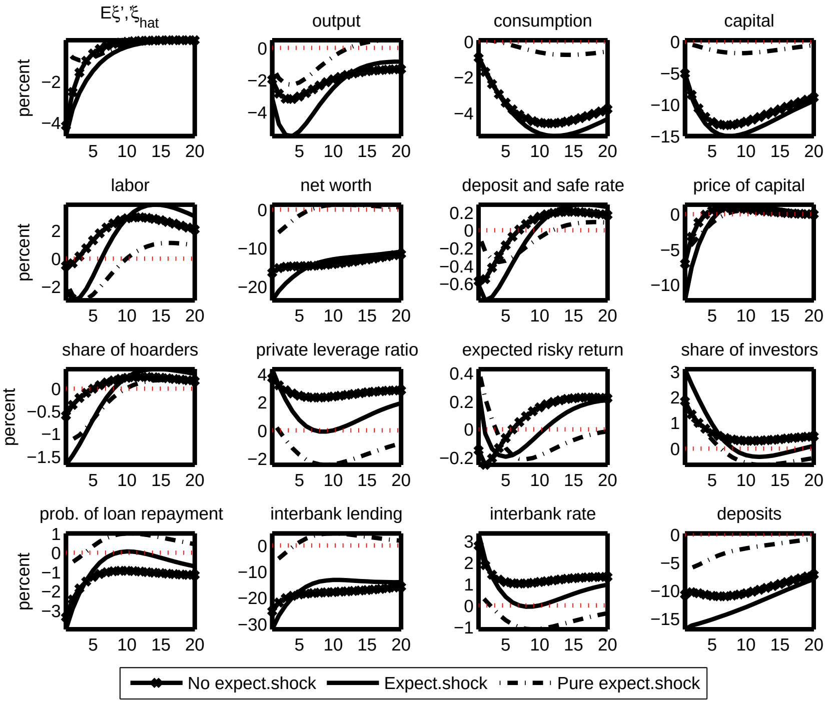
[22]For this purpose we first add a shock to the net worth equation to achieve the initial drop in the net worth of the same magnitude as in the baseline. This shock is a scaled shock to ξ. By doing this, we obtain policy functions of the model, reflecting the stronger reaction of variables to ξ shock. With these policy functions we calculate impulse responses while substituting values for the net worth from the baseline in the state variable. This exercise produces responses of the model as if the banks' net worth falls the same as in the baseline, but the model structure is unchanged.

a very similar initial drop in capital in both models. However, the private leverage ratio is smaller in our model as the expected gross rate of banks capital is lower (note that due to the agency problem between households and banks, this ratio depends on banks' continuation value, which is return on banks own capital). This is because only a share of banks invests in the risky asset (with higher return than the safe asset), and also due to slightly suppressed expectations of ξ, the return is underpredicted. That is why there are lower deposits, lower safe rate (and lower return on a safe asset), and lower labor and output.

Falling output contributes to a decline in the expected risky asset return despite the continuing fall in capital and growth of ξ. When the expected risky return falls, more bankers become hoarders, which reduces the expected return on banks' capital and leverage ratio, and deposits decline even further. The share of investors falls but is still above zero, with the share of hoarders rising over the steady state as the safe rate grows making hoarding more attractive. Thus the set of lenders grows slower, contributing to a high interbank rate. As the share of hoarders starts increasing, the deposits fall even more, adding to the continuing decline in capital. The fall in capital accumulation contributes to a further decline in output, consumption and welfare. That is, the interbank market serves as a shock amplifier: as lenders are those who would not invest themselves, but are willing to lend their funds to investors, a change in the set of lenders amplifies a fall or rise in credit to the real economy; when lenders hoard due to unfavorable interbank market conditions (low return on a risky asset and high counterparty risk), this leads to a larger fall in the real economy credit. This is visible in Figure 1.6 after period 10. In a model without the interbank market the capital returns to the steady state much faster. The resulting decline in output is more than 1.5 times more than in the baseline, being the result of both imperfect expectations and the interbank market adjustment to the shock.

Now consider the effect of different crisis shocks in our model : "fundamental" to ξ only, pure expectational shock to mean experts' beliefs about a permanent component of ξ, and a combination of these two shocks. A comparison of all shocks is shown in Figure 1.7. A model without expectational shock is described in Figure 1.5. Apart from 5% decline in ξ the model with expectational shock features a wave of pessimism among the investors, i.e. a 5% persistent decline in the average banks' prediction about μ_t. The pure expectational shock corresponds to the simulations in which only a pessimistic wave hits the economy, without an actual drop in ξ.

Figure 1.7: Crisis Simulations with and without Expectational Shocks



Comparing the responses to a crisis with and without the expectational shock, note that the expected risky asset return falls only without expectational shock because of a smaller decline in capital. Under the pure expectational shock, the expected risky asset return increases as the capital quality is not disturbed by the shock, but the capital investment declines. That increases the marginal return on capital enough to compensate for the pessimistic forecast of ξ. The difference in net worth is explained mostly by the price of capital as the shock to the actual ξ_t is the same. That is, net worth falls the most in the model with expectational shocks. Net worth affects banks' ability to attract deposits and influences the deposit rate. Smaller deposits result in a lower interest rate on them, making household savings less attractive. In the model, there are no frictions on the labor side, so it is labor that adjusts, with the fall in consumption being similar under both scenarios. Output falls in response to the fall in capital and labor, with the drop being twice as large as in the scenario with the expectational shock.

When a pure expectational shock hits the economy and there is no actual drop in ξ_t, banks underestimate ξ_t for some period of time. This generates a decline in net investment, a decrease in the price of capital, and a fall in the current return on capital, followed by a decline in net worth. Capital falls initially by 0.1%. The decline in net worth accelerates the fall in capital in the following periods, with the maximum decline being 1.244%. That is, a persistent pessimistic shock can generate a small recession, as investment falls, leading to a decline in output and consumption.

The probability of loan repayment (evaluated for marginal lender) reflects the changes in expected risky return and deposit rates, as borrowers' returns on the interbank market have to be enough to compensate for obligations to both household and lenders. Therefore the probability of loan repayment is largest with pure expectational shock and smallest with expectational and ξ shock. The drop and subsequent increases translate then into the interbank rate and market allocations.

Thus, if the crisis is interpreted as a combination of a shock to ξ and a shock to agents' expectations, the resulting responses look like the sum of the pure expectational shock and no expectational shock scenarios. Then the crisis of the observed magnitude could be simulated with a smaller size of the "fundamental" shock, depending on the size and persistence of a wave of pessimism. If one accepts the idea of sluggishness of investors' forecasts as in Coibion and Gorodnichenko (2015a) and Andrade and Le Bihan (2013), meaning overly pessimistic expectations after the crisis episodes, then our model with expectational shock could serve as an illustration of the crisis, generating the same 10% decline in capital over 2 years as in Gertler and Karadi (2011), but with the banks' net worth deviations matching the data due to a more realistic asset structure.

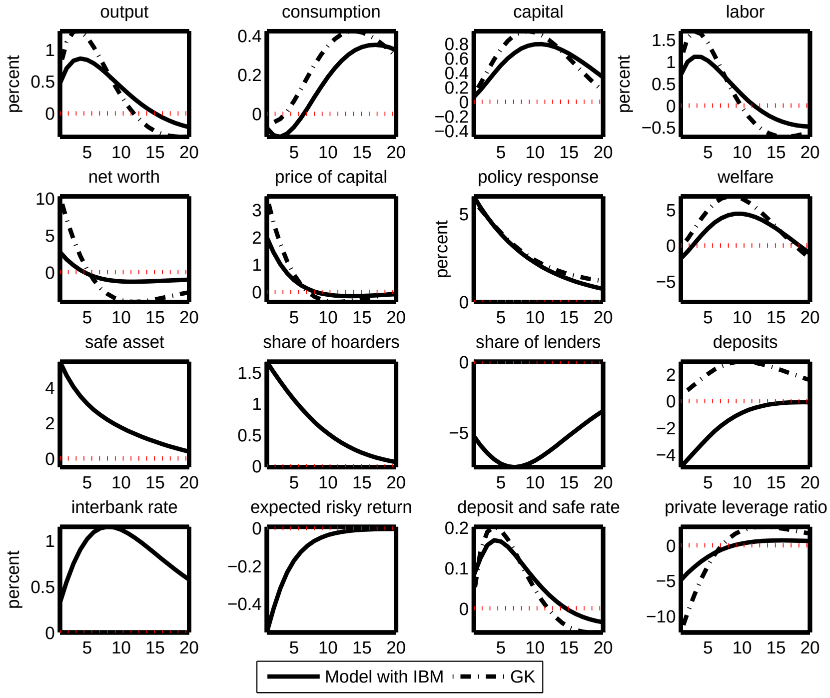
To summarize, the expectational shock alone can generate some need for a policy response by the central bank. Combined with the occurrence of an actual crisis, this leads to a more severe recession and a larger policy response. That is, investor sentiment can be an important factor for policy design and evaluation. Without the expectational shock, our model predicts a milder recession than Gertler and Karadi (2011), as banks in our model have an opportunity to diversify their assets and are thus less impacted by the crisis. With the expectational shock, our model has similar predictions to the baseline regarding the dynamics of output, capital, labor, and consumption.

1.4.3 Policy Results

The main difference from the simple model is the presence of feedback from expectations to asset prices and net worth, and from investment to the risky asset and the return on it. These differences will influence the model response to the policy actions described in the following section.

We start our analysis again with a comparison of the baseline model of Gertler and

Figure 1.8: Policy Effects vs Baseline



Karadi (2011) and our model with the interbank market without expectational shock. The comparison is complicated due to the different capital structure. If we control for the difference in the net worth as was done in Figure 1.6, then the crisis is much deeper in our model. Liquidity provision of the same scale leaves our economy in a worse recession than in the baseline model, but the effect relative to the simulation without policy is larger due to the larger initial drop. Therefore, for the comparison, we choose our crisis simulation with expectational shock. In this scenario, the economy is hit by a shock to ξ_t and a wave of pessimism. The resulting simulation gives a very similar drop in output and capital as in the baseline (note that the crisis there is only a shock to ξ_t), but the fall in the net wealth is two times smaller. However, the optimal policy rule as in (1.22) would be different in our model, requiring a larger policy response in our case. For this reason, we simulated our model using a vector of policy responses similar to the baseline. We present the comparison in Figure 1.8, where we plot the difference in variables with and without policy ($x = x_{policy} - x_{nopolicy}$). As the figure shows, the policy effects in our model with interbank market are lower lower and delayed. Moreover, the policy even has a negative effect on deposits and interbank market holdings. The policy increases the share of hoarders by decreasing the expected return on capital and increasing the safe rate. As more lenders leave the market to become hoarders, the interbank rate rises, depressing lending. As there are fewer investors (and lower return on banks' aggregate capital), the private leverage ratio falls together with deposits. In a certain sense, the policy "crowds out" interbank lending and borrowing from the households. Also note that the safe asset holding is increased by almost 6 percent, while capital asset less than 1 percent. This hoarding effect undermines the impact of the liquidity provision in our model.

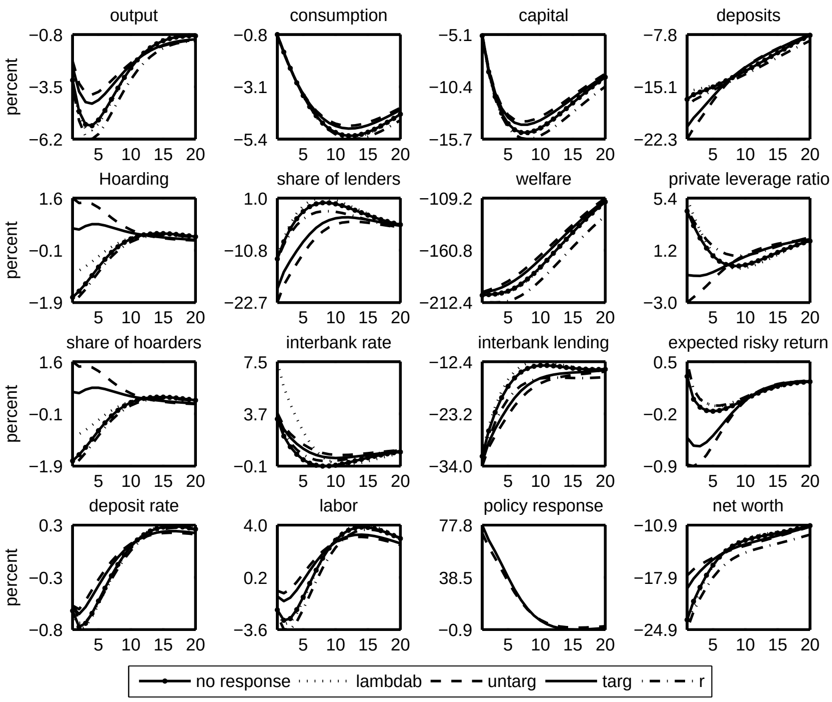
For an analysis of different policies in our model, we consider a "crisis" shock defined as a decline in capital quality, ξ_t, in combination with a wave of investor pessimism. The results are presented in Figure 1.9. The policy exercises without the expectational shock are presented in Appendix E and have the same qualitative results. In the latter case, the recession is smaller and so are the differences between policies.

First, consider the two types of liquidity provision: targeted, solid line; and untargeted, dashed line. The two policies have a very similar effect on output, consumption, and capital, and mitigate the crisis relative to the simulation with no policy response. The policy response is the total amount of funds supplied to banks – $\psi\left(QK + Res\right)$ and $\psi^{tar}\left(QK\right)$ for untargeted and targeted policies, respectively. The main difference between the two policies is in the share of hoarded assets, $Hoarding$, which is almost twice as large in the case of the untargeted policy. The absolute holdings of a safe asset are also almost twice as large under the untargeted policy. These predictions are in line with the simple model results: the liquidity provision helps restore credit to the real economy, but also increases reserve holdings.

With targeted credit support, banks expect a share of their risky asset purchases to be financed by the central bank. For those with high expectations about the risky asset return, this means less need to borrow from households and on the interbank market. Note that this is only true for banks with high return expectations. In the case of untargeted liquidity provision, where both assets are financed by the central bank, all banks have less need for household funding. Banks' deposits therefore fall more with untargeted support, together with the private leverage ratio and deposit rate. This explains the small differences in output and labor supply.

Thus, the two policies have a very similar effect on capital and output (the effect could be even more similar if the model featured labor rigidities), but the untargeted policy results in larger safe asset holdings. Interest rate policy is the least efficient policy in our simulations. It is modeled as a decline in the reserve rate, R^{res}, below the deposit rate R, meaning that banks are making negative returns on their reserves. In line with the

Figure 1.9: Policy Effects



simple model results, such a policy reduces the share of hoarders as the safe asset becomes less attractive, and lowers the share of hoarded assets in banks' portfolios. However, it reduces banks' net worth, leading to a large drop in investment. This drop in banks' net worth leads to even worse outcomes than in the case of no policy action.

Relaxing the collateral constraint on the interbank market by raising λ_b allows borrowers to borrow a larger fraction of their net worth. The larger demand for interbank credit drives up the interbank market rate, reducing the number of banks willing to borrow. Thus, there are fewer borrowers on the market, but they borrow more. As a result, interbank market lending increases. The high interbank market rate makes interbank lending more attractive relative to risky asset investment, and thus some potential investors become lenders on the interbank market and the share of those investing in the risky asset falls. As a result, despite the larger volume on the interbank market, credit supply to the real economy is almost unchanged, as are safe asset positions.[23]

To conclude, liquidity provision policies help mitigate the simulated crisis. However, relative to the baseline model of Gertler and Karadi (2011), our model with imperfect

[23]In alternative simulations we considered different response parameters for relaxing the collateral constraint: from 0.4 to 2.5. The difference in the output and capital responses is negligible. In addition, with stronger easing, there is less investment.

information and a storage asset displays low efficiency of liquidity provision, with higher costs, delayed responses and liquidity hoarding. The policy of targeted and untargeted liquidity provisions have a very similar effect in the general equilibrium content because of the feedback from household deposits and labor, with the latter policy resulting in larger hoarding. The policy of low reserve rates makes hoarding less attractive but has a negative impact on banks' net worth, leading to worse outcomes than in the case of no policy.

1.5 Conclusion

In this paper we address the role of imperfect market expectations in interbank lending and in amplifying economic fluctuations. In particular, we show that in addition to a liquidity shortage, assessment of counterparty risk can be one of the factors contributing to a credit crunch. In a simple finite-horizon model, we consider an expectations-driven credit crunch and show that policy effects in this case are very limited.

We then develop a linearized DSGE model with interbank lending and consider responses around the steady state. To study market expectations, we incorporate a heterogeneous banking sector with a continuum of risky asset return expectations. The heterogeneity of expectations gives rise to an interbank market where lenders take into account the possibility of a borrower failing to repay the loan. Imperfect information among the bankers results in higher assessment of counterparty risk after crisis episodes, as bankers are not sure how persistent the negative shock is. The interbank market serves as a shock propagating mechanism as a set of lenders shrinks.

To study how imperfect expectation and/or waves of pessimism amplify crisis shocks we consider several types of crises: with and without pessimism shocks, and purely driven by pessimism. We show that even a pure pessimism shock alone can generate a small recession. Imperfect expectations result in an underestimation of investment opportunities even when the crisis is not accompanied by the waves of pessimism, because agents overestimate the persistence of the shock. The combination of a crisis and a wave of pessimism results in a crisis producing roughly a 10% decline in capital stock over two years (as documented in Gertler and Karadi (2011)).

We consider several types of central bank policy responses, including unlimited liquidity provision, targeted credit support, and varying the interest rate on reserves. Our model predicts that the efficiency of the policy depends on market confidence. Market pessimism dampens the positive effects of policies, making banks hoard central bank funds in reserves instead of transferring them through the bank lending channel. Compared to the model of Gertler and Karadi (2011), the policy effects in our model are smaller and delayed because we allow banks to hoard liquidity and additional propagation from the interbank market.

A low reserve rate in our model devastates banks' balance sheets and results in a worse recession than in the case of no policy response. Even though it stimulates the interbank market and increases the number of investors, the wealth effect dominates.

Appendix

1.A Derivations for the Simple Model

Interbank market clearing.

Proof. The market clearing condition for the interbank market with a uniform beliefs distribution is:

$$F_{m,\sigma_v^2}\left(E^m\hat{R}^k\right) - F_{m,\sigma_v^2}\left(E^l\hat{R}^k\right) = \lambda_b\left(1 - F_{m,\sigma_v^2}\left(R^{ib}\right)\right)$$

The cumulative distribution function for the continuous uniform distribution is $\frac{x-a}{b-a}$. Then the market clearing condition is rewritten as:

$$\begin{aligned} \frac{E^m\hat{R}^k - a}{b-a} - \frac{E^l\hat{R}^k - a}{b-a} &= \lambda_b\left(1 - \frac{R^{ib} - a}{b-a}\right) \\ \Rightarrow E^m\hat{R}^k - a - E^l\hat{R}^k + a &= \lambda_b\left(b - a - R^{ib} + a\right) \\ \Rightarrow E^m\hat{R}^k - E^l\hat{R}^k &= \lambda_b\left(b - R^{ib}\right) \end{aligned}$$

where b is the upper bound on the beliefs distribution, denoted as $\bar{R}$ in the text. □

Proposition 1. The necessary condition for the interbank market to exist is that there is a unique R_t^{ib}, solving

$$a * (R^{ib})^3 + b * (R^{ib})^2 + c * (R^{ib}) + d = 0, \tag{1.32}$$

that is real and non-negative.[24] With $a > 0$, $b < 0$, and $d > 0$, if a positive root exists, it is unique. This positive root exists only if:

$$R^{res} > A + \frac{R}{\lambda_b + 1} < 0, \tag{1.33}$$

where [25] $A < 0$.

The sufficient conditions for the equilibrium with the interbank market are:

$$R^{res} < R^{ib} < \frac{1+\lambda_b}{\lambda_b}\sigma\sqrt{3}$$

for $\sigma > \frac{\lambda_b R^{res}}{(1+\lambda_b)\sqrt{3}}$.

Proof. Denote $\Delta = b^2 - 3ac$ the disriminant of the cubic equation 1.32 and $rib^{(1)}$, $rib^{(2)}$ and $rib^{(3)}$ its 3 roots, where the first one is always real, and the rest can be real. With $a > 0$, $b < 0$ and $d > 0$, the first root is always negative. The last two roots are real and distinct from the first only if $\Delta > 0$. The condition for $\Delta > 0$ is: 1.33. If the parameters are such that $\Delta > 0$, the second root is always negative and the third one is always positive. Therefore, for the positive real solution to exist the necessary condition is 1.33.

However, for the interbank market to exist, the marginal lender's belief from $p^l R^{ib} = R^{res}$ should be smaller than $p^l R^{ib}$. Otherwise, the marginal lender invests herself and the set of lenders vanishes. Using (1.3) the marginal lender's belief can be rewritten as:

$$\begin{aligned}
\left(\frac{1}{2} - \frac{\left(R + \lambda_b R^{ib}\right)}{(1+\lambda_b)2\sigma\sqrt{3}} + \frac{E^l \hat{R}^k}{2\sigma\sqrt{3}}\right) R^{ib} &= R^{res} \\
&\text{or} \\
\left(\frac{1}{2} - \frac{\lambda_b R^{ib}}{(1+\lambda_b)2\sigma\sqrt{3}}\right) R^{ib} &= R^{res} - \left(\frac{E^l \hat{R}^k}{2\sigma\sqrt{3}} - \frac{R}{(1+\lambda_b)2\sigma\sqrt{3}}\right) R^{ib}
\end{aligned}$$

Because $R^{ib} > 0$, it follows that either 1) $\left(\frac{1}{2} - \frac{\lambda_b R^{ib}}{(1+\lambda_b)2\sigma\sqrt{3}}\right) > 0$ and $R^{res} - \left(\frac{E^l \hat{R}^k}{2\sigma\sqrt{3}} - \frac{R}{(1+\lambda_b)2\sigma\sqrt{3}}\right) R^{ib} > 0$ or 2) $\left(\frac{1}{2} - \frac{\left(\lambda_b R^{ib}\right)}{(1+\lambda_b)2\sigma\sqrt{3}}\right) < 0$ and

$R^{res} - \left(\frac{E^l \hat{R}^k}{2\sigma\sqrt{3}} - \frac{R}{(1+\lambda_b)2\sigma\sqrt{3}}\right) R^{ib} < 0$. However, in this case $\frac{1}{2} - \frac{\left(R+\lambda_b R^{ib}\right)}{(1+\lambda_b)2\sigma\sqrt{3}} < 0$, which violates $p^i > 0$ for small $E^i R^k$

[24] $a = \sqrt{3}\lambda_b(1 + \lambda_b)$, $b = -\lambda_b\left(\sqrt{3}(\lambda_b+1)m + 9\lambda_b\sigma + 3\sigma\right)$, $c = 6\sigma\left(\lambda_b\left(\lambda_b m + \sqrt{3}\lambda_b\sigma + m - R^{res}\right) + R - R^{res} - \sqrt{3}\sigma\right)$, and $d = 12\sqrt{3}(\lambda_b+1)R^{res}\sigma^2$

[25] $A = \frac{-3\left(3\lambda_b^3+7\lambda_b+6\right)\sigma^2 - \lambda_b(\lambda_b+1)^2 m^2 + 4\sqrt{3}\lambda_b(\lambda_b+1)m\sigma}{6\sqrt{3}(\lambda_b+1)^2\sigma}$

For 1)

$$R^{ib} < \frac{(1+\lambda_b)\sigma\sqrt{3}}{\lambda_b} \tag{1.34}$$

and

$$R^{ib} < \frac{(1+\lambda_b)2\sigma\sqrt{3}R^{res}}{(1+\lambda_b)E^l\hat{R}^k - R} \tag{1.35}$$

Suppose that $E^l\hat{R}^{k26} = R^{res} = R$, then the second inequality is

$$R^{ib} < \frac{(1+\lambda_b)2\sigma\sqrt{3}}{\lambda_b} \tag{1.36}$$

If $E^l\hat{R}^k < R^{res}$, the upper bound in (1.36) increases, leaving (1.34) as the most restrictive. The lower bound for R^{ib} is the rate on the residuals. That is, the interbank market rate must satisfy:

$$R^{res} < R^{ib} < \frac{(1+\lambda_b)\sigma\sqrt{3}}{\lambda_b}$$

Finally, for the interbank market to exist, $R^{res} < \frac{(1+\lambda_b)\sigma\sqrt{3}}{\lambda_b}$. This gives the condition for σ and λ_b :

$$\frac{\lambda_b R^{res}}{\sqrt{3}(1+\lambda_b)} < \sigma$$

Also, with $\frac{(1+\lambda_b)}{\lambda_b} > 2$, we can write $R^{ib} < 2\sigma\sqrt{3}$ □

Figure 1.10: Existence of the Interbank Market

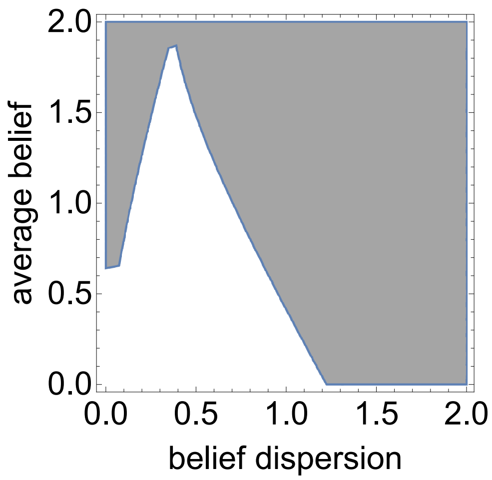


Figure is drawn for $R^{res} = 1.01$, $R = 1.01$, $\lambda_b = 0.1$

[26] R^{res} is the upper bound on E^lR^k and is set to be equal to R in the steady state.

Proposition 2. Low market beliefs result in a lower interbank rate and lower lending.

Proof. In the simple model, lending is given by $E^m R^k - E^l R^k$. Deriving with respect to the average market belief, m:

$$\begin{aligned}\frac{\partial\left(E^m R^k - E^l R^k\right)}{\partial m} &= \frac{\partial E^m R^k}{\partial R^{ib}}\frac{\partial R^{ib}}{\partial m} - \frac{\partial E^l R^k}{\partial R^{ib}}\frac{\partial R^{ib}}{\partial m} = \\ &= \frac{\partial R^{ib}}{\partial m}\left(\frac{\partial E^m R^k}{\partial R^{ib}} - \frac{\partial E^l R^k}{\partial R^{ib}}\right) = \frac{\left(\frac{\partial E^m R^k}{\partial R^{ib}} - \frac{\partial E^l R^k}{\partial R^{ib}}\right)}{\left(1 + \frac{1}{\lambda}\left(\frac{\partial E^m R^k}{\partial R^{ib}} - \frac{\partial E^l R^k}{\partial R^{ib}}\right)\right)}\end{aligned}$$

where the last equality is derived from the interbank market clearing condition

$$R^{ib} = m + \sqrt{3}\sigma - \frac{1}{\lambda}\left(E^m R^k - E^l R^k\right)$$

with the derivative with respect to the average market belief being

$$\frac{\partial R^{ib}}{\partial m} = 1 - \frac{1}{\lambda}\left(\frac{\partial E^m R^k}{\partial R^{ib}}\frac{\partial R^{ib}}{\partial m} - \frac{\partial E^l R^k}{\partial R^{ib}}\frac{\partial R^{ib}}{\partial m}\right)$$

or

$$\frac{\partial R^{ib}}{\partial m} = \frac{1}{\left(1 + \frac{1}{\lambda}\left(\frac{\partial E^m R^k}{\partial R^{ib}} - \frac{\partial E^l R^k}{\partial R^{ib}}\right)\right)}$$

With the marginal lender and the marginal investor defined, respectively, as $E^m R^k = R^{ib} p^m$ and $R^{res} = R^{ib} p^l$, and $p^i = \frac{1}{2} - \frac{\left(R + \lambda_b R^{ib}\right)}{2\sigma\sqrt{3}(1+\lambda_b)} + \frac{E^i \hat{R}^k}{2\sigma\sqrt{3}}$, we get

$$\begin{aligned}\frac{\partial p^l}{\partial R^{ib}} + \frac{\partial E^l R^k}{\partial R^{ib}} &= -\frac{R^{res}}{(R^{ib})^2} \text{ and} \\ \frac{\partial E^l R^k}{\partial R^{ib}} &= \frac{\lambda_b}{(1+\lambda_b)} - \frac{2\sigma\sqrt{3}R^{res}}{(R^{ib})^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial E^m R^k}{\partial R^{ib}} &= \frac{\partial p^m}{\partial R^{ib}} R^{ib} + \frac{\partial E^m R^k}{\partial R^{ib}} R^{ib} + p^m \text{ and} \\ \frac{\partial E^m R^k}{\partial R^{ib}} &= \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)} R^{ib}}{2\sigma\sqrt{3} - R^{ib}}\end{aligned}$$

$\frac{\partial E^m R^k}{\partial R^{ib}} - \frac{\partial E^l R^k}{\partial R^{ib}}$ then can be rewritten as

$$\frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3} - R^{ib}} - \frac{\lambda_b}{(1+\lambda_b)} + \frac{2\sigma\sqrt{3}R^{res}}{\left(R^{ib}\right)^2} > 0$$

To prove that this expression is positive, we use the result from proposition 1 that $R^{ib} < \frac{(1+\lambda_b)\sigma\sqrt{3}}{\lambda_b}$ and $\lambda < 1$, so that $\frac{1+\lambda}{\lambda} > 2$, $R^{ib} < 2\sigma\sqrt{3}$. Then the above expression can be negative only with $p^m < \frac{\lambda}{1+\lambda} + \frac{R^{res}\left(R^{ib}-2\sigma\sqrt{3}\right)}{\left(R^{ib}\right)^2}$ and $\lambda > \frac{R^{res}}{R^{ib}-R^{res}}$. With $\frac{\lambda}{1+\lambda}$ increasing in λ, $\frac{\lambda}{1+\lambda} > \frac{R^{res}}{Rib}$, so $p^m < \frac{R^{res}}{R^{ib}} - \frac{R^{res}\left(2\sigma\sqrt{3}-R^{ib}\right)}{\left(R^{ib}\right)^2}$. Multiplying both sides by R^{ib} we get $R^{ib}p^m < R^{res} - \frac{R^{res}\left(2\sigma\sqrt{3}-R^{ib}\right)}{\left(R^{ib}\right)}$, meaning that $E^m R^i = R^{ib}p^m < R^{res}$, which contradicts $E^m R^i > R^{res}$. □

Proposition 3 *A low policy rate increases lending and lowers the interbank market rate.*

Proof.

$$\frac{\partial R^{ib}}{\partial R^{res}} = -\frac{1}{\lambda}\left(\frac{\partial E^m R^k}{\partial Rib}\frac{\partial R^{ib}}{\partial R^{res}} - \frac{\partial E^l R^k}{\partial R^{res}}\right)$$

with

$$\frac{\partial E^m R^k}{\partial R^{ib}} = \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3} - R^{ib}}$$

and

$$\begin{aligned} E^l R^k &= \frac{R}{1+\lambda_b} + \frac{\lambda_b R^{ib}}{1+\lambda_b} - \sqrt{3}\sigma + \frac{2\sqrt{3}\sigma R^{res}}{R^{ib}} \\ \frac{\partial E^l R^k}{\partial R^{res}} &= \frac{\lambda_b}{1+\lambda_b}\frac{\partial R^{ib}}{\partial R^{res}} + \frac{2\sqrt{3}\sigma}{R^{ib}} - \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2}\frac{\partial R^{ib}}{\partial R^{res}} \end{aligned}$$

Then

$$\frac{\partial R^{ib}}{\partial R^{res}} = \frac{\frac{2\sqrt{3}\sigma}{R^{ib}}}{\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3}-R^{ib}} - \frac{\lambda_b}{1+\lambda_b} + \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2}}$$

With $\lambda > 0$, the size of the derivative is determined by the denominator. $\lambda_b > \frac{\lambda_b}{1+\lambda_b}$. Consider if $\frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3}-R^{ib}} + \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2} > 0$ with $\frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2} = \frac{2\sqrt{3}\sigma p^l}{R^{ib}}$:

$$\frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3} - R^{ib}} + \frac{2\sqrt{3}\sigma p^l}{R^{ib}} =$$

$$= \frac{(1+\lambda_b)2\sigma\sqrt{3}\left(R^m - R^{res}\right) + (1+\lambda_b)\left(2\sqrt{3}\sigma\right)^2 p^l - \lambda_b R^{ib}}{(1+\lambda_b)\left(2\sigma\sqrt{3} - R^{ib}\right)}$$

The first term is positive for $E^m R^k > R^{res}$ and the second term is positive with $2\sqrt{3}\sigma > R^{ib}$ and $\left(2\sqrt{3}\sigma\right)^2 p^l > R^{ib} p^l = R^{res}$

$$\frac{(1+\lambda_b)\left(2\sqrt{3}\sigma\right)^2 p^l}{\lambda_b} = \frac{(1+\lambda_b)2\sqrt{3}\sigma * kR^{res}}{\lambda_b} > R^{ib}$$

That is, $\frac{\partial R^{ib}}{\partial R^{res}} > 0$.

Consider $\frac{\partial E^m R^k}{\partial R^{res}} - \frac{\partial E^l R^k}{\partial R^{res}}$.

$$\frac{\partial E^m R^k}{\partial R^{res}} - \frac{\partial E^l R^k}{\partial R^{res}} = \frac{\partial E^m R^k}{\partial R^{ib}}\frac{\partial R^{ib}}{\partial R^{res}} - \frac{\partial E^l R^k}{\partial R^{res}} =$$

$$= \frac{2\sqrt{3}\sigma}{R^{ib}}\left(\frac{\left(\frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3} - R^{ib}} - \frac{\lambda_b}{1+\lambda_b} + \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2}\right)}{\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3} - R^{ib}} - \frac{\lambda_b}{1+\lambda_b} + \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2}} - 1\right) < 0$$

□

Proposition 4 *Relaxing the collateral constraint increases lending and the interbank market rate.*

Proof. The proof is based on deriving the interbank market rate:

$$\begin{aligned}
\frac{\partial R^{ib}}{\partial \lambda} &= \frac{1}{\lambda^2}\left(\frac{\partial E^m R^k}{\partial \lambda} - \frac{\partial E^l R^k}{\partial \lambda}\right) \\
\frac{\partial E^m R^k}{\partial \lambda} &= \frac{\frac{R - R^{ib}}{(1+\lambda_b)^2} R^{ib} + \left(\sigma\sqrt{3} - \frac{\left(R + 2\lambda_b R^{ib}\right)}{(1+\lambda_b)} + E^m \hat{R}^k\right)\frac{\partial R^{ib}}{\partial \lambda}}{2\sigma\sqrt{3} - R^{ib}} \\
\frac{\partial E^l R^k}{\partial \lambda} &= \frac{R^{ib} - R}{(1+\lambda_b)^2} + \left(\frac{\lambda_b}{1+\lambda_b} - \frac{2\sqrt{3}\sigma p^l}{R^{ib}}\right)\frac{\partial R^{ib}}{\partial \lambda}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial R^{ib}}{\partial \lambda} &= \frac{A}{B} \\
A &= \frac{R - R^{ib}}{(1+\lambda_b)^2}\left(\frac{R^{ib}}{\left(2\sigma\sqrt{3} - R^{ib}\right)} + 1\right)
\end{aligned}$$

$$B = \frac{\lambda}{\left(2\sigma\sqrt{3}-R^{ib}\right)}\left(\lambda\left(2\sigma\sqrt{3}-R^{ib}\right)-\frac{\left(E^{m}\hat{R}^{k}-E^{l}\hat{R}^{k}\right)}{\lambda}\frac{2\sigma\sqrt{3}}{R^{ib}}\right)$$
$$+\frac{2\sigma\sqrt{3}}{\left(2\sigma\sqrt{3}-R^{ib}\right)R^{ib}}\left(\frac{\lambda R^{ib}-p^{l}\left(2\sigma\sqrt{3}\right)^{2}(1+\lambda_b)}{(1+\lambda_b)}\right)$$

$\left(R-R^{ib}\right)<0$ and $2\sigma\sqrt{3}-R^{ib}>0$, that is, the nominator is negative. The denominator consists of two elements: $\frac{2\sigma\sqrt{3}}{\left(2\sigma\sqrt{3}-R^{ib}\right)R^{ib}}\left(\frac{\lambda R^{ib}-p^{l}\left(2\sigma\sqrt{3}\right)^{2}(1+\lambda_b)}{(1+\lambda_b)}\right)<0$ (as was shown in the proof of proposition 3, $\lambda R^{ib}-p^{l}\left(2\sigma\sqrt{3}\right)^{2}(1+\lambda_b)<0$). To see that the first term is also negative, recall that from interbank market clearing $\frac{\left(E^{m}\hat{R}^{k}-E^{l}\hat{R}^{k}\right)}{\lambda}=\left(\bar{R}-R^{ib}\right)>\left(2\sigma\sqrt{3}-R^{ib}\right)>\lambda\left(2\sigma\sqrt{3}-R^{ib}\right)>\lambda\left(2\sigma\sqrt{3}-R^{ib}\right)\frac{R^{ib}}{2\sigma\sqrt{3}}$

$$\frac{\partial E^{m}R^{k}}{\partial\lambda}-\frac{\partial E^{l}R^{k}}{\partial\lambda}=\frac{R-R^{ib}}{(1+\lambda_b)^{2}}\left(\frac{2\sigma\sqrt{3}}{2\sigma\sqrt{3}-R^{ib}}\right)$$
$$+\left(\frac{\sigma\sqrt{3}-\frac{\left(R+2\lambda_b R^{ib}\right)}{(1+\lambda_b)}+E^{m}\hat{R}^{k}}{2\sigma\sqrt{3}-R^{ib}}-\frac{\lambda_b}{1+\lambda_b}+\frac{2\sqrt{3}\sigma p^{l}}{R^{ib}}\right)\frac{\partial R^{ib}}{\partial\lambda}=$$
$$\frac{R-R^{ib}}{\left(1+\lambda_b\right)^{2}}\left(\frac{2\sigma\sqrt{3}}{2\sigma\sqrt{3}-R^{ib}}\right)\left(\frac{\lambda^{2}}{\lambda^{2}-\frac{\left(\sigma\sqrt{3}-\frac{\left(R+2\lambda_b R^{ib}\right)}{(1+\lambda_b)}+E^{m}\hat{R}^{k}\right)}{2\sigma\sqrt{3}-R^{ib}}+\left(\frac{\lambda_b}{1+\lambda_b}-\frac{2\sqrt{3}\sigma p^{l}}{R^{ib}}\right)}\right)>0$$

Also note that $\frac{\partial E^{l}R^{k}}{\partial\lambda}$ consists of two elements, both of them positive. That is, when λ is raised, the marginal lender must have higher return expectations, and those with lower expectations hoard. □

Proposition 5 *The effect of a policy rate reduction is limited by the mean market belief.*

Proof. Suppose that $E^{l}R^{k}>R^{res}$. Then, for the policy rate reduction to restore lending, the change should be such that $E^{l}R^{k}<R^{res}$

$$\frac{\partial E^{l}R^{k}}{\partial R^{res}}=\frac{\lambda_b}{1+\lambda_b}\frac{\partial R^{ib}}{\partial R^{res}}+\frac{2\sqrt{3}\sigma}{R^{ib}}-\frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^{2}}\frac{\partial R^{ib}}{\partial R^{res}}=$$
$$=\frac{\frac{2\sqrt{3}\sigma}{R^{ib}}}{\lambda_b+\frac{2\sigma\sqrt{3}p^{m}-\frac{\lambda_b}{(1+\lambda_b)}R^{ib}}{2\sigma\sqrt{3}-R^{ib}}-\frac{\lambda_b}{1+\lambda_b}+\frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^{2}}}\left(\frac{\lambda_b}{1+\lambda_b}-\frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^{2}}\right)+\frac{2\sqrt{3}\sigma}{R^{ib}}$$

$$= \frac{2\sqrt{3}\sigma}{R^{ib}} \left(\frac{\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)} R^{ib}}{2\sigma\sqrt{3} - R^{ib}}}{\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)} R^{ib}}{2\sigma\sqrt{3} - R^{ib}} - \frac{\lambda_b}{1+\lambda_b} + \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2}} \right) =$$

The nominator is smaller than the denominator:

$$\frac{2\sqrt{3}\sigma}{R^{ib}} \left(\frac{\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)} R^{ib}}{2\sigma\sqrt{3} - R^{ib}}}{\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)} R^{ib}}{2\sigma\sqrt{3} - R^{ib}} - \frac{\lambda_b}{1+\lambda_b} + \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2}} \right) > 1$$

$$2\sqrt{3}\sigma \left(\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)} R^{ib}}{2\sigma\sqrt{3} - R^{ib}} \right)$$
$$> R^{ib} \left(\lambda_b + \frac{2\sigma\sqrt{3}p^m - \frac{\lambda_b}{(1+\lambda_b)} R^{ib}}{2\sigma\sqrt{3} - R^{ib}} - \frac{\lambda_b}{1+\lambda_b} + \frac{2\sqrt{3}\sigma R^{res}}{\left(R^{ib}\right)^2} \right)$$

$$\lambda_b \left(2\sqrt{3}\sigma - R^{ib}\right) + 2\sigma\sqrt{3}p^m - \frac{2\sqrt{3}\sigma R^{res}}{R^{ib}} - \frac{\lambda_b}{(1+\lambda_b)} R^{ib} > -\frac{R^{ib}\lambda_b}{1+\lambda_b}$$
$$\lambda_b \left(2\sqrt{3}\sigma - R^{ib}\right) + 2\sigma\sqrt{3}p^m - \frac{2\sqrt{3}\sigma R^{res}}{R^{ib}} > 0$$

with $\frac{R^{res}}{R^{ib}} = p^l$

$$\lambda_b \left(2\sqrt{3}\sigma - R^{ib}\right) + 2\sigma\sqrt{3}\left(p^m - p^l\right) > 0$$

This is true, as $2\sqrt{3}\sigma - R^{ib} > 0$ (the result from propositions 1 and 2) and $p^m - p^l > 0$. That is, the derivative $\frac{\partial E^l R^k}{\partial R^{res}} > 1$.

Now consider how the difference $E^l R^k - R^{res}$ changes with respect to R^{res} :

$$\frac{\partial E^l R^k}{\partial R^{res}} - 1 > 0$$

That is, the function is increasing in R^{res} and is increasing faster than R^{res}. A downward shift in reserves reduces both the right and left-hand sides of the inequality $E^l R^k > R^{res}$, with $E^l R^k$ declining faster than R^{res}. Thus, if the difference between the marginal lender's belief and the policy rate is small, it is possible to reverse this inequality and restore lending: there will be some banker who would be better off lending on the interbank market at the low policy rate than investing herself or hoarding. However, with a large difference between $E^l R^k$and R^{res}, which happens with very low market expectations (see proposition 2), it is not possible to restore lending with a positive policy rate $\square$

Proposition 6 *Relaxing the collateral constraint does not restore the functioning of the interbank market or credit to the real economy*

Proof. Suppose that no one lends in the interbank market. This means that the marginal lender is better off investing herself than lending: $E^l R^k > p^l R^{ib} = R^{res}$. To restore lending, the policy should bring about $E^l R^k \leq p^l R^{ib} = R^{res}$. Proposition 4 showed that the marginal lender's belief increases with increasing collateral constraint λ_b. In this case, an increase in λ_b means an increase in $E^l R^k$, that is, lending is not restored. $\square$

1.B Correlation of Experts' Opinions, the Mean Market Belief, and Its Variance

Expert opinions are defined in the text as

$$\theta_t = \rho_\theta \theta_{t-1} + \eta_t^i$$

where η_t^i is the noise in the opinion of bank i's expert, with $\eta_t^h \sim N(\mu_t, \sigma_\eta)$. We assume that the noise in experts' opinions is correlated. That is, when one expert overestimates/underestimates the value of a persistent shock, others tend to do the same. Technically, we model correlated draws in the following way. First, there are N[27] independent draws from $N(\mu_t, \sigma_\eta)$. Then, each of the independent draws is rescaled:

$$\bar{\eta}_t^i = \rho^c \eta_t^1 + \sqrt{1-(\rho^c)^2}\eta_t^i, \ h \neq 1 \tag{1.37}$$

where η_t^i is one of the independent draws and ρ^c is the correlation coefficient

$$\rho^c = \frac{Cov(\bar{\eta}_t^i, \bar{\eta}_t^j)}{\sqrt{Var\left(\eta_t^i\right) Var\left(\eta_t^j\right)}}, i \neq j$$

where $Var\left(\eta_t^i\right) = Var\left(\eta_t^j\right) = Var\left(\bar{\eta}_t\right) = \sigma_\eta^2$. The last equality comes with the observation that $Var\left(\bar{\eta}_t^h\right) = (\rho^c)^2 Var\left(\eta_t^1\right) + \left(1-(\rho^c)^2\right) Var\left(\eta_t^h\right)$. With η_t^h and η_t^1 being drawn from

[27]In the text we assume the existence of a continuum of H banks, normalized to 1. Here, for computational purposes, we use N as the number of banks and set it equal to a "large number": $N = 100$.

the same distribution, $Var\left(\bar{\eta}_t^h\right) = \left((\rho^c)^2 + 1 - (\rho^c)^2\right) Var\left(\eta_t^h\right) = Var\left(\eta_t^h\right)$.

Using (1.37), we thus obtain a sequence of random variables, correlated with each other with correlation coefficient ρ^c. Because in equilibrium only the average shock to market beliefs matters, we now proceed to derive its properties. The expected average belief shock can be defined as:

$$\frac{1}{N}E\left(\eta_t^1 + \sum_{h=2}^{N}\bar{\eta}_t^h\right) = \frac{1}{N}E\left(\eta_t^1 + (N-1)\,\eta_t^1\rho^c + \sqrt{1-(\rho^c)^2}\sum_{h=2}^{N}\eta_t^h\right) \tag{1.38}$$

Note that η_t^1 and $\eta_t^{h,h\neq 1}$ are independent and drawn from the same distribution. This means that the expectation of their sum equals the sum of their expectations, which are unconditional expectations μ_t. The expected average belief shock is then:

$$\mu_t\frac{1}{N}\left(1 + (N-1)\left(\rho^c + \sqrt{1-(\rho^c)^2}\right)\right)$$

Note that with $\rho^c = 1$ in the case of perfect correlation and with $\rho^c = 0$ in the case of no correlation, the expected average of the correlated draws corresponds to the unconditional mean. Also, unless μ_t is zero, the average belief shock is not equal to the distributional mean.

The variance of the average belief shock is then:

$$\sigma_\eta^2\left(1 + (N-1)^2\left((\rho^c)^2 + 1 - (\rho^c)^2\right)\right) = \sigma_\eta^2\frac{(2 + N^2 - 2N)}{N^2} \tag{1.39}$$

1.C The Bank's Filtering Problem

The state-space representation of the filtering problem is given by the following equations.

The state equation is:

$$(\mu_t) = \left(\ \rho_\mu\ \right) \times (\mu_{t-1}) + (v_t) \tag{1.40}$$

where q is the variance of the i.i.d. Gaussian shock v_t.

The measurement vector consists of two types of signals: data on ξ_t and the expert opinion, $\tilde{\xi}_t^{ex}$. The measurement equation is:

$$\begin{pmatrix} \xi_t \\ \tilde{\xi}_t^{ex} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \times \mu_t + \begin{pmatrix} \rho_\xi \\ \rho_\theta \end{pmatrix} \times \xi_{t-1} + \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix},$$

where ε_t is Gaussian and η_t is uniformly distributed. The measurement equation can be rewritten as:

$$\xi_t = C\mu_t + D\xi_{t-1} + \varpi_t$$

where

$$C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; D = \begin{pmatrix} \rho_\zeta \\ \rho_\theta \end{pmatrix}, \; \tilde{\xi}_t = \begin{pmatrix} \xi_t \\ \tilde{\xi}_t^{ex} \end{pmatrix}$$

and $\varpi_t = (\nu_t, \; \eta_t)'$ is a vector of measurement errors with the variance-covariance matrix:

$$R = \begin{pmatrix} \sigma_\varepsilon^2 & \sigma_{\varepsilon\eta}^2 \\ \sigma_{\varepsilon\eta}^2 & \sigma_\eta^2 \end{pmatrix}$$

where $\sigma_{\varepsilon\eta}^2$ is the covariance of errors in econometric and expert forecasts.

1.D The Agency Problem

Recall that our agency problem differs from that of Gertler and Karadi (2011) in several respects. First, in our model banks have the possibility to put their funds in reserves. Second, banks are heterogeneous, with a share of them investing in a risky asset. Last but not least, some banks participate in the interbank market, transferring some funds from pessimistic to optimistic banks.

The banking family maximizes the terminal worth of each member, discounted by the stochastic discount factor $\beta^j \Omega_{t,t+j}$ arising from the household problem. The value

$$\begin{aligned} V_t &= \max E_t \sum_{j=0}^{\infty} (1-\theta)\, \theta^j \beta^{j+1} \Omega_{t,t+1+j} \left(N_{t+1+j}\right) = \\ &= \max E_t \sum_{j=0}^{\infty} (1-\theta)\, \theta^j \beta^{j+1} \Omega_{t,t+1+j} \left\{ \left(R_{t+1+j}^k - R_{t+j}\right) Q_{t+j} S_{t+j} + \left(R_{t+j}^{res} - R_{t+j}\right) Res_t + R_{t+j} N_{t+j} \right\} \end{aligned} \tag{1.41}$$

Equation (1.41) resembles the terminal worth equation in Gertler and Karadi (2011), the only difference being that we are applying it on the average level. Note that the banks' family budget constraint is

$$Q_t S_t + Res_t = N_t + B_t$$

Also, only those banks with the lowest return expectations hoard funds in reserves (others either invest themselves or lend funds to be invested by others): $Res_t = s_t^h \left(N_t + B_t\right) =$

$s_t^h\left(Q_tS_t+Res_t\right)$ with s_t^h being the share of hoarders. And $Q_tS_t=\left(1-s_t^h\right)\left(Q_tS_t+Res_t\right)$. The terminal worth is:

$$\begin{aligned}
&E_t\sum_{j=0}^{\infty}\left(1-\theta\right)\theta^j\beta\Omega_{t,t+1+j}\{\left(R^k_{t+1+j}-R_{t+j}\right)\left(1-s^h_{t+j}\right)\left(Q_tS_{t+j}+Res_{t+j}\right)+\\
&\qquad+\left(R^{res}_{t+j}-R_{t+j}\right)s^h_{t+j}\left(Q_tS_{t+j}+Res_{t+j}\right)+R_{t+j}N_{t+j}\}=\\
&=E_t\sum_{j=0}^{\infty}\left(1-\theta\right)\theta^j\beta^{j+1}\Omega_{t,t+1+j}\{\left(\left(1-s^h_{t+j}\right)R^k_{t+1+j}+s^h_{t+j}R^{res}_{t+j}-R_{t+j}\right)\left(Q_tS_{t+j}+Res_{t+j}\right)+\\
&\qquad+R_{t+j}N_{t+j}\}
\end{aligned}$$

We then have to restrict banks from borrowing from the household. Otherwise, for a non-negative $\beta^j\Omega_{t,t+j}\left(R^k_{t+1+j}-R_{t+j}\right)$ a bank would like to borrow indefinitely from the household. To avoid this, a moral hazard problem is introduced. At the beginning of the period, a banker can choose to divert a fraction λ of its assets. The depositors can recover the remaining fraction $(1-\lambda)$ of the banks' assets. For a depositor willing to participate, the banks must meet the incentives constraint:

$$V_t\geq\lambda\left(Q_tS_t+Res_t\right)$$

where V_t is the worth the banker would lose by diverting, and $\lambda\left(Q_tS_t+Res_t\right)$ is the gain from diverting. That is, the continuation value should be larger than the gain from deviating. We rewrite (1.41) as

$$V_t=\upsilon_t\left(Q_tS_t+Res_t\right)+\eta_tN_t$$

where

$$\begin{aligned}
\upsilon_t&=E_t\left\{\left(1-\theta\right)\beta\Omega_{t,t+1}\left(\left(1-s^h_t\right)R^k_{t+1}+s^h_tR^{res}_t-R_t\right)+\beta\Omega_{t,t+1}\theta\chi_{t,t+1}\upsilon_{t+1}\right\}\\
\eta_t&=E_t\left\{\left(1-\theta\right)+\beta\Omega_{t,t+1}\theta z_{t,t+1}\eta_{t+1}\right\}\\
\chi_{t,t+1}&=\frac{Q_{t+1}S_{t+1}+Res_{t+1}}{Q_tS_t+Res_t}\\
z_{t,t+1}&=\frac{N_{t+1}}{N_t}
\end{aligned}$$

and finally we have the expression for the financial accelerator:

$$Q_tS_t+Res_t=\frac{\eta_t}{\lambda-\upsilon_t}N_t=\varphi_tN_t$$

where φ_t is the leverage ratio, limiting the amount of assets an intermediary can acquire as a proportion of net worth.

To determine the leverage ratio, the household needs to form expectations about the future risky asset return. We assume that the household has a belief equal to the mean market belief.

1.E Household, Capital Producers and Retailers

Household The first order conditions for household problem are

$$[C_t]\ \rho_t = (C_t - hC_{t-1})^{-1} - \beta h E_t \left(C_{t+1} - hC_t\right)^{-1} \tag{1.42}$$

$$[L_t]\ \ \rho_t W_t - \chi L_t^{\phi} = 0 \tag{1.43}$$

$$\Omega_{t,t+1} \equiv \frac{\rho_{t+1}}{\rho_t} \tag{1.44}$$

$$[B_t]\ \ E_t \beta \Omega_{t,t+1} R_{t+1} = 1 \tag{1.45}$$

where $\Omega_{t,t+1}$ is a stochastic discount factor and ρ_t is the marginal utility of consumption.

Capital Producers The first-order conditions for investment give the price of capital, Q_t:

$$[I_t] : Q_t = 1 + f\left(\cdot\right) + \frac{In_k + I_{ss}}{In_{k-1} + I_{ss}} f'\left(\cdot\right) - E_t \beta \Omega_{t,t+1} \left(\frac{In_k + I_{ss}}{In_{k-1} + I_{ss}}\right)^2 f'\left(\cdot\right) \tag{1.46}$$

Retailers Firms are monopolistic competitors and maximize their profit:

$$\max_{P_t^*} \sum_{i=0}^{\infty} \gamma^i \beta^i \Omega_{t,t+1} \left[\frac{P_t^*}{P_{t+i}} \prod_{k=1}^{i} (1 + \pi_{t+k-1})^{\gamma_p} - P_{m,t+i}\right] Y_{ft+i} \tag{1.47}$$

subject to demand from households:

$$Y_{ft} = \left(\frac{P_{ft}^*}{P_t}\right)^{-\varepsilon} Y_t \tag{1.48}$$

where P_t^* is the optimal price set in period t, γ is the fraction of firms which cannot reset their prices but only index to inflation, and $\pi_t = \frac{P_t}{P_{t-1}} - 1$ is the one-period inflation rate.

The problem results in the first-order condition:

$$\sum_{i=0}^{\infty} \gamma^i \beta^i \Omega_{t,t+i} \left[\frac{P_t^*}{P_{t+i}} \prod_{k=1}^{i} (1 + \pi_{t+k-1})^{\gamma_p} - \mu P_{m,t+i}\right] Y_{ft+i} = 0 \tag{1.49}$$

where $\mu \equiv \frac{1}{1-\frac{1}{\varepsilon}}$ is a monopolistic mark-up.

The resulting equation for the price dynamics takes the form:

$$P_t = \left[\int_0^1 P_{ft}^{\frac{1}{1-\varepsilon}} df \right]^{1-\varepsilon} \tag{1.50}$$

$$P_t = \left[(1-\gamma) \left(P_t^*\right)^{1-\varepsilon} + \gamma \left\{ (1+\pi_{t+k-1})^{\gamma_p} P_{t-1} \right\}^{1-e} \right]^{\frac{1}{1-\varepsilon}} \tag{1.51}$$

1.F Calibrated Parameters from Gertler and Karadi (2011)

Table 1.2: Calibrated Parameters from Gertler and Karadi (2011)

β	0.99	household's discount rate
h	0.815	habit parameter
χ	3.409	relative utility weight of labor
ϕ	0.276	inverse Frisch elasticity of labor supply
λ	0.381	fraction of capital to be diverted
θ	0.972	survival rate of bankers
α	0.33	capital share
U	1	steady-state capital utilization rate
$\delta(U)$	0.025	steady-state depreciation rate
ζ	7.2	elasticity of marginal depreciation with respect to utilization rate
η_i	1.728	inverse elasticity of net investment to price of capital
ε	4.167	elasticity of substitution
γ	0.779	probability of keeping prices fixed
γ_p	0.241	measure of price indexation
κ_π	1.5	inflation coefficient of Taylor rule
κ_y	0.125	output gap coefficient of Taylor rule
ρ_i	0.8	smoothing parameter of Taylor rule
$\frac{G}{Y}$	0.2	steady-state proportion of government expenditure
τ	0.001	cost of government policy
κ	10	reaction parameter for government policy

Table 1.3: Comparing Model Generated Moments to the Data

	No Crisis Shock		Crisis Shock		Data
	Our Model	Baseline	Our Model	Baseline	
Output, Y	0.038	0.041	0.109	0.17	0.034
Consumption, C	0.039	0.036	0.222	0.28	0.041
Net Worth, N	0.062	0.108	0.783	1.54	0.817

1.G Comparing Model Generated Moments to the Data

In the table for the output we use GDP per capita, for the consumption - final consumption per capita, for the net worth - net financial assets of financial corporations. All data are from Eurostat and for the Euro area. The standard deviations are calculated for the log differences of the series. The first two columns show the standard deviations from simulations without crisis shocks, but with all other standard shocks in the literature. Namely, monetary policy, shock to government spending, technology shock, shock to banks' net worth, and a shock to the government policy. All shocks with 1 percent standard deviation. The other two columns show the results of simulations with crisis shock - which is 5 percent standard deviation of ξ. The last column corresponds to the Euro area data from 1995Q1 to 2016Q3. When the models are simulated without the crisis shock, the standard deviations of output and consumption are comparable to the moments in the data. The deviation of the net worth, however, is much smaller in the models. When we simulate the models with the crisis shock, conversely, models result in deviations of output and consumption that are several times larger. The deviation in the net worth is matched rather closely by our model, while the baseline model generates twice as large deviation than the observed one. Thus, when the models are simulated with the crisis shock, they overestimate the deviations of output and consumption, as they seem to overestimate the reliance of manufactures on credit.

Chapter 2

Information Acquisition and Excessive Risk: Impact of Policy Rate and Market Volatility[1]

Recent central bank policies of low interest rates have stimulated a debate as to whether these policies contribute to excessive risk taking. In this paper we build a theoretical model where risk taking has the form of decreased incentives to acquire information. We show that when safe interest rate or market volatility falls, agents choose to learn less about a risky asset even though they buy more of it. We study information decisions with entropy and linear learning rules in a general equilibrium model. As a result, a fall in volatility results in an increase in portfolio risk with the linear learning rule, but in a decline with the entropy learning rule. A fall in the interest rate leads to a rise in portfolio risk under both learning rules.

2.1 Introduction

The paper is motivated by the debate about whether a low policy rate has contributed to the recent financial crisis and if the ongoing policy of low interest rates is contributing to the building up of a new financial bubble. There are voices among policy-makers and academics suggesting that one could observe worrying tendencies of risky asset accumula-

[1]This chapter was published as V. Audzei (2015) "Information Acquisition and Excessive Risk: Impact of Policy Rate and Market Volatility", CERGE-EI Working Paper no. 536. We are greatful to Sergey Slobodyan, Filip Matejka, Michal Kejak, Mirko Wiederholt, Vincent Sterk and Jaume Ventura for comments and suggestions. The work was supported by Charles University Grant Agency grant number 528314 and by the Czech Science Foundation project No. P402/12/G097 DYME Dynamic Models in Economics. The research in this paper is an extension of a project published in ACTA VSFS, Prague.

tion[2]. There is evidence of an increased risk appetite, which is believed to be attributed to accommodative monetary policy conditions and subdued market volatility (for the evidence see, e.g., Bank for International Settlements 2014).

The question asked in this paper is if endogenous information acquisition can drive overaccumulation of risk when safe interest rates or market volatility is reduced. Financial agents invest in information to reduce the variance of their forecasts. We show that when market volatility declines, agents invest into information less and acquire more of a risky asset. This results in a portfolio risk comparable to that in the economy with higher market volatility. With interest rates being lowered, our model not only captures the standard "search-for-yield " effect, where financial intermediaries invest more into risky assets. We also show an increase in agents' ignorance about the asset quality. With low information investment and large risky asset holdings it implies a larger portfolio risk accumulation.

We show that average risk monitoring declines with lower interest rates despite the growth in excess return on a risky asset. Another result is overaccumulation of risky assets in a low risk environment. That is to say with low variance of risky asset return, agents take as much or more risk in their portfolio than they would have with high risky asset variance. This effect is explained in our model with just one deviation from rational expectations: agents do not know the future return, but only its distribution, i.e. there is no assumption of agents' irrationality. In our model, this result is driven by a decline in risk monitoring in low risk environment.

To check the robustness of the results, in the spirit of Nieuwerburgh and Veldkamp (2010) we consider two alternative learning functions, a linear and an entropy-based. The rise in portfolio risk when the safe interest rate falls is robust to a learning rule specification. The increase in risk with falling market volatility is more pronounced in a linear learning rule.

2.2 Related Literature

Our study relates to several stands of literature. First, there is the literature on the role of interest rates in mitigating or stimulating asset booms, in particular papers provid-

[2]For the evidence see Stein (2013); the recent examples of uncertainty among policy makers could be found in articles by Chris Giles "Central Bankers Say They Are Flying Blind " and "IMF warns on risks of excessive easing" in The Financial Times, April 17, 2013.

ing empirical evidence that easier monetary policy is associated with higher risk-taking. Maddaloni (2011) concludes that, for the euro area and US, low short-term interest rates cause softening of the banks' lending standards. Additional support for a risk-taking channel of monetary policy can be found in Gambacorta (2009) and Ongena and Peydro (2011). Adrian, Estrella, and Shin (2010) find empirical support for the notion that monetary policy effects the supply of credit, operating through the term spreads; and that monetary policy can influence risk appetite. Ahrend (2010) focuses on a different aspect of the financial imbalances - on excessive asset prices growth, and finds that low interest rates cause growth in some asset prices in OECD countries, particularly on the housing market. Detken and Smets (2004) come to the similar conclusion that low policy rates coincide with asset price booms. The evidence on the dynamic interaction between stock prices and Federal Reserve policy rate is provided by Laopodis (2010). White (2012) discusses the "unintended consequences" of easy monetary policy, among which are misallocation of credit and structural changes in the financial sector, e.g. movements from traditional banking model to shadow banking. Statistical evidence that a long period of low interest rate and low market volatility have contributed to excessive risk-taking is summarized in the Annual Report of the Bank for International Settlements (2014).

There are theoretical studies focusing on the channels through which monetary policy affects risk-taking or asset prices. Taylor (Taylor 2007 and Taylor 2010) suggests that the Fed's low rates stimulated a house price boom through credit growth. The several mechanisms through which the risk-taking channel of monetary policy could work are mentioned in Borio and Zhu (2008). In particular, search-for-yield implies that low interest rates result in a low return on the safe assets, which pushes investors to accumulate more of the risky ones in the search for an acceptable portfolio return. Also low interest rates imply a lower discount factor for evaluation of assets or income flows, causing higher risk tolerance. Our model incorporates both of these channels within the banks' portfolio choice problem. In this paper we tackle this question using rational inattention framework.

The banks' risk monitoring incentives in connection with monetary policy are studied in the model of Dell'Ariccia, Laeven, and Marquez (2010). Their findings depend on the banks' capital structure and the possibility of adjusting it. They conclude that with a flexible capital structure monetary policy easing leads to higher leverage and risk-taking. Their approach, however, is different from that pursued in this paper in several respects. They concentrate on a partial equilibrium model, where banks choose the probability of loan repayment subject to costs. Therefore, in their model banks do not learn about the asset quality, but invest to increase return probability. We build a general equilibrium model where banks are uncertain about the risky asset return, but might invest in reducing their uncertainty. That is, learning does not influence the return probability, but makes banks more informed. Therefore, we capture two aspects of risky behavior - investment in an asset known to be risky and investment into learning about the asset quality.

Another strand of literature our study is related to is dedicated to the learning and expectation formation and relaxation of the assumption of rational expectations. Among the papers to support the importance of imperfect expectations and learning are Boz and Mendoza (2014), Bullard, Evans, and Honkapohja (2010), Kurz and Motolese (2010), Lorenzoni (2009), Adam and Marcet (2010). Empirical support for the role of imperfect expectations can be found in Fuhrer (2011) and Beaudry, Nam, and Wang (2011). In this paper we incorporate the idea that agents do not have perfect foresight and have to form subjective expectations about risky asset return. We use the approach of Nieuwerburgh and Veldkamp (2010) to model the banks' decisions to invest in learning about the risky asset. In Nieuwerburgh and Veldkamp (2010), the investor draws an additional signal about asset return, and pays for an increase in the signal precision before observing it. We modify their formulation for information acquisition, so that in our model agents select the information budget depending on risk premia and market volatility. Freixas and Laux (2011) conclude that information acquisition is different during the business cycle: in the boom market participants' have less incentives to use or demand information. In the bust, the problem is then lack of information itself. Our study addresses the incentives to acquire information during the boom stage.

To conclude, our study is motivated by rich empirical evidence. Our model explores causalities between monetary policy and agents' risk-taking. We also show that prolonged periods of low policy rates or low risk lead to excessive accumulation of risk.

The remainder of the paper begins the description of policy environment during the recent years. Section 4 starts with analysis of a partial equilibrium model to describe the intuition for the main results. The financial sector is described, and the intuition for excessive risk-taking is presented in section 5 within a partial equilibrium. In section 6 we complete the model for general equilibrium and then proceed with the calibration, simulations and discussion in section 7. The last section concludes.

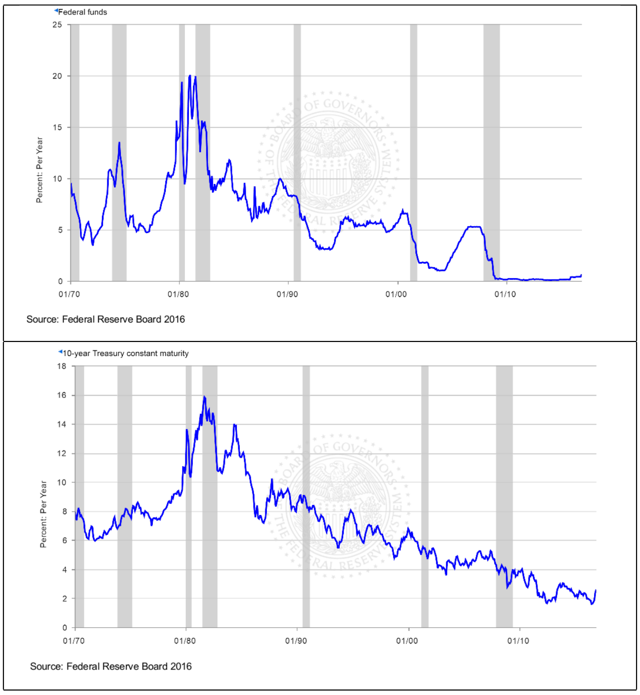


Figure 2.1: US Rates

2.3 Recent Policy Rates and Risk Taking

At the onset of the recent financial crisis in the US there appeared voices stating that monetary policy had created favorable conditions for the housing boom. An example is Taylor (2007), who in a simple model show that the Federal Funds Rate was well below the one suggested by the Taylor rule, and is partially responsible for the housing price boom (Taylor (2007), chart 1 and 2). Some, e.g. Bank for International Settlements (2014) or Stein (2013), consider that the current highly accommodative policy is contributing to another asset price boom. The historical values of policy and safe rates in the US n figure 2.1 show that the Great Moderation period was indeed characterized by very low interest rates. The rates went even lower after the crisis accompanied by accommodative monetary policy. The interest rates in the Eurozone display a similar pattern.

In addition to interest rates, monetary policy has succeeded in lowering market volatility during the post-crisis recession, for some periods even below the 2004 level (Bank for International Settlements 2014, p.38). In figure 2.2 there is a dynamic of VIX index often used as a measure of market uncertainty. There is a significant spike associated with the financial crisis in the US in 2009, somewhat smaller spikes in 2011 and 2012 could be attributed to the sovereign crisis in the Eurozone. The index falls after ECB commitment to do "whatever it takes" to save the euro.

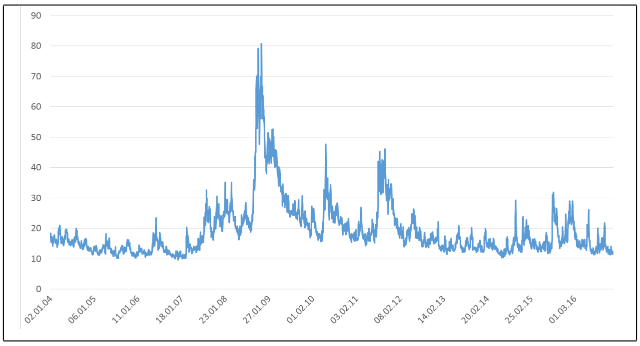


Figure 2.2: VIX Index

The question remains if such an accommodative environment was accompanied by excessive to risk-taking and if it continues to contribute to risk accumulation. The credit-to-GDP gap was positive during the pre-crisis period in most of the developed world. This can support the story that low policy rates contributed to the over-supply of credit. After the crisis, as the balance sheets of credit institutions were impaired, the credit-to-GDP gap remained negative. Yet, it does not mean that monetary accommodation or suppressing market volatilities does not spur risk-taking. For documentation of over accumulation of portfolio risk see Bank for International Settlements (2014), Graph II.2, or Stein (2013).

A low policy rate environment can stimulate investment in risky assets as financial institutions are "searching-for-yield". In this paper, we go beyond search-for-yield and look into agents incentives to gather information in the low-volatility and/or low-policy-rate economy. In the next section we develop a simple model with investors searching-for-yield and gathering information about the risky asset. We show that information acquisition contributes to the accumulation of risk on agents' balance sheets.

2.4 The Model of Financial Sector

Consider a model with a financial intermediary, bank, a manufacturing firm and a household. The assets in the economy are manufacturer claims (a risky asset) and reserves (a safe asset). The risk in manufacturer claims comes from the uncertainty about future productivity. All the agents in the economy know the productivity distribution. The household puts savings in the bank (in the form of investment), and the bank transfers all its profit back to the household. The safe and risky interest rates are set by the market.

The bank is risk-averse, which is motivated by the fact that banks are often subject to regulations and have reputational concerns for the safety of their deposits. We then expand the model and grant financial intermediary access to a noisy signal about future productivity. This signal helps the agents to reduce the variance of their forecast. Yet they have to pay for it. Banks are Bayesian, they form forecasts of risky returns as a weighted average of their prior and the signal.

We abstract from any nominal variables in the model. All the prices and returns are real. In what follows, we present the model set-up. We start with a partial equilibrium model to illustrate the mechanism of the excessive risk-taking and information acquisition. Then we simulate a simple general equilibrium model to study the model dynamic and potential role of interest rates feedback[3]. We start with a description of the financial sector.

Bank The bank is risk-averse and has mean-variance utility in its next period net return:

$$\max_{k_t^b} E_t \Pi_{t+1} - \frac{1}{\rho} Var\left(\Pi_{t+1}\right), \tag{2.1}$$

where ρ is the risk aversion parameter, k_t^b is the bank's risky asset holdings and Π_{t+1} stands for the next period return. That is, portfolio variance is costly and the bank, therefore, has incentives to reduce it. The next period return consists of the return on the bank's portfolio minus the information budget:

$$\Pi_{t+1}^b = d_t R_t^s + k_t^b \left(R_{t+1}^r - R_t^s\right) - b_t, \tag{2.2}$$

where d_t is household investment, R^r, R^s are respectively gross returns from risky and safe assets, b_t is the information budget selected by the bank. The bank's future return depends on the amount of funds it has for investment - d_t and from a composition of its portfolio - quantity of risky asset, k_t^b. Note that the return is reduced by the information investment, b_t.

The bank's objective is to maximize (2.1), and the choice variables are information budget, b_t , and risky asset quantity k_t^b. Compared to the strand of literature on rational inattention with exogenous capacity constraint, here we endogenize capacity and formulate it in budget terms.

[3] In our model a risky interest rate could be viewed as a reverse of the asset price. With larger demand for a risky asset, it drops, potentially offsetting higher risk appetite.

Maximizing the bank's utility, we get its holdings of the risky asset:

$$k_t^b = \frac{E_t R_{t+1}^r - R_t^s}{\rho \cdot \hat{\sigma}_t^2}, \tag{2.3}$$

where $\hat{\sigma}_t^2$ is risky asset return variance. Sign '$\hat{}$' stands for posterior variance, updated after information decisions. As is typical in the literature, the amount of risky assets bought is increasing with excess return $R_{t+1}^r - R_t^s$, and is decreasing with risk aversion, ρ, and risky asset return variance $\hat{\sigma}_t^2$.

For simplicity, we make the bank transfer all its profit to the household in return to their savings, d_t.

Information Acquisition The information acquisition is modeled similar to Nieuwerburgh and Veldkamp (2010). In their paper an investor allocates his/her exogenously limited capacity to learn between different assets depending on his/her portfolio decisions. In our model, we endogenize learning capacity by replacing it with the budget, b_t. The bank then chooses the budget to determine how much to learn subject to fixed learning costs, a.

Financial intermediaries can reduce the variance of their return forecast by investing in additional signal and pay costs proportional to the variance reduced. The decision to monitor is taken ex-ante signal realization. For this purpose, the period is decomposed into sub-periods. The timing is indicated in table 2.1 (in columns).

subperiod 1:	**subperiod 2:**
$\mu_t \sim N\left(R_{t+1}^r, \sigma_t^2\right)$	information signals are realized
expected posterior return is $E\hat{\mu} \sim N\left(\mu, \hat{\sigma}_t^2\right)$	$\hat{\mu}$ is formed using the Bayes rule,
budget, b_t and $\hat{\sigma}_t^2$ are chosen	and portfolio is chosen: k_t^b

Table 2.1: Timeline of Information Decisions

In table 2.1 μ_t is the bank's prior about future return, R_{t+1}^r, $E\hat{\mu}$ is the posterior the bank expects to obtain after observing the signal. $\hat{\sigma}_t^2$ is the posterior variance after observing the signal[4].

In the first subperiod the agent has prior variance, σ_t^2, and expected return, μ_t, both coinciding with true moments of return distribution. The agent decides what budget to

[4] All posterior variables are formes using Bayes rule.

allocate to information decision. The choice of the budget determines by how much the variance will be reduced. In the spirit of Nieuwerburgh and Veldkamp (2010) we interpret it as an investment into purchasing additional market data, when an agent does not have prior knowledge of what is in the data, but knows that this data will sharpen his/her forecast. We model this decision as a choice of budget that determines posterior variance, $\hat{\sigma}_t^2$. When choosing the budget and posterior variance, the agent takes into account what the return expectations will be after the signal is observed. In other words, the agent has to form expectations about return expectations: expected posterior $E\hat{\mu}$. Yet before paying for the signal and observing it, the expected posterior equals the prior $E\hat{\mu} = \mu$.

When taking decisions in subperiod 1, the agent rationally anticipates the demand for the risky asset in the subperiod 2 as in (2.3) where $\hat{\sigma}_t^2$ is posterior variance of the return. Thus, with the information investment - budget b_t, and (2.3), the banks utility is rewritten:

$$\max_{b_t, k_t^b} E_t \Pi_{t+1} - \frac{1}{\rho} Var\left(\Pi_{t+1}\right), \tag{2.4}$$

subject to the learning rule:

$$f\left(\sigma_t^2, \hat{\sigma}_t^2\right) \cdot a \leq b_t, \tag{2.5}$$

and non-forgetting constraint: $\sigma_t^2 - \hat{\sigma}_t^2 > 0$. a is cost of reducing the variance, and $f\left(\sigma_t^2, \hat{\sigma}_t^2\right)$ is the learning function. The function is continuous and monotone in both of its arguments[5], it is increasing in initial variance, σ_t^2, and is decreasing in posterior, $\hat{\sigma}_t^2$. Intuitively, the more we reduce the posterior variance relative to the prior, the more we should pay. We assume that the information budget is exhausted so that (2.5) becomes equality. Then with the properties of our learning function, the choice of the information budget, b_t, uniquely determines the posterior variance and captures the information decision of the bank.

In the following section we consider risk-taking decisions of the bank in a partial equilibrium to identify risk driving forces.

Aggregating Financial Markets The total investment into the safe asset, res, is given by the bank's financial resources not invested into the risky asset:

$$res_t = d_t - k_t^b.$$

[5]This implies for positive and finite variances.

The investment into the safe asset is determined as deposits, d_t, that was not invested in the risky asset, k_t^b.

Recall, that the risky asset in the model is the investment in the manufacturing firm, which uses it to built new capital. The manufacturing firm does not have funds for investment on its own. To invest it has to sell its claims to the bank. Thus, the total investment into the capital is then given by the bank's risky asset holdings:

$$I_t = k_t^b.$$

2.5 Excessive Risk-Taking and Information Acquisition

In this section we analyze the two channels through which a bank accumulates risk in the portfolio when the safe interest rate is reduced or market volatility declines. One of them is clear from (2.3): whenever the safe interest rate drops, it increases the risk-premium and makes the risky asset more attractive. Similarly, when asset variance is reduced, the bank rationally increases holdings of the risky asset. The other channel highlighted in this paper is a change in information acquisition: reduction in the information budget. Through this channel, the bank increases the riskiness of the asset per se by choosing to learn less about it. The portfolio risk then, as a product of risky asset holdings and return variance, increases with the lower interest rate and, in some cases, lower market volatility.

At first glance, the reduction in information acquisition with increase in risky asset holdings might seem counter-intuitive. It could be suggested that with larger asset holdings, agents would like to learn more about them. For example Nieuwerburgh and Veldkamp (2010) found that when allocating fixed learning capacity between the assets, agents allocate more to those assets they invest more into. Here, we should remind the reader that in our paper we are studying not the allocation of the fixed capacity, but the determination of this capacity: by how much agents are willing to reduce their expected income in order to reduce the income variance. Also this capacity, in the form of the information budget, is itself a function of expected return and initial variance. It describes a trade-off between the return the agent expects to obtain and variance he/she would like to reduce. Below, we study the properties of the information budget for specified learning functions.

As learning function choice could influence the results (and we show later that this is the case), we consider alternative functions. Nieuwerburgh and Veldkamp (2010) show that the choice of utility function and learning technologies influences results quantitatively and, sometimes, qualitatively. They consider mean-variance and exponential utility functions, and three learning rules: one linear and two entropy based measures. Below, we study mean-variance utility under linear and entropy learning functions.

Information Budget and Comparative Statics

We consider alternative learning functions, $f\left(\sigma_t^2, \hat{\sigma}_t^2\right)$ in (2.5): linear rule and entropy based. The linear function implies that the bank pays fixed costs, a, for each unit of the linear decline in the variance:

$$b_t = a \cdot \left(\sigma_t^2 - \hat{\sigma}_t^2\right). \tag{2.6}$$

Linear constraint is an intuitive rule and simple to work with. The one caveat is that it is marginally as costly for the agents to reduce the variance by 1% as by 100%. Agents potentially could choose to learn the whole truth and choose the posterior to be zero. This, of course, is very costly for them in absolute terms of linear costs, a, and this never happened in our simulations. But in the general case, one should consider this possibility.

The entropy based constraint implies that the agent pays for each unit of log variance decrease. One can find some variation in the definition of the entropy based learning rule. For example, in Nieuwerburgh and Veldkamp (2010) it is the simple ratio of prior to posterior variance. Mackowiak and Wiederholt (2009a) use the logarithm of base 2, while there are many papers on rational inattention using a natural logarithm (e.g. Matejka and McKay (2015) and Cabrales, Gossner, and Serrano (2013)) In our definition of entropy we follow Mackowiak and Wiederholt (2009a)[6]:

$$b_t = a \cdot \log_2 \left(\frac{\sigma_t^2}{\hat{\sigma}_t^2}\right). \tag{2.7}$$

The advantage of the entropy rule is that when the agent gets closer to learning the true state of the world (posterior variance goes to zero), the required budget goes to infinity. The entropy constraint is also well-motivated for analysis of processing the information subject to limited capacity. In our case, however, the agent's decision resembles more

[6]The results with a natural algorithm do not differ qualitatively, and there is a minor quantitative difference.

a choice of a quality of market report to buy or market expert to pay, than processing market data him/herself. That is, in our view, both types of constraints are well reasoned here.

To select the information budget the agent maximizes the utility as in (2.4), but the decision is now divided in two subperiods. The information budget is chosen in the first subperiod:

$$\max_{b_t,} E_{t,1}\left(E_{t,2}\Pi_{t+1} - \frac{1}{\rho}Var_{t,2}\left(\Pi_{t+1}\right)\right) \tag{2.8}$$

subject to (2.3) and posterior variance, $\hat{\sigma}_t^2$, given by one of the learning rules: (2.6) or (2.7).

Note that in (3.11) the agent chooses b_t in the first subperiod before knowing his expected return in the second subperiod (before the signal - market report - is realized). Adopting the formula from Nieuwerburgh and Veldkamp (2010), formula 14, we have:

$$\max_{b_t} -0.5 + \frac{\sigma_t^2}{\hat{\sigma}_t^2}\cdot\left(1+\frac{(\mu_t - R_t^s)^2}{\sigma_t^2}\right) - b_t.$$

For the learning rule considered we have the solutions in table 2.2.

	Linear rule	**Entropy rule**
Information budget, b_t	$a\sigma_t^2 - \sqrt{a(\sigma_t^2+(\mu_t-R_t^s)^2)}$	$0.5a\left(\frac{\log[\frac{a\sigma_t^2}{(\sigma_t^2+(\mu_t-R_t^s)^2)\log 2)}]}{\log 2}-1\right)$
Posterior variance, $\hat{\sigma}_t^2$	$\frac{\sqrt{a(\sigma_t^2+(\mu_t-R_t^s)^2)}}{a}$	$2\cdot\frac{(\sigma_t^2+(\mu_t-R_t^s)^2)\log 2)}{a}$
Portfolio variance, $\hat{\sigma}_t^2\cdot\left(k_t^b\right)^2$	$\frac{a(\mu_t-R_t^s)^2}{\rho^2\sqrt{a(\sigma_t^2+(\mu_t-R_t^s)^2)}}$	$\frac{a(\mu_t-R_t^s)^2}{2\rho^2(\sigma_t^2+(\mu_t-R_t^s)^2)\log 2)}$

Table 2.2: Solutions to Partial Equilibrium Model

It is instructive to analyze comparative statics of the resulting solutions. In the partial equilibrium model we take as given both assets' returns, R_{t+1}^r and its mean μ_t, and R_t^s. It will be convenient then to consider the model's response to the change in expected risk premia, $\mu_t - R_t^s$. In a general equilibrium, both returns will be determined by the market clearing condition, with a stochastic component influencing risk asset return. In table 2.3, the changes in the information budget with respect to variables of interest are described (for full description of the derivatives, the reader is referred to the appendix).

Information budget derivatives with respect to:	Linear rule	Entropy rule
risk premium $\frac{\partial b_t}{\partial(\mu_t - R_t^s)}$	negative	negative
initial variance $\frac{\partial b_t}{\partial \sigma_t^2}$	positive	positive
info. costs $\frac{\partial b_t}{\partial a}$	positive	positive

Table 2.3: Comparative Statics: Information Budget

Proposition 7. *Low safe interest rates and low risky asset volatility reduce the incentives to require information.*

Proof. Consider derivatives under both learning rules in table 2.3, we see the similar signs of the responses. The information budget rises when initial variance rises, so that with larger volatility in the market, agents are willing to sacrifice a larger budget to reduce uncertainty. Also, with a larger expected risk premium agents are willing to invest less in reducing the uncertainty, as the larger expected return compensates agents for taking a risk. □

Proposition 7 explains the information channel of an increase in risk-taking. When the safe interest rate falls, it increases the expected risk premium (which is $\mu_t - R_t^s$), and decreases the information budget. With a lower information budget, the agent has a larger posterior variance. Similarly, with a lower initial volatility (prior variance), the agent decides to have a smaller information budget. The initial effect of a reduction in interest rate or initial variance on the risky asset position is positive. It could be suggested that a small information budget and smaller initial variance may offset this effect. We show below that this is not the case in our model. The bank's risky position rises, and, together with a small information acquisition, drives up portfolio variance.

Risk Accumulation in Partial Equilibrium

To study risk accumulation we use the expression for posterior variance in table 2.2. Proposition 8 formalizes the findings.

Proposition 8. *Low safe rate and low volatility increase portfolio variance.*

Proof. calculating derivatives with respect to risk premium and prior variance, we find that risky asset holdings decrease in initial variance and increase in risk premium[7]. □

Figure 2.3 illustrates proposition 8. The graphs were drawn with fixed interest rates[8]. Later in the paper we analyze a general equilibrium model where interest rates are set by the market.

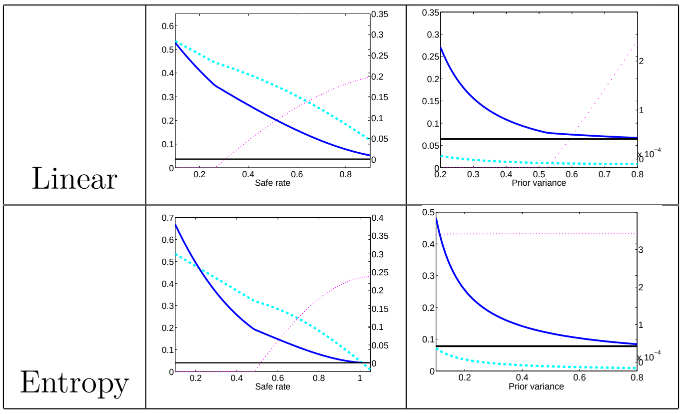


Figure 2.3: Risk Accumulation in a Partial Equilibrium

Note: the dotted line corresponds to information budget b, the dashed line to risky asset holdings kb, the blue solid line - to portfolio variance (right axis), and the black solid line to the steady state portfolio variance (right axis)

In figure 2.3 panels *a* and *b* correspond to a model with a linear learning rule; and *c* and *d* an entropy learning rule. The solid black line on all the graphs shows the initial (before reduction in safe interest rate and variance) portfolio variance. In the case of safe rate reduction the portfolio decisions under initial variance thus correspond to a baseline case as if agents do not reduce their information acquisition. In the case of prior variance reduction it shows the benchmark of portfolio variance with higher risky asset variance. The solid blue line represents portfolio variance, its rise over the initial level shows the increase in portfolio variance. The channels of portfolio variance increase are clear from the figure: there is a decline in information acquisition, b_t[9], and an increase in risky asset holdings, k_t^b.

Panels *a* and *c* in figure 2.3 show, that when the safe interest rate falls, there is a larger risk accumulated in the portfolio. The risky asset position increases and the information

[7]All derivatives are in the appendix.

[8]The parameters are as described in table 2.4, the safe and risky interest rates are fixed at the steady state level.

[9]At some point (panels *b*-*d*) the information budget hits zero. At this point, the model behaves the same as the one without information acquisition. Below this point, a sharper increase in risky asset holdings, k_t^b is observed.

budget falls. This resembles the debate that a low interest rate environment stimulated excessive risk-taking during the Great Moderation. In our model, we capture also lower incentives to get information about the risky asset - the agent becomes more ignorant about the asset quality.

A similar result is found for reduction in market volatility in panels b and d. Surprisingly, when the prior variance falls, the agent ends up with a larger portfolio risk than in a higher variance environment - which is the baseline black line. This result is, again, driven by the information channel: an agent is willing to pay less for variance reduction when it is already small; and by larger risky asset accumulation when the risk gets smaller. This finding could be also be applied to the Great Moderation period, when market volatility was perceived to be low and financial agents demonstrated a higher risk appetite.

Of course, when trying to explain overaccumulation of risk during the Great Moderation, other forces besides the low volatility, mentioned, and a low safe interest rate environment could be considered. We show in this paper, however, that market volatility and low policy rates could be contributing factors to increase in risk preferences. These are also important factors to consider when addressing central banks' current policy of low interest rates and suppressing market volatility.

Next, we complete the model and consider risk accumulation in a general equilibrium.

2.6 General Equilibrium Model

Here we briefly describe the rest of the model and general equilibrium. Then we consider the equilibrium impact of the interest rate change on risk preferences and information acquisition, when there is feedback between the agent's asset holdings and market interest rates.

Household There is a representative household which maximizes the following utility function:

$$\max_{\{c_{t+i},d_{t+i}\}_{i=0}^{\infty}} E_t \sum_{i=0}^{\infty} \beta^i u\left(c_{t+i}\right), \tag{2.9}$$

subject to a budget constraint:

$$d_t + c_t = \pi_t^{fin} + \pi_t^p - t_t, \tag{2.10}$$

where d is household investment into bank (recall, the household invests into the bank and gets all its profit in turn), π_t^{fin} is realized profit from the financial sector, π_t^p is realized profit from manufactures and t is a tax. The household decides how much to consume and to invest in the bank. Its income is generated by the bank's and manufacturer's profits net of lump-sum taxes. $u(c)$ is twice differentiable and concave. Note that we abstract from any labor decisions.

The consumption Euler equation looks standard and relates gross interest on savings to the stochastic discount factor:

$$u'(c_t) = R_{t+1}^d \beta E_t u'(c_{t+1}), \tag{2.11}$$

$$R_{t+1}^d = R_t^s + \frac{k_t^b}{d_t}\left(R_{t+1}^r - R_t^s\right). \tag{2.12}$$

Manufacturer On the production side there is a representative producer with a production function:

$$y_{t+1} = z_{t+1} k_t^{\alpha},$$

where z is stochastic productivity.

The producer needs to borrow money to finance investment (make new capital), and the law of motion for capital is then:

$$k_{t+1} = I_t + (1-\delta) k_t. \tag{2.13}$$

The producer maximizes one period profit, which consists of revenues minus payment on the loan for investment purposes:

$$\max_{k_{t+1}} E_t \pi_{t+1}^p = E_t\left(y_{t+1} - R_{t+1}^r * I_t\right) = E_t\left(z_{t+1} k_{t+1}^{\alpha} - R_{t+1}^r\left(k_{t+1} - (1-\delta) k_t\right)\right), \tag{2.14}$$

where R^r is the gross interest rate paid to investors in the capital. First order conditions with respect to capital determine R^r as

$$R_{t+1}^r = z_{t+1} \alpha \left(k_{t+1}\right)^{\alpha-1}. \tag{2.15}$$

That is, R^r depends on future productivity, is decreasing in capital, and is uncertain from the investors' point of view because of uncertain z. Productivity z is such that the expected return is as modeled in table 2.1: $\mu_t \sim N\left(R_{t+1}^r, \sigma_t^2\right)$.

Note that all variables are expressed in real terms - in the units of final output.

2.6.1 Central Bank and Government

It is assumed that the government pays gross interest on the safe asset, and finances expenditures by taxing the household. The government budget is balanced:

$$g_t = taxes_t = R^s_{t-1} res_{t-1} - res_t. \tag{2.16}$$

The role of the central bank in this economy is limited. Here we allow for a shock to the safe interest defined in (2.11) which is supposed to resemble monetary policy shock.

2.6.2 Equilibrium

Equilibrium in this model is a set of allocations: $\{c_{t,} d_t, y_t, k_t, k^b_t, res_t, b_{t,} \hat{\sigma}^2_t, g_t\}_0^\infty$ such that given prices $\left\{R^r_t, R^s_t, R^d_t\right\}$ and beliefs $\{\mu_t\}$ all agents solve their problems and markets clear:

- good market clears: $y_t = c_t + i_t + g_t$
- and capital market clears: $I_t = k^b_t$;, that is, new capital bought by the intermediaries, I_t, is equal to risky asset holding of the bank, k^b_t.
- and the government budget is balanced.

2.7 Simulations

2.7.1 Calibration and Parameter Values

In the model, most of the parameters are standard. Risk aversion is ρ, capital share in output α, depreciation δ, and household discount factor β. The only nonstandard parameters are learning costs, a, and initial variance of agent's beliefs, σ^2_t. This parameters was selected to match the steady-state risk-premium of 5.1%, what roughly corresponds to estimates of US risk-premium till [10]. Also, for alternative learning specifications, these parameters have to be slightly different.

[10]See, for example, dataset by prof. Damodaran http://pages.stern.nyu.edu/~adamodar/

		Linear	Entropy
ρ	risk-aversion	2	
α	capital share	0.33	
δ	depreciation	0.02	
β	discount factor	0.99	
a	information costs	1.8	1.9
σ_t^2	prior variance	0.81	0.91

Table 2.4: Parameter Values

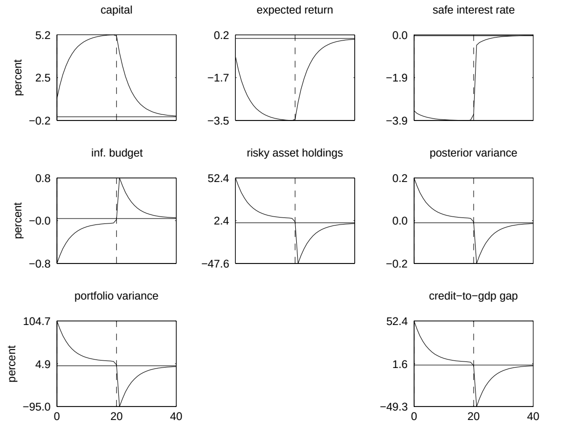
Table 2.4 shows the selected parameter values used for the simulation below. In the next subsection we show general equilibrium results for our model of information acquisition.

2.7.2 Simulations

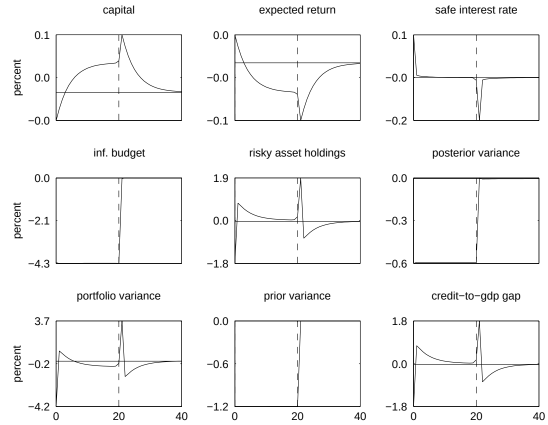
We start with a linear learning rule model. For the simulations[11], we lowered the initial variance or safe interest rate for 1 and 4 percentage points respectively for 20 periods. After 20 periods, both of the variables return to their steady state values. Figure 2.4 reports responses for a linear learning rule model. The vertical dashed lines mark the end of the decline in selected variables.

Figure 2.4a shows the reaction to a shock to the safe interest rate, which we here call "monetary policy". Recall that there is no money in the model, and this name is figurative to suggest that the shock to the safe interest rate resembles monetary authority action in a full-blown New Keynesian model. Following the decline in the safe interest rate, the bank's risky asset holdings increase. The risky asset is investment into capital in our economy, which is why additional capital is accumulated. Credit-to-GDP-ratio rises as banks invest more, and investment increases output in the next period. Larger capital accumulation reduces the expected return on capital. This is the force that returns the model to the steady state after the policy is removed. Before this, there is a drop in the information budget as a larger risk premium (expected return on risky asset falls less than safe interest rate) makes an agent tolerate larger risk. Lower information acquisition determines larger posterior variance. Both larger posterior variance and the risky asset position increase the bank's portfolio risk. For the change in initial variance, Figure

[11]The simulations are done using Dynare version 4.2.



(a) Accomodative "Mon. P." Shock, 4 pp



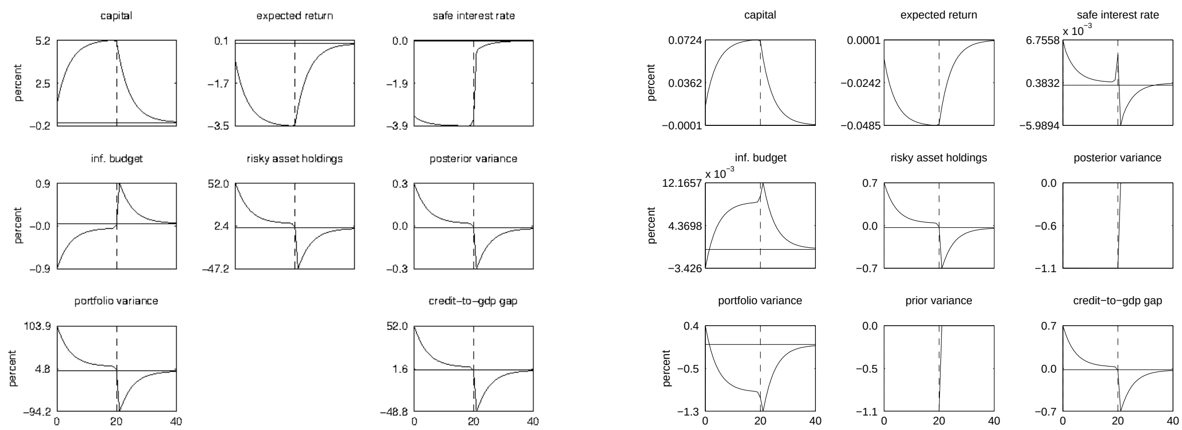
(b) Prior Variance Reduction by 1 pp

Figure 2.4: Linear Learing Rule

2.4b, accumulation of capital declines the return on capital, which is the risky asset in our model. When the initial variance falls, the information budget also falls. Posterior variance, being the difference of prior variance and the information budget, declines, but two times less than the prior because of a drop in information budget. A decline in the information budget here reduces the effect of initial volatility on the risk that agents are facing. This and a rise in risky asset portfolio holdings increase portfolio variance above the steady state level initially. When the expected return reaches its minimum value, risky asset holdings and portfolio variance start declining. After the policy is removed and the level of capital reduced, the increasing expected return returns the economy back to the steady state.

For the model with the entropy learning rule, Figure 2.5a, a very similar response to interest rate decline is found. A reduction in safe interest rates simultaneously reduces information acquisition and increases risky asset holdings. A combination of the two increases the bank's portfolio risk.

When considering a reduction in prior variance, Figure 2.5b, a different response of the information budget and safe interest rate is observed. Risky asset holdings are increased, raising capital and consumption and decreasing the expected return. At the same time there is a reduction in the information budget, but unlike in the linear model, this effect is short-lived, and is reversed in a couple of periods. This leads to short-lived increase in portfolio variance, which declines afterwards. If in the linear model the information budget is always below the steady state level for lower prior variance, it is not the case in entropy. With the entopy constraint, there is larger effect of falling expected return on



(a) Accomodative "Mon. P." Shock, 4 pp **(b)** Prior Variance Reduction by 1 pp

Figure 2.5: Entropy Learing Rule

the information budget (see appendix). In a simple model with fixed risky asset rate just a change in the prior variance decreases incentives to invest in information. With the expected return falling, the information budget starts to increase, decreasing posterior variance and portfolio risk.

2.7.3 Discussion

In the previous section it was shown that the effect of a low safe interest rate on an agent's portfolio risk is supported under both of the learning functions. In a low interest rate environment, the agent invests less in risk reduction and more in the risky asset. This results in a higher portfolio risk.

The effect of subdued market volatility is ambiguous. In a model with linear learning, the information budget falls in a low volatility environment. In a model with entropy learning, the information budget falls but for a very short time, and even rises afterwards, with the effect of falling return dominating the variance decline. This symmetrically implies that with a rise in market volatility, a decline in an agent's willingness to gain the information should be expected. At the same time, anecdotal evidence suggests that in times of higher volatility, people are trying to obtain more information. As anecdotal evidence we consider the index of search interest by Google trends. In figure 2.6 we show search trends from Google for three search terms: Federal Reserve, Mortgage backed securities and Quantitative Easing for the time period 2005 -2012[12]. The number of

[12]The numbers on the graph show search terms, relative to the total number of searches done on Google. They are further normalized by the largest number of searches and multiplied by 100.

searches in one of the leading search engines could serve as a demonstration of public interest in a particular topic.

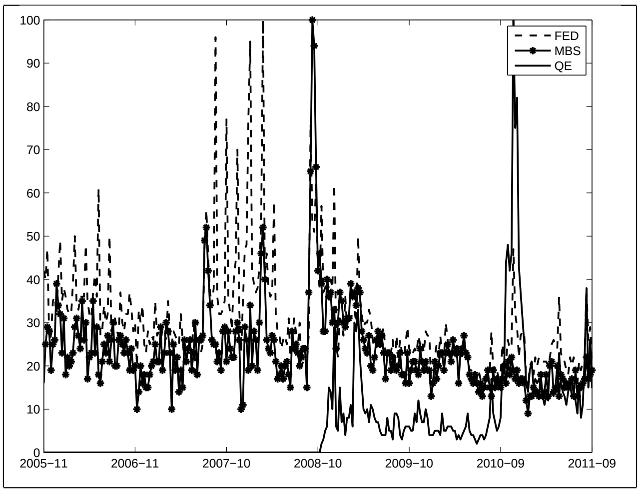


Figure 2.6: Search Trends

Data Source: Google Trends (www.google.com/trends)

Mortagage-backed securities (MBS), which were regarded as low risk before the financial crisis, attracted a lot of attention in the second half of 2007, when the uncertainty about them increased. There are also spikes in 2008 when holdings of MBS threatened viability of large financial players. After their riskiness is revealed, the interest for them is stabilized. An interesting example is search interest for the Federal Reserve System (FED). Before the second half of 2007, the Fed, like many other central banks, was fulfilling "routine" inflation targeting. After the onset of the crisis, the Fed started to act differently: as a lender of last resort, a conductor of unconventional policies. These actions were surrounded with large uncertainty around which institution is to be bailed out, and the design and implementation of unconventional policies. The introduction of quantitative easing in 2008 stimulated a spike in interest in the FED and in quantitative easing (QE) itself. An increase in searches for QE is also observed in late 2010 associated with the second round of QE. Of course, this is reflected in spikes in the search interest. After some time the uncertainty surrounding unconventional policies declines, as does the interest in them. One could object that spikes in internet searches may be driven by news releases, mentioning search terms intensively. But the news itself reflected the rise in uncertainty and importance of a particular topic. For instance, the Fed was as important at the onset of the crisis in 2007 as it was in the middle of 2009 when financial markets were not yet fully recovered and the economy was moving into deep recession.

But the interest in the Fed in 2009 was three times lower than at the end of 2007, when uncertainty about the Fed's actions was the largest. For this reason we suggest it is reasonable to interpret this anecdotal evidence as not contradicting our suggestion that a rise in volatility stimulates attention.

It is also reasonable to suggest that a decline in the interest in a particular term is associated with low volatility about this term. In the light of our results, this means that lower market volatility leads to lower incentives to acquire information about the risky asset, leading to larger portfolio risk than in a high volatility environment. The implications from the model with a linear learning rule are more in line with this suggestion.

2.8 Conclusion

This paper addresses the debate as to whether periods of low policy rates and low market volatility could lead to overaccumulation of risky assets. It is motivated by a number of empirical studies showing that an increase in risk appetite is associated with low policy rates.

We contribute to the literature by building a model with rationally inattentive financial agents who decide how much to invest in information acquisition subject to information costs. Information acquisition is modeled as paying for a decline in risky asset variance. We consider two basic learning functions: entropy and linear learning rule.

It is then shown that with a low safe interest rate there are two channels of increase in risk-taking: a standard in the literature search-for-yield, and a decline in the information budget. These two channels result in a high risky asset position and high risk of the asset per se, as an agent faces higher uncertainty about asset returns. As a result, the agent accumulates more risk in his or her portfolio when the safe asset rate falls. These findings are robust to the learning rule specification.

Another result is larger risk-taking with the decline in risky asset volatility under the linear learning rule. When the variance of a risky return falls, agents rationally increase their risky asset holdings. At the same time, they are willing to pay less for further reduction in return variance. Lower incentives for information acquisition partially offset the drop in initial variance, with posterior variance falling much less than the prior. In combination with larger risky asset holdings, it increases the agent's portfolio variance. Under the entropy learning rule, there is a larger influence of expected risk premium on the choice of the information budget. Then when prior variance is reduced and risky asset holdings are increased, there is a fall in expected risk premium.

2.A Comparative Statics

2.A.1 Linear Learning Rule

From table 2.2 the solution for information budget is positive when information costs are:

$$a > \frac{(\mu_t - R_t^s)^2}{\sigma_t^2} + 1. \tag{2.17}$$

That is, larger than one plus the expected return to variance ratio. In this interval, the derivative with respect to initial variance is positive:

$$a\left(1 - \frac{1}{2\sqrt{a\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)}}\right) > 0.$$

And the derivative with respect to risk premium is non-positive:

$$-\frac{a\left(\mu_t - R_t^s\right)}{\sqrt{a\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)}} \leq 0.$$

The impact of information costs increase is always positive on the interval with positive b_t :

$$\sigma_t^2 - \frac{\sqrt{a\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)}}{2a} > 0.$$

The effect on risky asset portfolio holdings is characterized by the following derivatives:

$$\begin{aligned}
\frac{\partial k_t^b}{\partial \sigma_t^2} &= -\frac{a^2\left(\mu_t - R_t^s\right)}{2\rho\left(a\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)\right)^{\frac{3}{2}}} < 0, \\
\frac{\partial k_t^b}{\partial\left(\mu_t - R_t^s\right)} &= \frac{a^2\sigma_t^2}{2\rho\left(a\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)\right)^{\frac{3}{2}}} > 0.
\end{aligned}$$

For the portfolio variance, $\left(k_t^b\right)^2 \hat{\sigma}_t^2$ derivative with respect to initial variance:

$$-\frac{a^2\left(\mu_t - R_t^s\right)^2}{2\rho^2\left(a\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)\right)^{\frac{3}{2}}} < 0.$$

2.A.2 Entropy Learning Rule

From the formula in table 2.2, b_t, is positive when

$$\log[\frac{a\sigma_t^2}{(\mu_t - R_t^s)^2 + \sigma_t^2}] > \log[\log[2]] - \log[2] = \log\left[\frac{\log[2]}{2}\right] = -1.0597.$$

The derivative of budget with respect to initial variance, σ_t^2 is always non-negative:

$$\frac{a(\mu_t - R_t^s)^2}{\sigma_t^2\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)\log(4)} \geq 0.$$

The derivative with respect to risk premia $(\mu_t - R_t^s)$ is always non-positive:

$$-\frac{a(\mu_t - R_t^s)}{\left((\mu_t - R_t^s)^2 + sigma\right)\log[2]} \leq 0.$$

The derivative of budget, b_t, with respect to information costs, a, is:

$$\frac{1 + \log[\frac{a\sigma_t^2}{(\mu_t - R_t^s)^2 + \sigma_t^2}] - \log[\log[4]]}{\log(4)}.$$

The sign of the derivative is determined by the nominator. The derivative is positive when:

$$\log[\frac{a\sigma_t^2}{(\mu_t - R_t^s)^2 + \sigma_t^2}] > \log[\log[4]] - 1 = \log\left[\frac{2\log 2}{e}\right] = -0.6703.$$

Since $-0.6703 > -1.0597$, there is a region where the derivative could be negative. The information budget is decreasing with information cost, when information costs are:

$$\frac{\log 2}{2}\left(\frac{(\mu_t - R_t^s)^2}{\sigma_t^2} + 1\right) < a < \frac{2\log 2}{e}\left(\frac{(\mu_t - R_t^s)^2}{\sigma_t^2} + 1\right).$$

Thus for relatively small information costs, an increase in information cost will reduce the information budget. For other, feasible values of a, an increase in information costs also increases the information budget.

The effect on risky asset portfolio holdings is characterized by the following derivatives:

$$\frac{\partial k_t^b}{\partial \sigma_t^2} = -\frac{a(\mu_t - R_t^s)}{\rho\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)^2\log(4)} < 0,$$

$$\frac{\partial k_t^b}{\partial\left(\mu_t - R_t^s\right)} = \frac{a\left(-\left(\mu_t - R_t^s\right)^2 + \sigma_t^2\right)}{\rho\left(\left(\mu_t - R_t^s\right)^2 + \sigma_t^2\right)^2 \log(4)} > 0 \text{ if } \sigma_t^2 > \left(\mu_t - R_t^s\right)^2.$$

For the portfolio variance, $\left(k_t^b\right)^2 \hat{\sigma}_t^2$ derivative with respect to initial variance:

$$-\frac{a^2\left(\mu_t - R_t^s\right)^2}{\rho^2\left(a\left(\left(\mu_t - R_t^s\right)^2 + \sigma_t^2\right)\right)^3 \log(4)} < 0.$$

To see how risk premium affects the impact of initial variance of information budget under both rules consider the following derivatives:

	Linear rule	Entropy rule
$\frac{\partial^2 b_t}{\partial \sigma_t^2 \partial(\mu_t - R_t^s)} =$	$\frac{a^2 \mu}{2\left(a\left[(\mu_t - R_t^s)^2 + \sigma_t^2\right]\right)^{\frac{3}{2}}}$	$\frac{a\mu}{\left((\mu_t - R_t^s)^2 + \sigma_t^2\right)^2 \log 2}$

The derivative under entropy rule is larger than under linear rule and for $\sigma_t^2 \leq \frac{4}{\log 4}$ and $a > \frac{(\mu_t - R_t^s)^2 + \sigma_t^2}{\sigma_t^2}$. The last condition is (3.14) for information budget to be positive, and the first condition is always satisfied in our simulation.

Chapter 3

Sparse Restricted Perception Equilibrium[1]

In this work we consider a concept of sparse rationality (developed by Gabaix, 2014) as a selection tool in a model with multiple equilibria. Under sparse rationality, paying attention to all possible variables is costly, and the agents could choose to over- or under-emphasize particular variables or even to fully exclude some. Our main question is whether an initially mis-specified equilibrium (the Restricted Perceptions Equilibrium, or RPE) is compatible with the equilibrium choice of sparse weights, describing allocation of attention to different variables by the agents inhabiting this RPE. In a simple business cycle model, we find that the agents stick to their initial mis-specified AR(1) forecasting model choice if the feedback from expectations in the model is strong or if inflation becomes more persistent. We also identify a region in the parameter space where the agents find it advantageous to pay attention to no variable at all.

3.1 Introduction

It has long been understood for a long time that the hypothesis of Rational Expectations (RE), while delivering a theoretically elegant, model consistent, and typically unique solution for agents' expectations, imposes on agents cognitive and computational demands that might be incompatible with reality. As a result, deviations from RE have been studied in a growing stream of theoretical and empirical literature, including Bounded Ra-

[1]This chapter is a joint work with Sergey Slobodyan. The work has benefited from the financial support by European Unions Seventh Framework Program (FP7) for research, technological development and demonstration under grant agreement number 612796

tionality, Marcet and Sargent (1989), Adaptive Learning, Evans and Honkapohja (2001), Sticky Information, Mankiw and Reis (2007b), Rational Inattention, Sims (2003), and Sparse Rationality, Gabaix (2014), among others.

In an adaptive learning approach to modeling deviations from RE, agents are assumed to not possess prior knowledge of the underlying structure of the economy and to be gradually learning the coefficients in their forecasting rules. A survey of this approach to learning in macroeconomics can be found in Evans and Honkapohja (2009). Several papers have shown that adaptive agents can persist in using forecasting rules that are mis-specified relative to the RE rules, cf., Molnar (2007) and Evans et al. (2012). In Molnar (2007), there is a class of agents who are learning what is the best forecasting model given past data. Even if their forecasting rules are mis-specified, such learners can survive competition with RE agents. In Evans et al. (2012), convergence to a mis-specified equilibrium happens when the expectation feedback is strong. Adam (2005) considers an economy where agents are restricted to proceeding only a certain number of variables in the regression and thus to use underparametrized forecasting rules. As the agents' expectations affect the data generating process of the model and induce a Restricted Perception Equilibrium (RPE), the restricted rule can outperform the rational expectation rule in equilibrium. Similarly to Evans et al. (2012), this happens for the large enough feedback from the expectations to the outcome variable, governed by the elasticity of labour supply.

One of the ways to justify the agents use of mis-specified forecasting rules is to assume that they have limited information processing capacity, as in the rational inattention literature, cf., Sims (2003), Mackowiak and Wiederholt (2009b), and Matejka and McKay (2015). In this literature, attention allocation is based on the concept of entropy. Gabaix (2014) pursues a different, less computationally demanding, approach, with agents allocating attention weights to variables based on their relative importance.

In this paper we contribute to this literature and make connections between two distinct concepts, adaptive learning and sparse rationality. We study an economy where adaptively learning agents choose a strict subset of variables forming the RE equilibrium for their forecasting functions, thus inducing a RPE. We then allow these agents, inhabiting the RPE, to reconsider their forecast rules, subject to the informational cost constraint modeled as in Gabaix (2014). We ask whether the model parameters which make the RPE stable are sufficient to ensure that the same subset of variables is selected by informationally constrained agents. In other words, we are interested in whether the initial mis-specification becomes self-perpetuating in case of informational constraints.

The paper is organized as follows. In the second section we study an economy where agents learn about a simple exogenous process. We derive analytical results and provide economic intuition. In the third section we move to a business cycle model as in Adam (2005) and study the conditions for a mis-specified rule to be used in equilibrium. The last section concludes.

3.2 Simple Model

We start our analysis with a simple process

$$y_t = \alpha + \beta E_t y_{t+1} + \gamma_1 w_t^1 + \gamma_2 w_t^2 + \eta_t, \tag{3.1a}$$

where w_t^1 and w_t^2 are persistent shocks such that

$$\begin{bmatrix} w_t^1 \\ w_t^2 \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix} \cdot \begin{bmatrix} w_{t-1}^1 \\ w_{t-1}^2 \end{bmatrix} + \begin{bmatrix} \epsilon_t^1 \\ \epsilon_t^2 \end{bmatrix}.$$

The (w_t^1, w_t^2) shocks are normally distributed around zero with variance-covariance matrix

$$\Sigma^w = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix},$$

with $\rho \in [-1, 1]$ being the correlation coefficient between the shocks, defined as $\frac{Cov\left(w_t^1, w_t^2\right)}{\sigma_1\sigma_2}$ and η_t is a white noise. The RE minimum state variable solution (MSV) of this model is given by

$$y_t = a + g_1 w_t^1 + g_2 w_t^2 + \eta_t,$$

with MSV coefficients:

$$g_1 = \frac{\gamma_1}{1 - \beta\rho_1}, \tag{3.2}$$

$$g_1 = \frac{\gamma_2}{1 - \beta\rho_2}. \tag{3.3}$$

We restrict the agents to using only one variable in their forecasting models, as in the framework of Adam (2005) and Adam (2007) .[2] In our model, their forecasting rule could use either w_t^1 or w_t^2:

$$\begin{aligned} y_t &= a_1 + b_1 w_t^1 + \eta_t, & (3.4) \\ y_t &= a_2 + b_2 w_t^2 + \eta_t. & (3.5) \end{aligned}$$

Without loss of generality, we assume that our agents use (3.4) as their Perceived Law of Motion (PLM), which induces the restricted perceptions equilibrium that we call RPE1 when agents use this PLM to form expectations about y_{t+1}. Substituting the forecast formed using (3.4) into (3.1a), we obtain the actual law of motion (ALM):

$$y_t = \alpha + \beta a_1 + \bar{b}_1 w_t^1 + \bar{b}_2 w_t^2 + \eta_t, \tag{3.6}$$

with

$$\begin{aligned} \bar{b}_1 &= \beta \rho_1 b_1 + \gamma_1, & (3.7) \\ \bar{b}_2 &= \gamma_2. & (3.8) \end{aligned}$$

We model our agents as econometricians who do not have prior knowledge of the underlying structure of the economy. They do the best they can using past data. In order for this learning process to converge, three conditions must hold. First, for the agents' PLM to be the equilibrium solution, the coefficient b_1 must be derived as a regression coefficient from ordinary least squares:

$$b_1 = \frac{Cov(y_t, w_t^1)}{Var\,(w_t^1)}. \tag{3.9}$$

Second, the equilibrium (3.4) must be expectationally stable (E-stable). Finally, the forecast errors produced by the rule of their choice, (3.4), must be smaller than those of the alternative, (3.5).

Proposition 9. *In RPE1 (RPE2), where the agents use as forecasting rule equation 3.4*

[2] Such a restriction could be motivated by empirical and experimental evidence, cf., Branch and Evans (2006), Adam (2007), Hommes (2014), and Pfajfar and Žakelj (2014), who show that very simple AR(1) rules might be used by subjects to forecast inflation in survey and experimental settings. Several papers that estimate DSGE models with adaptive expectations, for example, **?**) and Ormeno and Molnar (2015), show that assuming the agents are using very simple forecasting rules leads to superior model fit.

(3.5), the equilibrium coefficient $b_1(b_2)$ *is given by*

$$\begin{aligned} b_1 &= \frac{\gamma_1 + \gamma_2 \cdot \rho \frac{\sigma_2}{\sigma_1}}{1 - \beta\rho_1}, \\ b_2 &= (\beta\rho_1 b_1 + \gamma_1) \cdot \rho \frac{\sigma_1}{\sigma_2} + \gamma_2. \end{aligned}$$

Both RPE1 and RPE2 are E-stable. If the following condition holds, then the mean squared forecast error (MSFE) for an agent inhabiting RPE1 and using (3.4) as forecasting rule, $MSFE_1$*, is smaller than MSFE of agent using (3.5),* $MSFE_2$*, and thus RPE1 is an equilibrium:*

$$b_1^2\sigma_1^2 > b_2^2\sigma_2^2. \tag{3.10}$$

Proof. Appendix 3.A and 3.A.1. □

We next allow the agents to challenge their equilibrium forecasting rules and possibly to reconsider them. They know that other variables exist in the RPE1 (3.6), and w_t^2 is observable. They also know that using w_t^2 alone for forecasting is inferior to using only w_t^1, because (3.10) is true. However they may wonder whether *adding* w_t^2 to their forecasting rule is beneficial. The forecasting rule that includes both w_t^1 and w_t^2 would be clearly superior in this model, if the agents were allowed to learn its coefficients, coinciding with $\bar{b}_1$ and $\bar{b}_2$.[3] The agents, however, are subject to attention cost, modeled as in Gabaix (2014). They could attach weights to a variable according to its importance. The importance of a variable depends on its contribution to the variance of the process and to the agents' utility. The weights then determine how much attention is paid to a variable given the exogenously given cost of attention, and the loss stemmimg from inattention, which is reduced quality of the forecast. We let the agents choose the attention vector by maximizing the precision of their forecast of y_t as in (3.6):

$$u = -\frac{1}{2}\left(\hat{y}_t - y_t\right)^2. \tag{3.11}$$

For agents with rational expectations, the optimal forecast is equal to $\hat{y}_t = \widehat{\bar{b}}_1 w_t^1 + \widehat{\bar{b}}_2 w_t^2$, where $\widehat{\bar{b}}_1$ and $\widehat{\bar{b}}_2$ are OLS estimates of the coefficients in (3.6). That is, agents with rational expectations use both shocks in the forecasting rule. Sparse rational agents face

[3]If *all* agents start using both shocks in their forecasting rule, using least-squares learning from then onward, then their PLM coefficients will converge to those of an MSV solution (3.2), (3.3), as shown in Evans and Honkapohja (1994). However, we are asking whether an *atomistic* agent could find it advantageous to use both shocks for forecasting.

a trade-off between attention cost and increase in forecast precision. That is why they choose to allocate attention between the variables and form their optimal forecast rule as $\hat{y}_t = \widehat{\bar{b}}_1 \{m_1 w_t^1\} + \widehat{\bar{b}}_2 \{m_2 w_t^2\}$, where m_1, $m_2 \in [0,1]$ are attention weights, and $\widehat{\bar{b}}_1$, $\widehat{\bar{b}}_2$ are OLS estimates of the coefficients in the selected forecasting rule. Clearly, if m_2 or/ and m_1 are close to zero, estimates of $\widehat{\bar{b}}_1$, $\widehat{\bar{b}}_2$ are different from the rational expectation case. If both weights are non-neglible, the estimates of the coefficients converge to the RE case. Denoting $x = (w_t^1, w_t^2)^T$, $\mu = (\bar{b}_1, \bar{b}_2)$, and $a = \hat{y}_t = \sum_{i=1}^{2} \widehat{\bar{b}}_i m_i x_i$ the agent's action: their forecasting rule, we rewrite (3.6) as

$$\max_{m_1, m_2} u = -\frac{1}{2} \left(a - \bar{b}_1 w_t^1 - \bar{b}_2 w_t^2 - \eta_t\right)^2 .$$

Then, using the formula 6 from Gabaix 2014, we derive the optimal attention vector m :

$$m = \arg \min_{m \in [0,1]^n} \frac{1}{2} \sum_{i,j=1\ldots n} (1 - m_i) \Lambda_{ij} (1 - m_j) + \kappa \sum_{i,j=1\ldots n} m_i \tag{3.12}$$

for $n = 2$. The loss from inattention is given by $\Lambda_{ij} = -\sigma_{ij} a_{w_i} u_{aa} a_{w_j}$, while the parameter κ governs the attention cost. Loss from inattention reflects how much variation is lost in the process when we neglect the variable. In the formula (3.12), σ_{ij} denotes the (i,j) element of shocks' variance-covariance matrix Σ^w, $a_{w_i} = -u_{aa}^{-1} u_{aw_i}$ determines how much a change in a variable, w_i, changes the agent's action a, and

$$\begin{aligned} u_{aa} &= \frac{\partial^2 u_a}{\partial a^2} = \frac{\partial}{\partial a}\left(-(a - \bar{b}_1 w_t^1 - \bar{b}_2 w_t^2)\right) = -1, \\ u_{aw_i} &= \bar{b}_i. \end{aligned}$$

Then $a_{w_i} = \bar{b}_i$, and the cost of inattention is therefore given by $\Lambda_{ij} = \sigma_{ij} \bar{b}_i \bar{b}_j$. Taking the derivatives of (3.12) with respect to m_1 and m_2 gives the following expressions (details are in Appendix 3.A.3):

$$m_1 = 1 - \frac{\kappa}{\bar{b}_1 \bar{b}_2 \sigma_1 \sigma_2 (1 - \rho^2)} \frac{\bar{b}_2 \sigma_2 - \bar{b}_1 \rho \sigma_1}{\bar{b}_1 \sigma_1}, \tag{3.13}$$

$$m_2 = 1 - \frac{\kappa}{\bar{b}_1 \bar{b}_2 \sigma_1 \sigma_2 (1 - \rho^2)} \frac{\bar{b}_1 \sigma_1 - \rho \bar{b}_2 \sigma_2}{\bar{b}_2 \sigma_2}. \tag{3.14}$$

For the agents to stick to the initial mis-specified rule, the weight on the shock w_t^2 must be zero or negative. In the RE MSV equilibrium, the weights on both shocks must be significantly larger than zero. For the agents to choose the second mis-specified rule, RPE2, the weight on the first shock, m_1, must be zero or negative.

Proposition 10. *The condition for $m_1 > m_2$ coincides with the condition for $MSFE_1 < MSFE_2$:*

$$\bar{b}_1^2 \sigma_1^2 > \bar{b}_2^2 \sigma_2^2.$$

Proof. Appendix 3.A.3. □

Proposition 10 states that as long as using PLM (3.4) which is consistent with RPE1, produces smaller forecast errors than using PLM (3.5), sparsely rational agents optimally pay more attention to the variable w_t^1 than to w_t^2. Alternatively, we could say that as long as variable w_t^1 is responsible for a larger scale of total variance of y_t than w_t^2 in the data generating process (3.6), the agents should pay more attention to w_t^1 than to w_t^2. If such agents continue to learning adaptively, they could converge to a RE MSV solution (see footnote 3). In order for PRE1 to remain the equilibrium even under further learning, the agents must pay *no* attention to w_t^2. We are therefore interested in situations where $m_2 \geq 0$.

Proposition 11. *The second shock has a positive weight in the agents' forecast when:*

$$\kappa < \frac{\left(1-\rho^2\right) \bar{b}_2^2 \sigma_2^2}{1-\rho \frac{\bar{b}_2 \sigma_2}{\bar{b}_1 \sigma_1}}, \tag{3.15}$$

Proof. Re-arranging (3.14),

$$\frac{\kappa}{\bar{b}_1 \bar{b}_2 \sigma_1 \sigma_2 \left(1-\rho^2\right)} \frac{\bar{b}_1 \sigma_1 - \rho \bar{b}_2 \sigma_2}{\bar{b}_2 \sigma_2} < 1 \Rightarrow \kappa < \frac{\left(1-\rho^2\right) \bar{b}_2^2 \sigma_2^2}{1-\rho \frac{\bar{b}_2 \sigma_2}{\bar{b}_1 \sigma_1}}.$$

□

We could interpret the condition (3.15) as follows. Consider the ALM (3.6). It includes two normally distributed variables, $\tilde{w}_t^1 = \bar{b}_1 w_t^1$ and $\tilde{w}_t^2 = \bar{b}_2 w_t^2$. The correlation coefficient between $\tilde{w}_t^1$ and $\tilde{w}_t^2$ is equal to ρ. Then $\rho \frac{\bar{b}_2 \sigma_2}{\bar{b}_1 \sigma_1}$ represents the coefficient in a regression of $\tilde{w}_t^2$ on $\tilde{w}_t^1$, while $\left(1-\rho^2\right) \bar{b}_2^2 \sigma_2^2$ is the variance of the conditional distribution of $\tilde{w}_t^2$ given $\tilde{w}_t^1$. Thus, if the cost of attention, κ, corrected for the information about $\tilde{w}_t^2$ already contained in (3.4), is less than the variance of omitted information — variance of $\tilde{w}_t^2$ conditional on $\tilde{w}_t^1$, then the agents find it beneficial to include the second shock in their forecasts. Otherwise, the cost of attention is too large, and it pays to stick to

(3.4) and fully exclude w_t^2 from the forecasting rule. The condition (3.15), then, has an intuitive economic interpretation of equalizing costs and benefits of considering that portion of information contained in the second shock w_t^2 which is above and beyond that which is already evaluated, given that w_t^1 is taken in account in the forecast.

Next, we normalize the attention costs in (3.15) by $\bar{b}_2^2\sigma_{\varepsilon 2}^2$, where $\sigma_{\varepsilon 2}^2$ is the variance of ϵ_t^2, the innovation to the shock w_t^2. We define the cost-to-variance ratio as $f \equiv \frac{\bar{\kappa}}{\bar{b}_2^2\sigma_{\varepsilon 2}^2} = \frac{\left(1-\rho^2\right)}{\left(1-\rho\frac{\bar{b}_2\sigma_{\varepsilon 2}}{\bar{b}_1\sigma_{\varepsilon 1}}\sqrt{\frac{1-\rho_1^2}{1-\rho_2^2}}\right)(1-\rho_2^2)}$, where $\bar{\kappa}$ is the threshold for the second shock to be included in the agents' rule. We plot this ratio in figure 3.1 in the coordinates $\left(\rho, \log(\Gamma\tilde{\Sigma})\right)$, where $\Gamma\tilde{\Sigma} \equiv \frac{\gamma_1\sigma_{\varepsilon_1}}{\gamma_2\sigma_{e_2}}$. We do this for different values of ρ_1 and ρ_2, autocorrelation coefficients of the shocks w_t^1 and w_t^2. The agents include w_t^2 in the forecasting rule when the normalized attention cost, $\frac{\kappa}{\bar{b}_2^2\sigma_{\varepsilon 2}^2}$, is smaller than f. Thus, larger f means that the range of costs consistent with w_t^2 used for forecasting is wider. In the figure, the white area corresponds to the parameter values where RPE1 is not selected, as it produces larger forecast error than RPE2. Large persistence of the w_t^2, ρ_2, decreases the parameter space where $MSFE_1 < MSFE_2$, while increasing ρ_1 expands this area. Higher ρ_1 increases the share of the y_t variance explained by the first shock, and thus decreases the value of taking the second one into account. Higher ρ_1 has the opposite effect.

Large correlation between the shocks, $|\rho|$, also contributes to better forecasting performance of RPE1-consistent PLM (3.35) so that $MSFE_1 < MSFE_2$. As described in Appendix 3.A.2, the $MSFE_1 < MSFE_2$ condition is satisfied for positive ρ if $\rho > \frac{1-\beta\rho_1-\Gamma\Sigma}{\beta\rho_1}$, which is the upper bound for the white area in all the graphs. As in the figure for $\rho_1 = 0.9$, in case of large persistence of the first shock so that $\beta\rho_1 > 1/2$, RPE1 PLM outperforms RPE2 also for very negative correlation, such that $\rho < \frac{-(1-\beta\rho_1)-\Gamma\Sigma}{\beta\rho_1}$. When the MSFE criterion (3.10) is satisfied, large absolute correlation increases the variation in y_t explained by the first shock alone making right hand side of (3.15) smaller. When the threshold for cost-to-variance ratio is small, it means that the second shock is added to the PLM only for very small learning costs to variance ratio.

With correlation close to zero, the second shock is included in the forecasting rule even when cost-to-variance ratio exceeds unity — that is, even for large attention costs. Note that this effect is more pronounced for positive correlation. To gain an intuition for this result, consider that if $Cov\left(\tilde{w}_t^1, \tilde{w}_t^2\right) = \rho\sigma_1\sigma_2$, then $Cov\left(\tilde{w}_t^1, \tilde{w}_t^2 - \rho\frac{\sigma_2}{\sigma_1}\tilde{w}_t^1\right) = 0$. The pair of orthogonal variables $\tilde{w}_t^1$ and $\tilde{w}_t^2 - \rho\frac{\sigma_2}{\sigma_1}\tilde{w}_t^1$ contains all the information (variance) contained

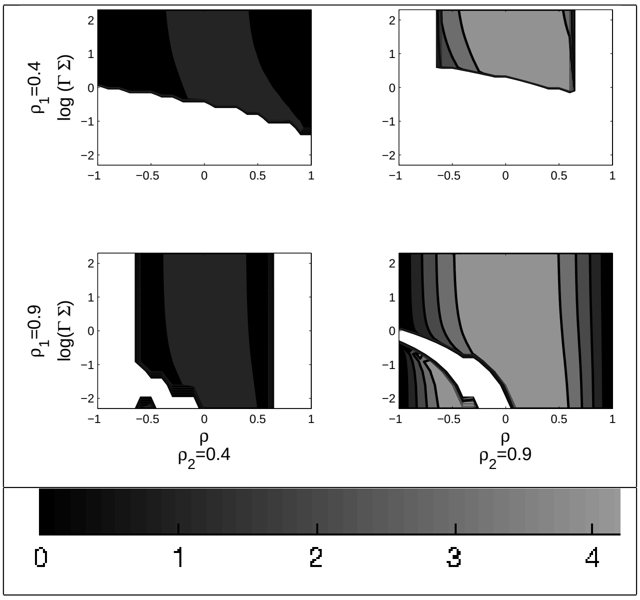


Figure 3.1: Threshold for the Cost-to-Variance Ratio

in the pair $\tilde{w}_t^1$ and $\tilde{w}_t^2$. However, their sum, $\tilde{w}_t^1 + \tilde{w}_t^2 - \rho\frac{\sigma_2}{\sigma_1}\tilde{w}_t^1 = \tilde{w}_t^1\left(1 - \rho\frac{\sigma_2}{\sigma_1}\right) + \tilde{w}_t^2$, have a smaller norm than $\tilde{w}_t^1 + \tilde{w}_t^2 = y_t$ for positive ρ. A consequence of the fact that we need a "shorter" vector to summarize all the information contained in $(\tilde{w}_t^1, \tilde{w}_t^2)$ is an effectively smaller attention cost, $\kappa\left(1 - \rho\frac{\sigma_2}{\sigma_1}\right)$. For negative ρ, the reasoning is the opposite: two orthogonal components sum up to a vector that is larger in norm than y_t, effectively increasing the attention cost. This effect of ρ leads to the asymmetry observed in Figure 3.1.

Finally, note that even under condition $\log(\Gamma\Sigma) > 0$, which means $\Gamma\Sigma > 1$, so that w_t^1 plays a more important role in (3.1a) then w_t^2, and both ρ_1 and ρ are large, so that taking into account w_t^1 is very informative (see the lower left panel of figure 3.1), then there could be still attention costs low enough for w_t^2 to be included into sparsely rational agents PLM. Therefore, unconstrained agents with $\kappa = 0$ will always find it beneficial to include w_t^2 in their forecasting rules.

Proposition 12. *When (3.10) is satisfied, the weight on the first shock, m_1 is non-positive when*

$$\begin{aligned} \kappa &\geq \frac{\bar{b}_1^2\sigma_1^2\left(1-\rho^2\right)}{1-\rho\frac{\bar{b}_1}{\bar{b}_2}\frac{\sigma_1}{\sigma_2}}, \\ \textit{and } \beta\rho_1 &> \frac{1}{2} \textit{ if } \bar{b}_1 < 0. \end{aligned}$$

Proof. Appendix 3.A.3. □

Proposition 12 states that if agents use PLM consistent with RPE1 and the cost of attention are large enough, agents choose not to pay attention to any variable. Note that according to the proposition 10, in RPE1 $m_1 > m_2$. That is, for if $m_1 \leq 0$, $m_2 < m_1 \leq 0$.

In the next section we move to a business cycle model and study cases in which the agents reconsider their forecasting rule.

3.3 A Model of Business Cycle

In this section we study a simple business cycle model as in Adam (2005). There is a unit mass of consumers - workers, entrepreneurs and the government. Workers choose consumption, c_t^j, and labor supply, n_t^j, when maximizing utility:

$$\max_{\{c_t^j,\, n_t^j\}} E_t \sum_{t=0}^{\infty} \beta^t \left(u\left(c_t^j\right) - v\left(n_t^j\right)\right),$$

where u and v are continuous, increasing and twice differentiable functions, $u" < 0$ and $v" \geq 0$. The coefficient of relative risk-aversion is smaller than one: $\frac{-u"(c)\cdot c}{u'(c)} < 1$ for all c. Subject to cash-in-advance-constraint and the budget:

$$c_t^j \leq \frac{m_{t-1}^j}{\Pi_t} + \tau_t, \tag{3.16}$$

$$m_t^j = \frac{m_{t-1}^j}{\Pi_t} - c_t^j + n_t^j w_t + \tau_t, \tag{3.17}$$

where w_t is wage, τ_t - lump sum transfers from the government, m_t^j - real money holdings, $\Pi_t = \frac{P_t}{P_{t-1}}$ is the inflation rate.

The maximization yields to labor supply function

$$n_t = n\left(w_t, E_t\left[\Pi_{t+1}\right]\right). \tag{3.18}$$

Entrepreneurs produce an intermediate consumption good q_t^i, and each intermediate good is an imperfect substitute in the construction of an aggregate consumption good:

$$c_t = \left(\int_{i\in[0,1]} \left(q^i\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, \tag{3.19}$$

with $0 \leq \sigma < 1$. The production technology is linear in labor costs. Because of imperfect competition among the entrepreneurs, they set the price P_t^i as a mark-up over expected production costs:

$$P_t^i = \frac{1}{1-\sigma} E_t \left\{ P_t w_t \right\}. \tag{3.20}$$

The government issues money and makes lump sum transfers to agents. The resulting money balancing equation is:

$$m_t = \frac{m_{t-1}}{\Pi_t} + \tau_t.$$

In equilibrium, output and inflation are functions of their past realization and of expectations about future values:

$$\Pi_t = \frac{1}{1-\sigma} E_{t-1} \left[\Pi_t w \left(\frac{y_{t-1}}{\Pi_t} + \tau_t, \Pi_{t+1} \right) \right], \tag{3.21}$$

$$y_t = \frac{(1-\sigma)\, y_{t-1}}{E_{t-1} \left[\Pi_t w \left(\frac{y_{t-1}}{\Pi_t} + \tau_t, \Pi_{t+1} \right) \right]} + \tau_t, \tag{3.22}$$

where $w\left(\frac{y_{t-1}}{\Pi_t} + \tau_t, \Pi_{t+1} \right)$ is an inverse of (3.18).

We further work with the linearized around the deterministic steady state system of (3.21) and (3.22)[4]:

$$\pi_t = -1 + (1 - \frac{1}{\varepsilon}) E_{t-1}\pi_t + E_{t-1}\pi_{t+1} + \frac{1}{\bar{y}\varepsilon} y_{t-1}, \tag{3.23}$$

$$y_t = 2\bar{y} - \bar{y}(1 - \frac{1}{\varepsilon}) E_{t-1}\pi_t - \bar{y} E_{t-1}\pi_{t+1} + (1 - \frac{1}{\varepsilon}) y_{t-1} + \tau_t, \tag{3.24}$$

where τ_t is white noise, and ε is the real wage elasticity of the labor supply. Parameter ε governs autocorrelation of inflation: with larger ε demand shock generates greater inflation (as labor costs do not change much) and influences feedback from expectations to outcome. With large ε past inflation has greater predictive power.

As proven in Adam (2005), the system has two solutions under rational expectations: the MSV solutions:

$$\pi_t = \frac{1}{\bar{y}} y_{t-1}, \tag{3.25}$$

and a non-stationary, non-zero inflation steady state:

$$\pi_t = 1 + \varepsilon - \frac{1}{\varepsilon \bar{y}} y_{t-1}.$$

[4]The deterministic steady state is $\bar{\Pi} = 1$, $\bar{\mathrm{y}}$=n$(1-\sigma, 1)$.

Adam (2005) considers two RPEs, in which agents use either lagged inflation or lagged output in their forecasting rules. In M_π agents use the rule

$$\hat{\pi}_t = \alpha_\pi + b_\pi \pi_{t-1}, \tag{3.26}$$

while in M_y it is:

$$\hat{\pi}_t = \alpha_y + c_y y_{t-1}. \tag{3.27}$$

M_y coincides with the MSV solution, while M_π is mis-specified as the lag of inflation does not enter the law of motion (3.23). Yet, if agents stick to M_π, the lag of inflation affects the actual law of motion through expectational terms. Adam (2005) shows that for large enough ε M_π-consistent PLM converges to RPE, and then compares forecast errors of rule (3.26) with those of (3.27). That is, if agents use (3.26), (3.27) results in larger forecast errors. We consider whether M_π would still be the equilibrium choice if the agents inhabiting the RPE induced by the forecasting rule (3.26) could consider adding y_{t-1} to their PLM, subject to non-zero attention costs.

We start by deriving the coefficients in agents forecast rules and in the actual law of motion for inflation and output:

$$\begin{aligned} \pi_t &= \bar{a}_\pi + \bar{b}_\pi \pi_{t-1} + \bar{c}_\pi y_{t-1}, \\ y_t &= \bar{a}_y + \bar{b}_{y\pi} \pi_{t-1} + \bar{c}_{y\pi} y_{t-1} + \tau_t, \end{aligned} \tag{3.28}$$

where the coefficients are the following:

$$\begin{aligned} \bar{a}_\pi &= \alpha_\pi \left(2 + b_\pi - \frac{1}{\varepsilon}\right) - 1, \\ \bar{b}_\pi &= b_\pi \left(1 + b_\pi - \frac{1}{\varepsilon}\right), \\ \bar{c}_\pi &= \frac{1}{\bar{y}\varepsilon}, \\ \bar{a}_{y\pi} &= \bar{y}\left(2 + a\left(\frac{1}{\varepsilon} - 2 - b_\pi\right)\right), \\ \bar{b}_{y\pi} &= \frac{b_\pi \bar{y}}{\varepsilon}\left(1 - \varepsilon - b_\pi \varepsilon\right), \\ \bar{c}_{y\pi} &= \frac{\varepsilon - 1}{\varepsilon}. \end{aligned}$$

Proposition 13. *In M_π (M_y), where the agents use equation 3.26 (3.27) as a forecasting* *rule, the equilibrium coefficients α_π, $b_\pi(\alpha_y,\, c_y)$ are given by*

$$\begin{aligned} \alpha_\pi &= 1 - b_\pi, \\ b_\pi &= \sqrt[3]{z} - \frac{3\,(\varepsilon - 1)}{9\varepsilon^2\sqrt[3]{z}} + \frac{1}{3\varepsilon}, \end{aligned} \tag{3.29}$$

$$\begin{aligned} a_y &= 1 - \frac{1}{\bar{y}\varepsilon} - b_\pi\left(1 + b_\pi - \frac{1}{\varepsilon}\right)\frac{\sigma_{\pi y}}{\sigma_y^2}, \\ c_y &= \frac{1}{\bar{y}\varepsilon} + b_\pi\left(1 + b_\pi - \frac{1}{\varepsilon}\right)\frac{\sigma_{\pi y}}{\sigma_y^2}. \end{aligned}$$

with $z = \frac{1}{54\varepsilon^3}(2 - 9\varepsilon - 27\varepsilon^2 + 27\varepsilon^3) + \frac{\sqrt{3}}{18\varepsilon^2}\sqrt{-5 + 26\varepsilon + 9\varepsilon^2 - 54\varepsilon^3 + 27\varepsilon^4}$. Both M_π and M_y are E-stable. If the following condition holds, then the mean squared forecast error (MSFE) for an agent inhabiting M_π and using (3.26) as a forecasting rule is smaller than MSFE of agent using (3.27), and thus M_π is an equilibrium:

$$\bar{b}_\pi^2\sigma_\pi^2 > \bar{c}_\pi^2\sigma_y^2.$$

M_π results in smaller mean forecast squared errors than M_y if:

$$\bar{\Gamma}^2\Sigma^2 > 1 \tag{3.30}$$

where $\Sigma = \frac{\sigma_\pi}{\sigma_y}$ and $\bar{\Gamma} = \frac{\bar{b}_\pi}{\bar{c}_\pi}$. That is, if M_π explains a larger share of variation in inflation in (3.28).

Proof. Appendix 3.B. □

Proposition 13 states that the criterion for a mis-specified PLM to be an equilibrium resembles the condition from the simple model, see proposition 10.

Our solution for b_π in (3.29) is continuous in ε for $\varepsilon > 1/3$ (see proof of proposition 13). We can thus plot it to see the range of values b_π can possibly take for all ranges of ε considered. Figure 3.2 shows that $|b_\pi|$<1 for the ranges of $\varepsilon \in (1/3, 3)$.

Now, if the actual law of motion is the same as in (3.28) and the agents could reconsider their forecasting rules, would they give significant weight to past output? In other words, would they stick to the mis-specified rule, or move to the RE solution[5]? Or, if attention

[5] Again, if agents start to include lagged output in their forecasting rules and employ least squares learning onward, they eventually learn the coefficient on past π and y, with the former being zero.

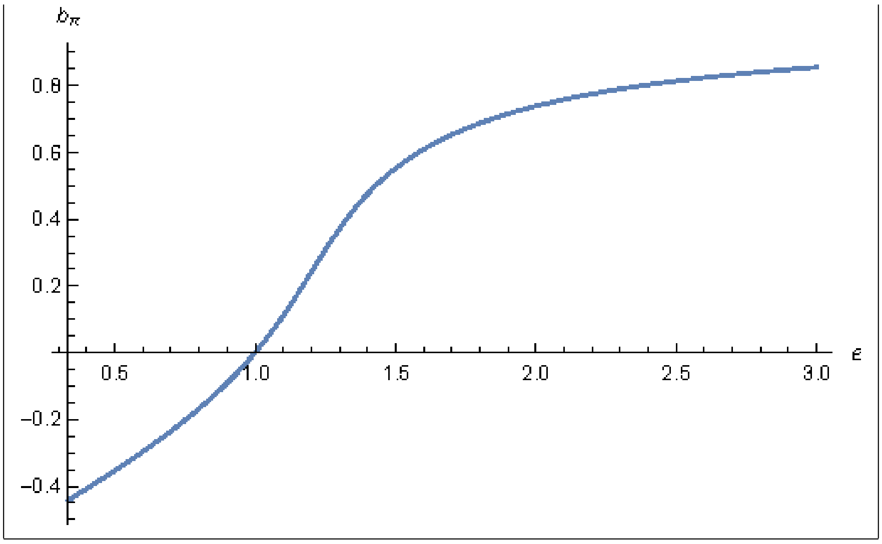


Figure 3.2: Coefficient on Past Inflation

costs are present, would agents choose not paying attention to any of the variables? To answer these questions, we allow the agents to select the sparse weights as in (3.11). The difference from the simple case above is the vector of variables, that are now endogenous to the system. That is, agents are minimizing $\hat{\pi}_t - \pi_t$ subject to attention cost. Denoting $a = \hat{\pi}_t = m_y \bar{c}_\pi y_{t-1} + m_\pi \bar{b}_\pi \pi_{t-1}$ to be the agents forecasting rule, $x = (\pi_{t-1}, y_{t-1})^T$, $\mu = \left(\bar{b}_\pi, \bar{c}_\pi\right)$, we rewrite utility as

$$u = -\frac{1}{2}(a - \mu x)^2 . \tag{3.31}$$

To find the sparse weights, agents minimize (3.12), where the cost of inattention, $\Lambda_{ij} = -\sigma_{ij} a_{w_i} u_{aa} a_{w_j}$, is now modified to account for new variables and covariance:

$$\begin{aligned}
u_{aa} &= \frac{\partial^2 u_a}{\partial a^2} = \frac{\partial}{\partial a}\left(-(a - \bar{b}_\pi \pi_{t-1} - \bar{c}_\pi y_{t-1})\right) = -1, \\
u_{a\pi_{t-1}} &= \bar{b}_\pi, \\
u_{ay_{t-1}} &= \bar{c}_\pi, \\
a_{\pi_{t-1}} &= -u_{aa} u_{a\pi_{t-1}} = -\bar{b}_\pi, \\
a_{y_{t-1}} &= -u_{aa} u_{ay_{t-1}} = -\bar{c}_\pi.
\end{aligned}$$

Then the cost of inattention is: $\Lambda = \begin{pmatrix} \sigma_\pi^2 \bar{b}_\pi^2 & \sigma_{\pi y} \bar{b}_\pi \bar{c}_\pi \\ \sigma_{\pi y} \bar{b}_\pi \bar{c}_\pi & \sigma_y^2 \bar{c}_\pi^2 \end{pmatrix}$, where σ_π^2, σ_y^2 and $\sigma_{\pi y}$ are the variance of inflation, output and covariance between them respectively under M_π, and are defined in (3.80), (3.81) and (3.79).

Taking first order conditions of (3.12) and solving for weights results in the following expressions:

$$\begin{aligned} m_y &= 1 - \frac{\kappa}{\bar{c}_\pi^2 \sigma_y^2 \sigma_\pi (1 - R^2)} \frac{\bar{b}_\pi \sigma_\pi - \bar{c}_\pi \sigma_y R}{\bar{b}_\pi}, & (3.32) \\ m_\pi &= 1 - \frac{\kappa}{\bar{b}_\pi^2 \sigma_\pi^2 \sigma_y (1 - R^2)} \frac{\bar{c}_\pi \sigma_y - \bar{b}_\pi \sigma_\pi R}{\bar{c}_\pi}, & (3.33) \end{aligned}$$

where $R = \frac{\sigma_{\pi y}}{\sigma_\pi \sigma_y}$ is the correlation between y_t and π_t in M_π

Proposition 14. *Whenever $MSFE_\pi < MSFE_y$, past inflation is given more weight than past output:*

$$m_\pi > m_y.$$

Proof. Consider (3.32) and (3.33) and using that $\bar{c}_\pi = \frac{1}{\bar{y}\varepsilon} > 0$ and $\bar{b}_\pi > 0$ for the range of $\varepsilon \in (1/3, 3)$ considered:

$$\begin{aligned} m_\pi &> m_y : \frac{\kappa}{\bar{b}_\pi^2 \sigma_\pi^2 \sigma_y (1 - R^2)} \frac{\bar{c}_\pi \sigma_y - \bar{b}_\pi \sigma_\pi R}{\bar{c}_\pi} < \frac{\kappa}{\bar{c}_\pi^2 \sigma_y^2 \sigma_\pi (1 - R^2)} \frac{\bar{b}_\pi \sigma_\pi - \bar{c}_\pi \sigma_y R}{\bar{b}_\pi}, \\ &: \frac{\bar{c}_\pi \sigma_y - \bar{b}_\pi \sigma_\pi R}{\bar{b}_\pi \sigma_\pi} < \frac{\bar{b}_\pi \sigma_\pi - \bar{c}_\pi \sigma_y R}{\bar{c}_\pi \sigma_y}, \\ &: \left(\bar{c}_\pi \sigma_y - \bar{b}_\pi \sigma_\pi R\right) \bar{c}_\pi \sigma_y < \bar{b}_\pi \sigma_\pi \left(\bar{b}_\pi \sigma_\pi - \bar{c}_\pi \sigma_y R\right), \\ &: \bar{c}_\pi^2 \sigma_y^2 - \bar{c}_\pi \sigma_y \bar{b}_\pi \sigma_\pi R < \bar{b}_\pi^2 \sigma_\pi^2 - \bar{b}_\pi \sigma_\pi \bar{c}_\pi \sigma_y R, \\ &: \bar{c}_\pi^2 \sigma_y^2 < \bar{b}_\pi^2 \sigma_\pi^2, \\ &: \bar{\Gamma}^2 \Sigma^2 > 1. \end{aligned}$$

The result is identical to condition (3.30). □

Proposition 15 summarizes the condition for output to be given a positive weight in the agents' forecast.

Proposition 15. *Lag of output is given positive weight in agents' forecast when*

$$\kappa < \frac{(1 - R^2)\, \bar{c}_\pi^2 \sigma_y^2}{1 - R \frac{\bar{c}_\pi \sigma_y}{\bar{b}_\pi \sigma_\pi}} \qquad (3.34)$$

Proof. Rearranging (3.32)

$$
\begin{aligned}
&: \quad \frac{\kappa}{\bar{c}_{\pi}^{2}\sigma_{y}^{2}\sigma_{\pi}\left(1-R^{2}\right)}\frac{\bar{b}_{\pi}\sigma_{\pi}-\bar{c}_{\pi}\sigma_{y}R}{\bar{b}_{\pi}}<1 \\
&: \quad \kappa<\frac{\bar{b}_{\pi}\bar{c}_{\pi}^{2}\sigma_{y}^{2}\sigma_{\pi}\left(1-R^{2}\right)}{\bar{b}_{\pi}\sigma_{\pi}-\bar{c}_{\pi}\sigma_{y}R}=\frac{\left(1-R^{2}\right)\bar{c}_{\pi}^{2}\sigma_{y}^{2}}{1-R\frac{\bar{c}_{\pi}\sigma_{y}}{\bar{b}_{\pi}\sigma_{\pi}}}
\end{aligned}
$$

where in the second step we take into account that $\bar{b}_{\pi} = b_{\pi}\left(1+b_{\pi}-\frac{1}{\varepsilon}\right) > 0$: when $\frac{1}{\varepsilon} < 1$, $b_{\pi} > 0$ (cf. figure 3.2), if $b_{\pi} < 0$ then $\varepsilon < 1$ and $\frac{1}{\varepsilon} > 1$. Then $1 + b_{\pi} - \frac{1}{\varepsilon} < 0$ and $b_{\pi}\left(1+b_{\pi}-\frac{1}{\varepsilon}\right) < 0$. □

Figure 3.3 shows the range of agents' choices under different learning costs, κ, and the labor supply elasticity, ε[6]. For ε smaller than 1.8 M_{π} produces larger forecast errors than M_{y}, therefore it cannot be the agents' equilibrium choice. This area is marked with white. When M_{π} explains a larger variation in inflation, the weight on lagged inflation is always larger than the weight on lagged output. The shades of dark grey on the figure show areas where, however, lagged output has non zero weight. That is, agents move to a forecasting rule with both past output and inflation present. The darkest area corresponds to small learning cost and both output and inflation are given high attention, with weight on lagged output larger than 0.9. In the extreme case with learning costs being zero, both weights are unity and agents move to a rational expectation equilibrium. The lighter area corresponds to smaller weight on output within the range of [0.000001, 0.5]. With learning costs growing larger, agents drop the lagged output from their rules, while attention weight on lagged inflation is different than zero. Note nonlinear interaction between learning cost and elasticity of labor supply, ε. Larger ε reflects larger prediction power of past inflation. For a given value of learning costs, larger ε results in lower weight on lagged output, and larger weight on lagged inflation. In the lightest area, agents do not allocate attention to any of the variables, and choose to use only constant in their forecasting rules. There the costs of learning are too high compared to the increased forecast precision.

Similar to the simple model case, we plot the threshold for the cost-to-variance ratio $\frac{\kappa}{\bar{c}_{\pi}^{2}\sigma_{y}^{2}} = \frac{\left(1-R^{2}\right)}{1-R\frac{\bar{c}_{\pi}\sigma_{y}}{\bar{b}_{\pi}\sigma_{\pi}}}$ in figure 3.4 (solid line). The threshold is plotted as a function of ε for $\varepsilon > 1.8$, where MSFE criterion (3.30) is satisfied. Again, the output is included in the forecasting rule only if the cost-to-variance ratio is smaller than the threshold plotted.

[6] Adam (2005) shows that the process is stationary for $0.35 \leq \varepsilon \leq 2.15$. That is why we consider the range of $\varepsilon \leq 2.15$. For $0.35 \leq \varepsilon < 1.8$ M_{π} has larger MSFE than M_{y}.

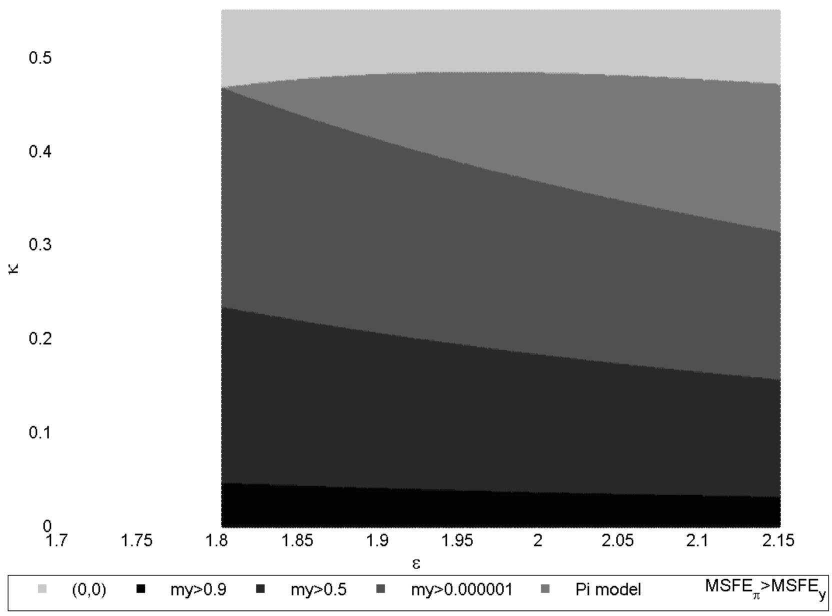


Figure 3.3: Model Selection under Sparse Weights

On the same figure we plot the correlation between output and inflation - R (dotted line), and $\bar{b}_\pi$ (dashed line) as a proxy for inflation persistence in the model. The figure shows that as ε increases, the cost-to-variance ratio falls, hitting zero lower bound. Therefore for large ε even for negligible costs, agents would not include past output in their forecasting rules. The dynamic of $Corr\left(\pi, y\right)$ and $\bar{b}_\pi$ as functions of ε provides intuition for this. $\bar{b}_\pi$ is increasing in ε, approaching unity asymptotically.The correlation is negative for most of the range, but becomes positive after $\varepsilon > 2.15$. As in the simple model, as inflation becomes persistent, addition of past output adds little to the performance of the forecasting rule. The positive sign of the correlation, combined with the negative coefficient of the past output, similarly to the simple model, leads to worse performance of the rules with output.

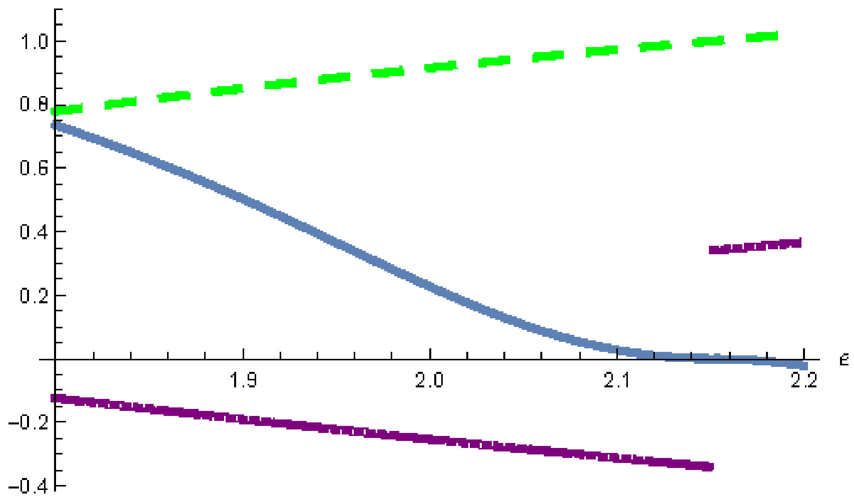


Figure 3.4: Threshold for the Cost-to-Variance Ratio
Note: the dotted line corresponds to the correlation between the output and inflation, the dashed line - to the coefficient on the past inflation, blue solid line - to the threshold for the cost-to-variance ratio

3.4 Discussion

Figure 3.3 shows that when attention costs are present, initially mis-specified forecasting rule can be supported under sparse-rationality. Comparing this result to the literature, support for a mis-specified rule is also found in a number of papers. In Hommes (2014) agents' mis-specified expectations become self-fulfilling in a New Keynesian model. In an experimental setting, Hommes (2014) and Heemeijer et al. (2009) emphasize the importance of expectations feedback parameter. In Hommes (2014), with a negative feedback parameter expectations converge to REE, but with a positive parameter agents coordinate on non-rational self-fulfilling equilibrium. Large feedback parameter results in a convergence to a mis-specified equilibrium in Evans et al. (2012). In our model, the feedback parameter, ε, affects the equilibrium choice of mis-specified forecasting rules through mean squared forecast errors. It also affects the selected sparse weights. There is experimental evidence on agents switching from their forecasting rule when the parameters of the process change. In Pfajfar and Žakelj (2014), Pfajfar and Žakelj (2017), and Assenza et al. (2013), agents choose forecasting rules of inflation under alternative monetary policy regimes, namely the aggressiveness of response to inflation in the Taylor rule. As monetary policy becomes more aggressive, agents switch to using forecasting rules compatible with adaptive expectations. As Pfajfar and Žakelj (2017) discuss, the aggressiveness of monetary policy response resembles the expectation feedback, thus supporting our finding that larger feedback from expectation leads to selection of mis-specified rules, relative to REE.

Another prediction of the paper is that agents will stick to the AR(1) model for inflation forecasting as persistence of inflation increases and/or the correlation between output and inflation is reversed, cf. figure.3.4. This result is in line with the observed behavior of professional forecasters after the recent financial crisis. There are number of studies (examples are Fendel, Lis, and Rülke 2011, Lopez-Perez 2017, and Frenkel, Lis, and Rülke 2011), showing that professional forecasters' predictions behave as though they are using Phillips curve. After the financial crisis, inflation became more persistent(cf. e.g. Watson 2014). Although the evidence on flattening of the Phillips curve is mixed due to different specifications[7] and time horizons considered, there are studies showing a decline in slope, examples being IMF (2013) and Kuttner and Robinson (2010). Donayre and Panovska (2016) document breaks in the wage Phillips curve during the recessions and the subsequent recoveries. As a result, Lopez-Perez (2017)[8] shows that forecasters' predictions started to react much less to unemployment after the financial crisis of 2007-2009, consistent with our model predictions.

3.5 Conclusion

In this paper we study if an initially mis-specified forecasting rule can be an equilibrium choice under sparse-rationality. We consider a simple process consisting of only exogenous variables and then generalize our results to a business cycle model with lagged endogenous variables.

For both models we find a region in the parameter space, where mis-specified RPE is selected by both minimum squared forecast error conditions and the sparse-weights. Agents could re-consider their initial choice of mis-specified RPE and either move towards an REE forecasting model or continue using an initial rule, depending on the learning costs and parameters of the process. If learning costs are very large, there is a region of parameter space where agents choose not to allocate attention to any of the variables. If learning costs are zero, agents switch to REE. For medium range of learning costs, an initial forecasting rule prevails for large persistence of the variable used in the rule and

[7]For example , Coibion and Gorodnichenko (2015b) conclude the disinflationary puzzle could be explained if firms expectations are substituted by household expectations.

[8]Frenkel, Lis, and Rülke (2011) use data up to 2010Q and do not find evidence of a change in forecasters behavior, while Lopez-Perez (2017) uses a longer data set and includes a forward looking inflation term in the Philips curve.

for the large correlation between the included and the omitted variables, especially if the omitted variable has low persistence.

The prediction from a business cycle model is that when inflation persistence increases, the survival of AR(1) rule for inflation forecasting is achieved for a broader area of the parameter space. The same is true for the reverse in correlation between inflation and output. This prediction is supported by the professional forecasters' behavior; their predictions are found to be consistent with paying less attention to output gap after the financial crisis (Lopez-Perez 2017), when inflation became more persistent and the correlation between output and inflation might have changed.

In line with the previous literature, our study supports the importance of the expectation feedback parameter in survival of a mis-specified rule. If this expectation feedback is large enough, the mis-specified forecasting rule prevails in equilibrium.

3.A A Simple Model

Proof. Proposition 9.

Deriving RPE1. The agents' PLM, consistent with this RPE, is

$$y_t = a_1 + b_1 w_t^1, \tag{3.35}$$

therefore, the ALM is given by

$$\begin{aligned} y_t &= \alpha + \beta\left(a_1 + b_1\rho_1 w_t^1\right) + \gamma_1 w_t^1 + \gamma_2 w_t^2 + \eta_t = & (3.36)\\ &= \alpha + \beta a_1 + \left(\beta\rho_1 b_1 + \gamma_1\right) w_t^1 + \gamma_2 w_t^2 + \eta_t = & (3.37)\\ &= \alpha + \beta a_1 + \bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t. & (3.38) \end{aligned}$$

In what follows, we will set $\alpha = 0$ and assume that the agents know this; therefore, $a_1 = 0$ as well.

In order for the agents to use (3.35) in equilibrium, it must be the case that b_1 is a coefficient in the regression of y_t on w_t^1, or

$$b_1 = \frac{Cov\left(y_t, w_t^1\right)}{Var\left(w_t^1\right)}. \tag{3.39}$$

Computing the above expression, we get

$$\begin{aligned} Cov\left(y_t, w_t^1\right) &= E_t\left[\bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t, w_t^1\right] = \bar{b}_1\sigma_1^2 + \gamma_2\cdot\rho\sigma_1\sigma_2, & (3.40)\\ b_1 &= \frac{\bar{b}_1\sigma_1^2 + \gamma_2\cdot\rho\sigma_1\sigma_2}{\sigma_1^2} = \bar{b}_1 + \gamma_2\cdot\rho\frac{\sigma_2}{\sigma_1} = & (3.41)\\ &= \beta\rho_1 b_1 + \gamma_1 + \gamma_2\cdot\rho\frac{\sigma_2}{\sigma_1} \Rightarrow b_1 = \frac{\gamma_1 + \gamma_2\cdot\rho\frac{\sigma_2}{\sigma_1}}{1-\beta\rho_1}. & (3.42) \end{aligned}$$

Deriving RPE2. Similarly to the RPE1 case, we have now the PLM

$$y_t = a_2 + b_2 w_t^2, \tag{3.43}$$

which implies that b_2 must be equal to the regression coefficient:

$$\begin{aligned} b_2 &= \frac{Cov\left(y_t, w_t^2\right)}{Var\left(w_t^2\right)} = \frac{E_t\left[\bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t, w_t^2\right]}{\sigma_2^2} = & (3.44)\\ &= \frac{\left(\beta\rho_1 b_1 + \gamma_1\right)\cdot\rho\sigma_1\sigma_2 + \gamma_2\sigma_2^2}{\sigma_2^2} = \left(\beta\rho_1 b_1 + \gamma_1\right)\cdot\rho\frac{\sigma_1}{\sigma_2} + \gamma_2. & (3.45) \end{aligned}$$

E-stability. For the solution in (3.42) and (3.45) to be E-stable, the following should hold:

$$\begin{aligned} \frac{\partial T_{b_1}}{\partial b_1} &< 1, \\ \frac{\partial T_{b_2}}{\partial b_2} &< 1. \end{aligned}$$

T_{b_1} is given by (3.42) and T_{b_2} by (3.45) . That is, $\frac{\partial T_{b_1}}{\partial b_1} = \frac{\partial\left[\beta\rho_1 b_1+\gamma_1+\gamma_2\cdot\rho\frac{\sigma_2}{\sigma_1}\right]}{\partial b_1} = \beta\rho_1$ and $\frac{\partial T_{b_2}}{\partial b_1} = \frac{\partial\left[(\beta\rho_1 b_1+\gamma_1)\cdot\rho\frac{\sigma_1}{\sigma_2}+\gamma_2\right]}{\partial b_1} = 0$. Thus, the only condition to be satisfied is:

$$\beta\rho_1 < 1. \tag{3.46}$$

With both β<1 and $\rho_1 < 1$ both solutions are E-stable. □

3.A.1 Comparing the Forecast Errors

Next, we want to ensure that the Mean Squared Forecast Error (MSFE) of the agent living in RPE1 and using (3.35) in lower than the MSFE of the agent using (3.43). Otherwise a small proportion of the latter could outperform the majority and lead to increasing deviations from the RPE1.

$$\begin{aligned} e_t^1 &= \bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t - b_1 w_t^1, & (3.47) \\ MSFE_1 &= E\left[e_t^1\right] = E\left[\left(\left(\bar{b}_1 - b_1\right) w_t^1 + \gamma_2 w_t^2 + \eta_t\right)^2\right]. & (3.48) \end{aligned}$$

Similarly, for $MSFE_2$ we have the following expression

$$\begin{aligned} e_t^2 &= \bar{b}_1 w_t^1 + \gamma_2 w_t^2 + \eta_t - b_2 w_t^2, & (3.49) \\ MSFE_2 &= E\left[e_t^2\right] = E\left[\left(\bar{b}_1 w_t^1 + \left(\gamma_2 - b_2\right) w_t^2 + \eta_t\right)^2\right]. & (3.50) \end{aligned}$$

We are looking for the conditions under which $MSFE_1 < MSFE_2$:

$$\begin{aligned} &: \; E\left[\left(\left(\bar{b}_1 - b_1\right) w_t^1 + \gamma_2 w_t^2 + \eta_t\right)^2\right] < E\left[\left(\bar{b}_1 w_t^1 + \left(\gamma_2 - b_2\right) w_t^2 + \eta_t\right)^2\right], \\ &: \; \left\{ \begin{array}{c} \left(\beta\rho_1 b_1 + \gamma_1 - b_1\right)^2 \sigma_1^2 + \gamma_2^2 \sigma_2^2 + \\ 2\gamma_2\left(\beta\rho_1 b_1 + \gamma_1 - b_1\right) \cdot \rho\sigma_1\sigma_2 \end{array} \right\} < \left\{ \begin{array}{c} \left(\beta\rho_1 b_1 + \gamma_1\right)^2 \sigma_1^2 + \left(\gamma_2 - b_2\right)^2 \sigma_2^2 + \\ 2\left(\beta\rho_1 b_1 + \gamma_1\right)\left(\gamma_2 - b_2\right) \cdot \rho\sigma_1\sigma_2 \end{array} \right\}, \end{aligned}$$

$$
\begin{aligned}
&: \left\{\begin{array}{c} (\beta\rho_1 b_1+\gamma_1)^2\sigma_1^2+b_1^2\sigma_1^2 \\ -2b_1(\beta\rho_1 b_1+\gamma_1)\sigma_1^2+\gamma_2^2\sigma_2^2 \\ +2\gamma_2(\beta\rho_1 b_1+\gamma_1)\cdot\rho\sigma_1\sigma_2- \\ -2\gamma_2 b_1\cdot\rho\sigma_1\sigma_2 \end{array}\right\} < \left\{\begin{array}{c} (\beta\rho_1 b_1+\gamma_1)^2\sigma_1^2+\gamma_2^2\sigma_2^2 \\ -2\gamma_2 b_2\sigma_2^2+b_2^2\sigma_2^2+ \\ +2\gamma_2(\beta\rho_1 b_1+\gamma_1)\cdot\rho\sigma_1\sigma_2- \\ -2b_2(\beta\rho_1 b_1+\gamma_1)\cdot\rho\sigma_1\sigma_2 \end{array}\right\}, \\
&: \left\{\begin{array}{c} b_1^2\sigma_1^2-2b_1(\beta\rho_1 b_1+\gamma_1)\sigma_1^2- \\ -2\gamma_2 b_1\cdot\rho\sigma_1\sigma_2 \end{array}\right\} < \left\{\begin{array}{c} -2\gamma_2 b_2\sigma_2^2+b_2^2\sigma_2^2- \\ -2b_2(\beta\rho_1 b_1+\gamma_1)\cdot\rho\sigma_1\sigma_2 \end{array}\right\}, \\
&: \left\{\begin{array}{c} b_1^2\sigma_1^2- \\ -2b_1\sigma_1^2\underbrace{\left(\beta\rho_1 b_1+\gamma_1+\gamma_2\rho\frac{\sigma_2}{\sigma_1}\right)}_{=b_1} \end{array}\right\} < \left\{\begin{array}{c} b_2^2\sigma_2^2- \\ -2b_2\sigma_2^2\underbrace{\left(\gamma_2+(\beta\rho_1 b_1+\gamma_1)\cdot\rho\frac{\sigma_1}{\sigma_2}\right)}_{=b_2} \end{array}\right\}, \\
&: -b_1^2\sigma_1^2 < -b_2^2\sigma_2^2 \Rightarrow b_1^2\sigma_1^2 > b_2^2\sigma_2^2. \qquad (3.51)
\end{aligned}
$$

The condition $b_1^2\sigma_1^2 > b_2^2\sigma_2^2$ has a very simple interpretation: in order to have MSFE of PLM1 lower than for PLM2, the share of variance of y_t explained by the PLM1 must be higher than that of PLM2. Alternatively, the R^2 of the regression (3.35) must be higher than the R^2 of regression (3.43).

3.A.2 Analyzing Conditions for $MSFE_1 < MSFE_2$

$$
\begin{aligned}
b_1^2\sigma_1^2 &> b_2^2\sigma_2^2 \Leftrightarrow b_1^2 > \left[(\beta\rho_1 b_1+\gamma_1)\cdot\rho\frac{\sigma_1}{\sigma_2}+\gamma_2\right]^2\frac{\sigma_2^2}{\sigma_1^2}, \\
b_1^2 &> \left[(\beta\rho_1 b_1+\gamma_1)\cdot\rho+\gamma_2\frac{\sigma_2}{\sigma_1}\right]^2 = \left[\bar{b}_1\cdot\rho+\gamma_2\frac{\sigma_2}{\sigma_1}\right]^2, \\
&: \left[\bar{b}_1+\gamma_2\cdot\rho\frac{\sigma_2}{\sigma_1}\right]^2 > \left[\bar{b}_1\cdot\rho+\gamma_2\frac{\sigma_2}{\sigma_1}\right]^2, \\
&: \bar{b}_1^2+2\bar{b}_1\gamma_2\rho\frac{\sigma_2}{\sigma_1}+\left(\gamma_2\frac{\sigma_2}{\sigma_1}\right)^2\rho^2 > \bar{b}_1^2\rho^2+2\bar{b}_1\rho\gamma_2\frac{\sigma_2}{\sigma_1}+\left(\gamma_2\frac{\sigma_2}{\sigma_1}\right)^2, \\
&: \bar{b}_1^2\left(1-\rho^2\right) > \left(\gamma_2\frac{\sigma_2}{\sigma_1}\right)^2\left(1-\rho^2\right) \Rightarrow \bar{b}_1^2 > \left(\gamma_2\frac{\sigma_2}{\sigma_1}\right)^2, \\
&: \left|\bar{b}_1\right| > \gamma_2\frac{\sigma_2}{\sigma_1}. \qquad (3.52)
\end{aligned}
$$

Theoretically, there could be 2 separate cases:

$$CASE\ I \quad : \quad \bar{b}_1 > \gamma_2\frac{\sigma_2}{\sigma_1}, \qquad (3.53)$$

$$CASE\ II \quad : \quad \bar{b}_1 < -\gamma_2\frac{\sigma_2}{\sigma_1}. \qquad (3.54)$$

As the CASE I is the most obvious, and will occur most easily (assuming $\gamma_{1,2} > 0$ which is what we impose; otherwise, just re-define variable w_t^i so that $\gamma_{1,2}$ become positive), we start with this case.

CASE I: $\bar{b}_1 > \gamma_2 \frac{\sigma_2}{\sigma_1}$

Coming back to the condition of $MSFE_1 < MSFE_2$, we get

$$\begin{aligned}
\bar{b}_1 &> \gamma_2 \frac{\sigma_2}{\sigma_1}, \\
\frac{\gamma_1 + \gamma_2 \cdot \rho \frac{\sigma_2}{\sigma_1} \cdot \beta\rho_1}{1 - \beta\rho_1} &> \gamma_2 \frac{\sigma_2}{\sigma_1}, \\
\gamma_1 + \gamma_2 \cdot \rho \frac{\sigma_2}{\sigma_1} \cdot \beta\rho_1 &> \gamma_2 \frac{\sigma_2}{\sigma_1} - \gamma_2 \frac{\sigma_2}{\sigma_1} \cdot \beta\rho_1, \\
\gamma_1 + \gamma_2 \frac{\sigma_2}{\sigma_1} \beta\rho_1 (1 + \rho) &> \gamma_2 \frac{\sigma_2}{\sigma_1} \\
\frac{\gamma_1}{\gamma_2} \frac{\sigma_1}{\sigma_2} + \beta\rho_1\rho + \beta\rho_1 - 1 &> 0. \qquad (3.55)
\end{aligned}$$

Denoting the ratio of coefficients at the observable shocks $\frac{\gamma_1}{\gamma_2}$ as Γ, and the ratio of standard deviations $\frac{\sigma_1}{\sigma_2}$ as Σ, we see that the condition for CASE I to be true is $\Gamma\Sigma + \rho \cdot \beta\rho_1 > 1 - \beta\rho_1$, or

$$\rho > \frac{1 - \beta\rho_1 - \Gamma\Sigma}{\beta\rho_1}. \qquad (3.56)$$

This condition is satisfied when $\rho_1 \to 1$ and $\Gamma\Sigma$ is large. Alternatively, when $\rho_1 \sim 0$ and $\Gamma\Sigma$ is small so that the numerator is positive, this condition may amount to $\rho > 1$ and thus be impossible to satisfy.

CASE II: $\bar{b}_1 < -\gamma_2 \frac{\sigma_2}{\sigma_1}$

In this case, we have

$$\begin{aligned}
\bar{b}_1 &< -\gamma_2 \frac{\sigma_2}{\sigma_1}, \\
\frac{\gamma_1 + \gamma_2 \cdot \rho \frac{\sigma_2}{\sigma_1} \cdot \beta\rho_1}{1 - \beta\rho_1} &< -\gamma_2 \frac{\sigma_2}{\sigma_1}, \\
\gamma_1 + \gamma_2 \cdot \rho \frac{\sigma_2}{\sigma_1} \cdot \beta\rho_1 &< -\gamma_2 \frac{\sigma_2}{\sigma_1} + \gamma_2 \frac{\sigma_2}{\sigma_1} \cdot \beta\rho_1, \\
\frac{\gamma_1}{\gamma_2} \frac{\sigma_1}{\sigma_2} + \beta\rho_1\rho &< -1 + \beta\rho_1. \qquad (3.57)
\end{aligned}$$

Using the notation just introduced, the condition $\bar{b}_1 < -\gamma_2 \frac{\sigma_2}{\sigma_1}$ amounts to $\Gamma\Sigma + \rho \cdot \beta\rho_1 + (1 - \beta\rho_1) < 0$. Consider an intersection of with horizontal axis where $\Gamma\Sigma = 0$. Then, $\rho < -\frac{(1-\beta\rho_1)}{\beta\rho_1}$. As $|\rho| < 1$, the area where the solution exists is $-\frac{(1-\beta\rho_1)}{\beta\rho_1} > -1$, meaning that $\beta\rho_1 > 1/2$. That is, for the MSFE condition satisfied for $\bar{b}_1 < 0$, $\beta\rho_1$ must be larger than $1/2$.

To sum up, combining to cases together, we see that we need

$$|\Gamma\Sigma + \rho \cdot \beta\rho_1| > 1 - \beta\rho_1. \tag{3.58}$$

3.A.3 Deriving the Sparse Weights

Sparse weights are derived by minimizing the following expression:

$$\min_{m\in[0,1]^n} \frac{1}{2} \sum_{i,j=1\ldots n} (1 - m_i)\, \Lambda_{ij}\, (1 - m_j) + \kappa \sum_{i,j=1\ldots n} m_i \tag{3.59}$$

with:

$$\begin{aligned}
\Lambda_{ij} &= -\sigma_{ij} a_{w_i} u_{aa} a_{w_j}, \\
a_{w_i} &= -u_{aa}^{-1} u_{aw_i}, \\
u_{aa} &= \frac{\partial^2 u_a}{\partial a^2} = \frac{\partial}{\partial a}\left(-(a - \bar{b}_1 w_t^1 - \bar{b}_2 w_t^2)\right) = -1, \\
u_{aw_i} &= \bar{b}_i.
\end{aligned}$$

Then the cost of inattention is

$$\Lambda_{ij} = \sigma_{ij} \bar{b}_i \bar{b}_j. \tag{3.60}$$

Plugging the cost of inattention as in (3.60) into (3.59) we get the following problem:

$$\min_{m\in[0,1]^n} \frac{1}{2} \left\{(1 - m_1)^2 \sigma_1^2 \bar{b}_1^2 + 2\,(1 - m_1)\,(1 - m_2)\, \sigma_{12} \bar{b}_1 \bar{b}_2 + (1 - m_2)^2 \sigma_2^2 \bar{b}_2^2\right\} + \tag{3.61}$$

$$+\kappa\,(m_1 + m_2)\,. \tag{3.62}$$

First order conditions of (3.61) with respect to m_1 and m_2:

$$[m_1] \;:\; \kappa + \frac{1}{2}(-2\bar{b}_1^2(1 - m_1)\sigma_1^2 - 2\bar{b}_1\bar{b}_2\,(1 - m_2)\,\sigma_{12} = 0, \tag{3.63}$$

$$[m_2] \;:\; \kappa + \frac{1}{2}(-2\bar{b}_2^2(1 - m_2)\sigma_2^2 - 2\bar{b}_1\bar{b}_2\,(1 - m_1)\,\sigma_{12} = 0. \tag{3.64}$$

Solving (3.63) and (3.64) for m_1 and m_2 gives the expressions (3.13) and (3.14) in the text.

Proof. Proposition 10. Consider when $m_1 > m_2$:

$$\begin{aligned}
-\frac{\kappa}{\bar{b}_1\bar{b}_2\sigma_1\sigma_2\left(1-\rho^2\right)}\frac{\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1}{\bar{b}_1\sigma_1} &> -\frac{\kappa}{\bar{b}_1\bar{b}_2\sigma_1\sigma_2\left(1-\rho^2\right)}\frac{\bar{b}_1\sigma_1-\rho\bar{b}_2\sigma_2}{\bar{b}_2\sigma_2} \\
-\frac{1}{\bar{b}_1}\frac{\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1}{\bar{b}_1\sigma_1} &> -\frac{1}{\bar{b}_1}\frac{\bar{b}_1\sigma_1-\rho\bar{b}_2\sigma_2}{\bar{b}_2\sigma_2} \\
\left(\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1\right)\bar{b}_2\sigma_2 &< \bar{b}_1\sigma_1\left(\bar{b}_1\sigma_1-\rho\bar{b}_2\sigma_2\right) \\
\bar{b}_2^2\sigma_2^2-\bar{b}_1\bar{b}_2\rho\sigma_1\sigma_2 &< \bar{b}_1^2\sigma_1^2-\bar{b}_1\bar{b}_2\rho\sigma_1\sigma_2 \\
\bar{b}_2^2\sigma_2^2 &< \bar{b}_1^2\sigma_1^2 \qquad (3.65)
\end{aligned}$$

□

Proof. Proposition 11. Assume that the (3.65) is satisfied. Then, we need

$$1-\kappa\frac{1-\rho\frac{\bar{b}_2}{\bar{b}_1}\frac{\sigma_2}{\sigma_1}}{\bar{b}_2^2\sigma_2^2\left(1-\rho^2\right)}<0.$$

We distinguish between the following cases depending on the sign of $1-\rho\frac{\bar{b}_2}{\bar{b}_1}\frac{\sigma_2}{\sigma_1}$.

CASE I: $\bar{b}_1 > \frac{\bar{b}_2\sigma_2}{\sigma_1}$. Then $\frac{\bar{b}_2}{\bar{b}_1}\frac{\sigma_2}{\sigma_1} < 1$ and $1-\rho\frac{\bar{b}_2}{\bar{b}_1}\frac{\sigma_2}{\sigma_1} > 0$. **CASE II:** $\bar{b}_1 < -\frac{\bar{b}_2\sigma_2}{\sigma_1} < 0$. Then $1-\rho\frac{\bar{b}_2}{\bar{b}_1}\frac{\sigma_2}{\sigma_1} > 0$. Note that for RPE1 consistent rule to be an equilibrium for $\bar{b}_1 < 0$, it must hold that $\beta\rho_1 > 1/2$ (see CASE II of Appendix 3.A.2). In terms of the parameter of the model (3.15) could be written as

$$\kappa < \gamma^2\sigma_2^2\left(1-\rho^2\right)\frac{\left(\Gamma\Sigma+\beta\rho\rho_1\right)}{\Gamma\Sigma+\rho\left(2\beta\rho_1-1\right)} > 0.$$

The condition also states that if RPE1 consistent rule is an equilibrium, that is (3.10) is, satisfied, there is no region in the parameter space such that for any κ this equilibrium survives. □

Proof. Proposition 12.

$$\begin{aligned}
&: \frac{\kappa\left(\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1\right)}{\bar{b}_1^2\bar{b}_2\sigma_1^2\sigma_2\left(1-\rho^2\right)} > 1, \\
&: \kappa\left(\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1\right) > \bar{b}_1^2\bar{b}_2\sigma_1^2\sigma_2\left(1-\rho^2\right).
\end{aligned}$$

We distinguish between two subcases. **Case I:** $\bar{b}_2\sigma_2 - \bar{b}_1\rho\sigma_1 > 0$, $\kappa > \frac{\bar{b}_1^2\bar{b}_2\sigma_1^2\sigma_2\left(1-\rho^2\right)}{\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1} = \frac{\bar{b}_1^2\sigma_1^2\left(1-\rho^2\right)}{1-\rho\frac{\bar{b}_1}{\bar{b}_2}\frac{\sigma_1}{\sigma_2}} = \frac{\gamma_1+\frac{\beta\rho_1(\gamma_1+(\gamma_2\rho\frac{\sigma_2}{\sigma_1}))}{1-\beta\rho_1}}{1-\frac{\rho(\Gamma\Sigma+\beta\rho\rho_1)}{1-\beta\rho_1}}$. Consider $1-\frac{\rho(\Gamma\Sigma+\beta\rho\rho_1)}{1-\beta\rho_1}$ with $\Gamma\Sigma = 0$, then $\rho^2 < \frac{1-\beta\rho_1}{\beta\rho_1}$, and it must be $\frac{1-\beta\rho_1}{\beta\rho_1} < 1$, which means $\beta\rho_1 > \frac{1}{2}$.

Case II: $\bar{b}_2\sigma_2 - \bar{b}_1\rho\sigma_1 < 0$, implies $\kappa < \frac{\bar{b}_1^2\bar{b}_2\sigma_1^2\sigma_2\left(1-\rho^2\right)}{\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1}$. With $\bar{b}_1^2\bar{b}_2\sigma_1^2\sigma_2\left(1-\rho^2\right) > 0$ and $\bar{b}_2\sigma_2 - \bar{b}_1\rho\sigma_1 < 0$, $\frac{\bar{b}_1^2\bar{b}_2\sigma_1^2\sigma_2\left(1-\rho^2\right)}{\bar{b}_2\sigma_2-\bar{b}_1\rho\sigma_1}$ is < 0, which contradicts $\kappa > 0$. □

3.B A Simple Business Cycle

Proof. Proposition 13. **Deriving $\mathbf{M}_\pi$.** As in the case of a simple model, we allow our agents to be econometricians who estimate the coefficients for their learning rule as regression coefficients. Then for M_π the coefficient is:

$$b_\pi = \frac{Cov\left(\pi_t, \pi_{t-1}\right)}{Var\left(\pi_{t-1}\right)} = \frac{Cov\left(\bar{a}_\pi + \bar{b}_\pi\pi_{t-1} + \bar{c}_\pi y_{t-1}, \pi_{t-1}\right)}{Var\left(\pi_{t-1}\right)},$$

denoting $\sigma_{\pi y} \equiv Cov(y,\pi)$, $\sigma_\pi^2 \equiv Var\left(\pi\right)$, $\sigma_y^2 \equiv Var(y)$:

$$\begin{aligned} b_\pi = \frac{\bar{b}_\pi\sigma_\pi^2 + \bar{c}_\pi\sigma_{\pi y}}{\sigma_\pi^2} = \bar{b}_\pi + \bar{c}_\pi\frac{\sigma_{\pi y}}{\sigma_\pi^2} = b_\pi\left(1 + b_\pi - \frac{1}{\varepsilon}\right) + \frac{1}{\bar{y}\varepsilon}\frac{\sigma_{\pi y}}{\sigma_\pi^2} = \\ = \frac{\varepsilon - 1 + b_\pi\left(\varepsilon + b_\pi\varepsilon - 1\right)}{\varepsilon + b_\pi\left(\varepsilon + b_\pi\varepsilon - 1\right)}. \end{aligned} \tag{3.66}$$

$$\alpha_\pi = \bar{\pi} - b_\pi\bar{\pi},$$

with $\bar{\pi} = \bar{a}_\pi + \bar{b}_\pi\bar{\pi} + \bar{c}_\pi\bar{y}$, $\bar{\pi} = \frac{\bar{a}_\pi + \bar{c}_\pi\bar{y}}{1-\bar{b}_\pi}$, and variances and covariances defined in subsection 3.B below. The expression for α_π could be simplified as:

$$a_\pi = \frac{\bar{a}_\pi + \bar{c}_\pi\bar{y}}{1 - \bar{b}_\pi}\left(1 - b_\pi\right) = \left(1 - b_\pi\right). \tag{3.67}$$

Solving for b_π in terms of the parameters of the model[9]:

$$\begin{aligned} b_\pi &= \sqrt[3]{z} - \frac{3\left(\varepsilon - 1\right)}{9\varepsilon^2\sqrt[3]{z}} + \frac{1}{3\varepsilon}, \qquad (3.68) \\ z &= \frac{1}{54\varepsilon^3}(2 - 9\varepsilon - 27\varepsilon^2 + 27\varepsilon^3) + \frac{\sqrt{3}}{18\varepsilon^2}\sqrt{-5 + 26\varepsilon + 9\varepsilon^2 - 54\varepsilon^3 + 27\varepsilon^4}. \end{aligned}$$

[9] Appendix 9.4 in Adam (2005).

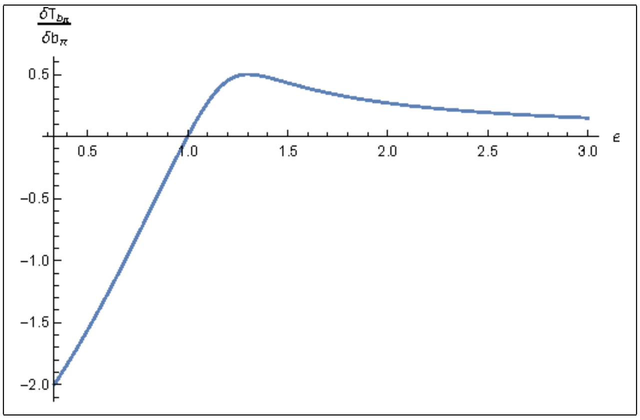


Figure 3.5: E-Stability Condition for Model of Inflation

Note that the solution does not exist for $-5 + 26\varepsilon + 9\varepsilon^2 - 54\varepsilon^3 + 27\varepsilon^4 < 0$, that is, for $0 < \varepsilon < 0.192897$, and for $\varepsilon = 1/3$ when $z = 0$.

Deriving M$_y$. Coefficient c_y is then derived as regression coefficient

$$\begin{aligned} c_y &= \frac{Cov\left(\pi_t, y_{t-1}\right)}{Var\left(y_{t-1}\right)} = \frac{Cov\left(\bar{a}_\pi + \bar{b}_\pi \pi_{t-1} + \bar{c}_\pi y_{t-1}, y_{t-1}\right)}{Var\left(y_{t-1}\right)} = \\ &= \frac{\bar{b}_\pi \sigma_{\pi y} + \bar{c}_\pi \sigma_y^2}{\sigma_y^2} = \bar{c}_\pi + \bar{b}_\pi \frac{\sigma_{\pi y}}{\sigma_y^2} = \frac{1}{\bar{y}\varepsilon} + b_\pi \left(1 + b_\pi - \frac{1}{\varepsilon}\right) \frac{\sigma_{\pi y}}{\sigma_y^2}, \qquad (3.69) \\ a_y &= 1 - c_y = 1 - \frac{1}{\bar{y}\varepsilon} - b_\pi \left(1 + b_\pi - \frac{1}{\varepsilon}\right) \frac{\sigma_{\pi y}}{\sigma_y^2}. \qquad (3.70) \end{aligned}$$

E-stability. For M$_\pi$ to be E-stable under least-squares learning, the following must be satisfied

$$eig \begin{pmatrix} \frac{\partial T_{b_\pi}}{\partial b_\pi} & \frac{\partial T_{b_\pi}}{\partial \alpha_\pi} \\ \frac{\partial T_{\alpha_\pi}}{\partial b_\pi} & \frac{\partial T_{\alpha_\pi}}{\partial \alpha_\pi} \end{pmatrix} < 1,$$

with $T_{b_\pi}$and T_{α_π} defined in (3.66) and (3.67). As $\frac{\partial T_{\alpha_\pi}}{\partial \alpha_\pi} = 0$, the conditions collapses for $\frac{\partial T_{b_\pi}}{\partial b_\pi} < 1$. Substituting in (3.66) (3.79), (3.80) and (3.81), the derivative is:

$$\frac{\partial}{\partial b_\pi} \left[\frac{\varepsilon - 1 + b_\pi \left(\varepsilon + b_\pi \varepsilon - 1\right)}{\varepsilon + b_\pi \left(\varepsilon + b_\pi \varepsilon - 1\right)} \right] = \frac{\varepsilon + 2b_\pi \varepsilon - 1}{\left(\varepsilon + b_\pi \left(\varepsilon + b_\pi \varepsilon - 1\right)\right)^2}. \qquad (3.71)$$

Keeping in mind that (3.71) is continuous in ε on the interval $\varepsilon > \frac{1}{3}$ and plotting it, we see that the value of the function is below unity for all the parameter ranges considered:

For M_y to be E-stable under least-squares learning the following must be satisfied

$$eig\begin{pmatrix} \frac{\partial T_{c_y}}{\partial c_y} & \frac{\partial T_{c_y}}{\partial \alpha_y} \\ \frac{\partial T_{\alpha_y}}{\partial c_y} & \frac{\partial T_{\alpha_y}}{\partial \alpha_y} \end{pmatrix} < 1.$$

With T_{c_y} and T_{α_y} defined in (3.69) and (3.70), the above matrix is a zero matrix and the condition is satisfied.

As shown in Adam (2005) level of average output does not have influence on the relevant features of the model. Therefore, we set the average level of output equal to 1. Then it is easy to verify that:

$$1 - \bar{b}_\pi - \bar{a}_\pi - \bar{c}_\pi = \frac{\bar{y} - 1}{\bar{y}\varepsilon} = 0.$$

In the calculations below, we denote the correlation between output and inflation as R. The correlation is then $R = \frac{\sigma_{\pi y}}{\sigma_y \sigma_\pi}$ with $\sigma_{\pi y} \equiv Cov(\pi, y)$, $\sigma_\pi \equiv \sqrt{Var(\pi)}$, $\sigma_y \equiv \sqrt{Var(y)}$. We start with a mean forecast error of M_π. The forecast error of M_y:

$$: \quad e_t^y = (\alpha_y - \bar{a}_\pi) + (c_y - \bar{c}_\pi) y_{t-1} - \bar{b}_\pi \pi_{t-1} =$$

$$: \quad = \left(1 - \bar{c}_\pi - \bar{b}_\pi \frac{\sigma_{\pi y}}{\sigma_y^2} - \bar{a}_\pi - \bar{b}_\pi + \bar{b}_\pi\right) + \bar{b}_\pi \frac{\sigma_{\pi y}}{\sigma_y^2} y_{t-1} - \bar{b}_\pi \pi_{t-1} = \tag{3.72}$$

$$: \quad = \bar{b}_\pi \left\{\left(1 - \frac{\sigma_{\pi y}}{\sigma_y^2}\right) + \frac{\sigma_{\pi y}}{\sigma_y^2} y_{t-1} - \pi_{t-1}\right\} = \tag{3.73}$$

$$: \quad = \bar{b}_\pi \left\{\left(1 - R\frac{\sigma_\pi}{\sigma_y}\right) + R\frac{\sigma_\pi}{\sigma_y} y_{t-1} - \pi_{t-1}\right\} \tag{3.74}$$

$$: \quad MSFE_y = Var[\bar{b}_\pi \left\{\left(1 - R\frac{\sigma_\pi}{\sigma_y}\right) + R\frac{\sigma_\pi}{\sigma_y} y_{t-1} - \pi_{t-1}\right\}] +$$

$$: \quad + \left(E[\bar{b}_\pi \left\{\left(1 - R\frac{\sigma_\pi}{\sigma_y}\right) + R\frac{\sigma_\pi}{\sigma_y} y_{t-1} - \pi_{t-1}\right\}]\right)^2 = \tag{3.75}$$

$$: \quad = \bar{b}_\pi^2 [\left(R\frac{\sigma_\pi}{\sigma_y}\right)^2 \sigma_y^2 + \sigma_\pi^2 - 2R\frac{\sigma_\pi}{\sigma_y}\sigma_{\pi y}] + \tag{3.76}$$

$$: \quad + \left([\bar{b}_\pi \left\{1 - R\frac{\sigma_\pi}{\sigma_y} + R\frac{\sigma_\pi}{\sigma_y} - 1\right\}]\right)^2 = \bar{b}_\pi^2 [\sigma_\pi^2 - R^2 \sigma_\pi^2] \tag{3.77}$$

We are looking for the conditions under which $MSFE_\pi < MSFE_y$:

$$\bar{c}_\pi^2 \left[\sigma_y^2 - R^2 \sigma_y^2\right] < \bar{b}_\pi^2 \left[\sigma_\pi^2 - R^2 \sigma_\pi^2\right],$$

denoting $\Sigma = \frac{\sigma_\pi}{\sigma_y}$and $\bar{\Gamma} = \frac{\bar{b}_\pi}{\bar{c}_\pi}$.And the criterion is simply:

$$\bar{\Gamma}^2\Sigma^2 > 1. \tag{3.78}$$

$\square$

Variances and covariance

With the actual law of motion as in (3.28) we derive variances and covariance of output and inflation.

$$\begin{aligned}
\sigma_{y\pi} &= Cov\left(\bar{a}_{y\pi} + \bar{b}_{y\pi}\pi_{t-1} + \bar{c}_{y\pi}y_{t-1} + \tau_t, \bar{a}_\pi + \bar{b}_\pi\pi_{t-1} + \bar{c}_\pi y_{t-1}\right) = \\
&\quad \bar{b}_{y\pi}\bar{b}_\pi\sigma_\pi^2 + \bar{b}_{y\pi}\bar{c}_\pi\sigma_{y\pi} + \bar{c}_{y\pi}\bar{b}_\pi\sigma_{y\pi} + \bar{c}_{y\pi}\bar{c}_\pi\sigma_y^2, \\
\sigma_{y\pi} &= \frac{\bar{b}_{y\pi}\bar{b}_\pi\sigma_\pi^2 + \bar{c}_{y\pi}\bar{c}_\pi\sigma_y^2}{1 - \bar{b}_{y\pi}\bar{c}_\pi - \bar{c}_{y\pi}\bar{b}_\pi}, &(3.79)\\
\sigma_\pi^2 &= \bar{b}_\pi^2\sigma_\pi^2 + \bar{c}_\pi^2\sigma_y^2 + 2\bar{c}_\pi\bar{b}_\pi\sigma_{y\pi}, &(3.80)\\
\sigma_y^2 &= \bar{b}_{y\pi}^2\sigma_\pi^2 + \bar{c}_{y\pi}^2\sigma_y^2 + 2\bar{c}_{y\pi}\bar{b}_{y\pi}\sigma_{y\pi} + \sigma_\tau^2. &(3.81)
\end{aligned}$$

Solving the above system:

$$\begin{aligned}
\sigma_\pi^2 &= \frac{\bar{c}_\pi^2\left(\bar{b}_{y\pi}\bar{c}_\pi - \bar{b}_\pi\bar{c}_{y\pi} - 1\right)\sigma_\tau^2}{\left(1 + \bar{b}_{y\pi}\bar{c}_\pi - \bar{b}_\pi\bar{c}_{y\pi}\right)\left(-1 + \bar{b}_\pi + \bar{c}_{y\pi} + \bar{b}_{y\pi}\bar{c}_\pi - \bar{b}_\pi\bar{c}_{y\pi}\right)\left(1 + \bar{b}_\pi + \bar{c}_{y\pi} - \bar{b}_{y\pi}\bar{c}_\pi + \bar{b}_\pi\bar{c}_{y\pi}\right)}, \\
\sigma_y^2 &= \frac{\left(1 - \bar{b}_{y\pi}\bar{c}_\pi + \bar{b}_\pi\bar{c}_{y\pi}\right)\left(\bar{b}_{y\pi}\sigma_\pi^2 + \sigma_\tau^2\right)}{\left(\bar{b}_\pi\bar{c}_{y\pi} - 1\right)\left(\bar{c}_{y\pi}^2 - 1\right) - \bar{b}_{y\pi}\bar{c}_\pi\left(1 + \bar{c}_{y\pi}^2\right)}, \\
\sigma_{\pi y} &= \frac{\left(\bar{b}_\pi\bar{b}_{y\pi}\sigma_\pi^2 + \bar{c}_{y\pi}\bar{c}_\pi\sigma_y^2\right)}{\left(1 - \bar{b}_\pi\bar{c}_{y\pi} - \bar{b}_{y\pi}\bar{c}_\pi\right)}.
\end{aligned}$$

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