Appendix to

"Growth Uncertainty, Rational Learning, and Option Prices"

Abstract

This appendix reviews the numerical solution methodology for the model with full information and parameter uncertainty. We focus on the learning about parameters economy with unknown transition probabilities, mean growth rates and volatilities of productivity growth.[1] For the unknown parameter case, we further provide numerical solutions for anticipated utility pricing and priced parameter uncertainty.

[1]It might be instructive to consider simpler models with learning about transition probabilities or learning about transition probabilities and mean growth rates. Please refer to Babiak and Kozhan (2020) for details about the numerical solution methodology in these two cases.

Contents

A Full Information

Here, we offer details of how the continuation utility and the levered equity claim are computed for the economy in which all parameters are known. This case simplifies to solving a standard rational expectations model in which the agent knows the true parameters of the economy.

A.1 Solving for the Continuation Utility

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma_{s_t} \cdot \varepsilon_t,$$

where $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$, s_t is a two state Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0,1)$. The regime switches in s_t are independent of the Gaussian shocks ε_t. We define the following stationary variables:

$$\{\tilde{C}_t, \tilde{I}_t, \tilde{Y}_t, \tilde{K}_t, \tilde{U}_t\} = \left\{\frac{C_t}{A_t}, \frac{I_t}{A_t}, \frac{Y_t}{A_t}, \frac{K_t}{A_t}, \frac{U_t}{A_t}\right\}$$

The household's problem is:

$$\tilde{U}_t = \max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}^{1-\gamma} \cdot \left(\frac{A_{t+1}}{A_t}\right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}} \tag{A.1}$$

subject to the constraints:

$$\tilde{C}_t + \tilde{I}_t = \tilde{K}_t^{\alpha} \bar{N}^{1-\alpha} \tag{A.2}$$

$$e^{\Delta a_{t+1}} \tilde{K}_{t+1} = (1-\delta)\tilde{K}_t + \varphi\left(\frac{\tilde{I}_t}{\tilde{K}_t}\right)\tilde{K}_t \tag{A.3}$$

$$\Delta a_t = \mu_{s_t} + \sigma_{s_t} \cdot \varepsilon_t, \ \varepsilon_t \sim N(0,1) \tag{A.4}$$

$$\tilde{C}_t \geq 0, \ \tilde{K}_{t+1} \geq 0 \tag{A.5}$$

where the subscript t indicates the time, $E_t(\cdot)$ denotes the expectation conditional on the information available at time t. Because the true parameters are assumed to be known, s_t

and $\tilde{K}_t$ are the only state variables in the economy. Ultimately, the recursive equation (A.1) can be rewritten as:

$$\tilde{U}_t(s_t, \tilde{K}_t) \tag{A.6}$$
$$= \max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}\left(s_{t+1}, \tilde{K}_{t+1}\right)^{1-\gamma} \cdot e^{(1-\gamma)\Delta a_{t+1}} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}}$$

To solve the recursion (A.6), we use the the value function iteration algorithm. The numerical algorithm proceeds as follows:

1. We find the de-trended steady state capital $\tilde{K}_{ss}$, assuming the productivity growth equals the steady state level predicted by a Markov-switching model. The state space for capital normalized by technology is set at $[0.1\tilde{K}_{ss}, 2.6\tilde{K}_{ss}]$. We further use $n_k = 100$ points on a grid for capital in the numerical computation. A denser grid does not lead to significantly different results.

2. For any level of capital $\tilde{K}_t$ at time t, we construct a grid for $\tilde{I}_t$ with uniformly distributed points between 0 and $\tilde{K}_t^{\alpha}\bar{N}^{1-\alpha}$. Specifically, we use $n_i = 100$ points.

3. For the expectation, we use Gauss-Hermite quadrature with $n_{gh} = 8$ points. Using the quadrature weights and nodes, we can calculate the expression on the right hand side of the recursion (A.6).

4. We solve the optimization problem in the Bellman equation (A.6) subject to constraints (A.2)-(A.5) and update a new value function $\tilde{U}_t = \tilde{U}_t(s_t, \tilde{K}_t)$ given that the old one is $\tilde{U}_{t+1} = \tilde{U}_{t+1}(s_{t+1}, \tilde{K}_{t+1})$.

5. We iterate Steps 2-4 by updating the continuation utility at each iteration until a suitable convergence is achieved. Specifically, the stopping rule is that the distance between the new value function and the old value function satisfies $|\tilde{U}_{t+1} - \tilde{U}_t| / |\tilde{U}_t| < 10^{-12}$.

A.2 Solving for a Dividend Claim

The results of this paper are based on pricing an equity claim to calibrated stock market dividends given by:

$$\Delta d_t = g_d + \lambda \Delta c_t + \sigma_d \varepsilon_t^d, \quad \varepsilon_t^d \overset{\text{iid}}{\sim} N(0,1),$$

in which $\varepsilon_t^d \overset{\text{iid}}{\sim} N(0,1)$, λ is the leverage factor, and g_d and σ_d are the dividend growth rate and volatility. The equilibrium condition for the price-dividend ratio is given by:

$$PD_t = E_t\left[\beta\left(\frac{\tilde{C}_{t+1}}{\tilde{C}_t}\right)^{-\frac{1}{\psi}}\left(\frac{A_{t+1}}{A_t}\right)^{-\frac{1}{\psi}}\left(\frac{\tilde{U}_{t+1}\cdot\left(\frac{A_{t+1}}{A_t}\right)}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot\left(\frac{A_{t+1}}{A_t}\right)\right)}\right)^{\frac{1}{\psi}-\gamma}\left(\frac{D_{t+1}}{D_t}\right)(PD_{t+1}+1)\right].$$

Rewriting dividend growth in terms of the stationary consumption growth series, we can rewrite the previous recursion in the following form:

$$PD_t = E_t\left[\beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} e^{g_d+0.5\sigma_d^2}\cdot(PD_{t+1}+1)\right]. \quad \text{(A.7)}$$

To solve the recursion (A.7), we use the the value function iteration algorithm. The numerical algorithm proceeds as follows:

1. We find the de-trended steady state capital $\tilde{K}_{ss}$, assuming the productivity growth equals the steady state level predicted by a Markov-switching model. The state space for capital normalized by technology is set at $[0.1\tilde{K}_{ss}, 2.6\tilde{K}_{ss}]$. We further use $n_k = 100$ points on a grid for capital in the numerical computation. A denser grid does not lead to significantly different results.

2. For any level of capital $\tilde{K}_t$ at time t, we construct a grid for $\tilde{I}_t$ with uniformly distributed points between 0 and $\tilde{K}_t^\alpha \bar{N}^{1-\alpha}$. Specifically, we use $n_i = 100$ points.

3. For the expectation, we use Gauss-Hermite quadrature with $n_{gh} = 8$ points. Using the quadrature weights and nodes, we can calculate the expression on the right hand side of the recursion (A.7).

4. We solve the optimization problem in the Bellman equation (A.7) subject to constraints (A.2)-(A.5) and update a new value function $\tilde{PD}_t = \tilde{PD}_t(s_t, \tilde{K}_t)$ given an old one $\tilde{PD}_{t+1} = \tilde{PD}_{t+1}(s_{t+1}, \tilde{K}_{t+1})$.

5. We iterate Steps 2-4 by updating the continuation utility at each iteration until a suitable convergence is achieved. Specifically, the stopping rule is that the distance between the new value function and the old value function satisfies $|\tilde{PD}_{t+1} - \tilde{PD}_t|/|\tilde{PD}_t| < 10^{-12}$.

B Anticipated Utility

In the anticipated utility case, the representative household learns about unknown parameters but ignores parameter uncertainty when making decisions. The numerical solution proceeds as follows. At each time t, the household holds his current beliefs and solves for the continuation utility and the levered equity claim in the rational expectations model in which the true parameter values in the productivity growth process are centered at the time t posterior means. In the next period $t+1$, the household updates his beliefs upon observing new data and resolves the rational expectations economy in which the true parameters are centered at the time $t+1$ posterior means. In sum, the numerical algorithm reduces to applying the methodology for the full information case with a set of model parameters, which are equal to the mean beliefs at each point in time t.

C Priced Parameter Uncertainty

The numerical solution for the case of priced parameter uncertainty consists of two main steps.[2] First, we solve for the equilibrium pricing ratios when true parameters are actually known by the household (by assumption, these are learned at $T = \infty$). We find the solution for this simplest limiting economy on a dense set of state variables by applying the methods outlined in Appendix A. Second, we use the known parameters boundary economy as a terminal value in the backward recursion to obtain the equilibrium model solution at each time t. Here, we outline the details of the numerical solution for the model with the unknown transition probabilities, mean growth rates and the volatility of productivity growth.

C.1 Solving for the Continuation Utility

Productivity growth is given by:

$$\Delta a_t = \mu_{s_t} + \sigma_{s_t} \cdot \varepsilon_t,$$

[2] Johnson (2007) uses this solution methodology in a case with parameter learning and power utility. Johannes, Lochstoer, and Mou (2016) and Collin-Dufresne, Johannes, and Lochstoer (2016) extend this approach to the case of Epstein-Zin utility in the endowment economy. We further extent the numerical solution to the case of Epstein-Zin utility in the production economy.

where $\varepsilon_t \overset{\text{iid}}{\sim} N(0,1)$, s_t is a two state Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} \pi_{11} & 1-\pi_{11} \\ 1-\pi_{22} & \pi_{22} \end{bmatrix},$$

where $\pi_{ii} \in (0,1)$. The regimes switches in s_t are independent of the Gaussian shocks ε_t.

The representative household does not know the true values of the transition probabilities (π_{11}, π_{22}), the mean growth rates (μ_1, μ_2) and the volatilities (σ_1, σ_2) but observes states (s_t) of the economy. At time $t = 0$, the household holds prior beliefs about unknown parameters and updates beliefs each period upon the realization of new series and regimes. We assume a conjugate prior for all parameters: the Beta distributed prior and the truncated normal-inverse-gamma prior for the transition probabilities, the mean growth rates and volatilities, respectively.

The Beta distribution has the probability density function of the form:

$$p(\pi|a,b) = \frac{\pi^{a-1}(1-\pi)^{b-1}}{B(a,b)},$$

where $B(a,b)$ is the Beta function (a normalization constant), a and b are two positive shape parameters. We are particularly interested in the expected value of the Beta distribution defined by:

$$E[\pi|a,b] = \frac{a}{a+b}.$$

We use two pairs of hyper-parameters (a_1, b_1) and (a_2, b_2) for unknown transition probabilities π_{11} and π_{22}, respectively. At time t, the household uses Bayes' rule and the fact that states are observable to update hyper-parameters for each state i as follows:

$$a_{i,t} = a_{i,0} + \#(\text{state } i \text{ has been followed by state } i), \quad \text{(C.8)}$$

$$b_{i,t} = b_{i,0} + \#(\text{state } i \text{ has been followed by state } j), \quad \text{(C.9)}$$

given the initial prior beliefs $a_{i,0}$ and $b_{i,0}$. Once we find the limiting boundary economies in the first step, we perform a backward recursion using the following state variables

$$\tau_{1,t} = a_{1,t} - a_{1,0} + b_{1,t} - b_{1,0} \tag{C.10}$$

$$\lambda_{1,t} = E_t[\pi_{11}] = \frac{a_{1,t}}{a_{1,t} + b_{1,t}} \tag{C.11}$$

$$\tau_{2,t} = a_{2,t} - a_{2,0} + b_{2,t} - b_{2,0} \tag{C.12}$$

$$\lambda_{2,t} = E_t[\pi_{22}] = \frac{a_{2,t}}{a_{2,t} + b_{2,t}} \tag{C.13}$$

Next, we denote hyper-parameters of the truncated normal-inverse-gamma distributed prior for the mean and variance of productivity growth in each state i by $(\mu_{i,t}, A_{i,t})$ and $(b_{i,t}, B_{i,t})$. Formally, at time t the joint prior over the mean μ_i and variance σ_i^2 conditional on the data Δa^t and the states of the economy s^t is:

$$p(\mu_i, \sigma_i^2 | \Delta a^t, s^t) = p(\mu_i | \sigma_i^2, \Delta a^t, s^t) p(\sigma_i^2 | \Delta a^t, s^t),$$

where

$$p(\sigma_i | \Delta a^t, s^t) = IG\left(\frac{b_{i,t}}{2}, \frac{B_{i,t}}{2}\right),$$

$$p(\mu_i | \sigma_i^2, \Delta a^t, s^t) = N(\mu_{i,t}, A_{i,t}\sigma_i^2).$$

We update these hyper-parameters using the Bayes' rule as follows:

$$\mu_{i,t+1} = \mu_{i,t} + \mathbf{1}_{s_{t+1}=i} \frac{A_{i,t}}{A_i + 1} (\Delta a_{t+1} - \mu_{i,t}) \tag{C.14}$$

$$A_{i,t+1}^{-1} = A_{i,t}^{-1} + \mathbf{1}_{s_{t+1}=i}, \tag{C.15}$$

$$b_{i,t+1} = b_{i,t} + \mathbf{1}_{s_{t+1}=i}, \tag{C.16}$$

$$B_{i,t+1} = B_{i,t} + \mathbf{1}_{s_{t+1}=i} \frac{(\Delta a_{t+1} - \mu_{i,t})^2}{1 + A_{i,t}} \tag{C.17}$$

where $i \in \{1, 2\}$, $\mathbf{1}$ is an indicator function that equals 1 if the condition in the subscript is true and 0 otherwise.

Note that since the hyper-parameters $A_{i,t}$'s and $b_{i,t}$'s are a function of the time spent in each period, the following 8-dimensional vector

$$X_t \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}, \mu_{1,t}, \mu_{2,t}, B_{1,t}, B_{2,t}\}$$

is sufficient statistics for the priors. Thus, we can define X_{t+1} using the equations (C.8)-(C.13), (C.14)-(C.17), the next period regime, and sufficient statistics at time t :

$$X_{t+1} = f(s_{t+1}, s_t, X_t).$$

We further define $X_t^s \equiv \{\tau_{1,t}, \lambda_{1,t}, \tau_{2,t}, \lambda_{2,t}\}$ and $X_t^{\Delta a} \equiv \{\tilde{K}_t, \mu_{1,t}, \mu_{2,t}, B_{1,t}, B_{2,t}\}$, where the superscripts s and Δa indicate that variables in the vectors X_t^s and $X_t^{\Delta a}$ are a function only of the observed state realization s_t and a function of the realized productivity growth as well. Thus, $X_t = \left[X_t^s, X_t^{\Delta a}\right]$. Using these notations, we can rewrite

$$\tilde{U}_{t+1}(s_{t+1}, X_{t+1}) = \tilde{U}_{t+1}(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a})$$

to better indicate the dependence of state variables on specific shocks. Ultimately, the recursive equation (A.1) is of the same form:

$$\begin{aligned} & \tilde{U}_t(s_t, X_t) \qquad \text{(C.18)} \\ = \quad & \max_{\tilde{C}_t, \tilde{I}_t} \left\{ (1-\beta)\tilde{C}_t^{1-\frac{1}{\psi}} + \beta \left(E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_t, X_t \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\psi}}, \end{aligned}$$

where the expectation on the right hand side is equivalent to:

$$\begin{aligned} & E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_t, X_t \right] \\ = & \sum_{s_{t+1}=1}^{2} E_t(\pi_{s_{t+1}, s_t} | s_t, X_t^s) \quad \ldots \\ \times & E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_{t+1}, s_t, X_t \right]. \qquad \text{(C.19)} \end{aligned}$$

In this case, we compute the conditional expectation in (C.19) by integrating over conditional distribution of mean growth rates and volatilities as well as Gaussian distribution of the error term in productivity growth. In particular:

$$\begin{aligned} & E_t \left[\tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a_{t+1}} \middle| s_{t+1}, s_t, X_t \right] \\ \approx & \sum_{j=1}^{J} \omega_\varepsilon(j) \left[\sum_{k=1}^{K} \omega_{\sigma^2_{s_{t+1}}}(k) \sum_{l=1}^{L} \omega_{\mu_{s_{t+1}}}(l) \cdot \tilde{U}_{t+1}^{1-\gamma}\left(s_{t+1}, s_t, X_t^s, \Delta a(j,k,l), X_t^{\Delta a}\right) \cdot e^{(1-\gamma)\Delta a(j,k,l)} \middle| s_{t+1}, s_t, X_t \right], \end{aligned}$$

(C.20)

where $\omega_\varepsilon(j)$ is the quadrature weight corresponding to the quadrature node $n_\varepsilon(j)$ used for the integration of a standard normal shock ε_{t+1} in productivity growth, $\omega_{\sigma^2_{s_{t+1}}}(k)$ and $\omega_{\mu_{s_{t+1}}}(l)$ are the quadrature weights corresponding to the quadrature nodes $n_{\sigma^2_{s_{t+1}}}(k)$ and $n_{\mu_{s_{t+1}}}(l)$ used for the integration of a truncated inverse gamma variable $\sigma^2_{s_{t+1}}$. and a truncated standard normal variable $\mu_{s_{t+1}}$, respectively. The observed realized productivity growth, $\Delta a(j,k,l)$, and a state variable, $\tilde{K}_{t+1}(j,k,l)$, are updated as follows:

$$\Delta a(j,k,l) = n_{\mu_{s_{t+1}}}(l) + \sqrt{n_{\sigma^2_{s_{t+1}}}(k)} \cdot n_\varepsilon(j) \tag{C.21}$$

$$e^{\Delta a(j,k,l)}\tilde{K}_{t+1}(j,k,l) = (1-\delta)\tilde{K}_t + \varphi\left(\frac{\tilde{I}_t}{\tilde{K}_t}\right)\tilde{K}_t, \tag{C.22}$$

where

$$\tilde{I}_t = \tilde{K}_t^\alpha \bar{N}^{1-\alpha} - \tilde{C}_t. \tag{C.23}$$

Finally, the numerical backward recursion can be performed by using (C.18)-(C.23). The boundary conditions are defined by the limiting economies $\tau_{1,\infty}$ and $\tau_{2,\infty}$, in which the transition probabilities π_{11} and π_{22}, mean growth rates μ_1 and μ_2, volatilities σ_1 and σ_2, are known.

C.2 Solving for a Dividend Claim

We also solve for the price-dividend ratio of the equity claim written on aggregate dividends, which are defined as a leverage to aggregate consumption. Exogenous aggregate dividends are given by:

$$\Delta d_{t+1} = g_d + \lambda \Delta c_{t+1} + \sigma_d \varepsilon_{d,t+1},$$

where $g_d = \Big(1-\lambda\Big)\Big(E(\mathbb{P}(s_\infty = 1|\pi_{11},\pi_{22}))\mu_1 + E(\mathbb{P}(s_\infty = 2|\pi_{11},\pi_{22}))\mu_2\Big)$ and $\mathbb{P}(s_\infty = i|\pi_{11},\pi_{22})$ is the ergodic probability of being in state i conditional on the transition probabilities π_{11} and π_{22}. Note that the long run mean of dividend growth, g_d, is changing under the household's filtration, though the true long run growth is constant. The subjective beliefs about the true parameter values induce fluctuations in g_d, which can be expressed as $g_d = g_d(s_{t+1}, s_t, X_t)$.

The equilibrium condition for the price-dividend ratio is given by (A.7). Similarly to the solution for the value function, we rewrite all variables in the recursion (A.7) as a function of the state variables and further use quadrature-type numerical methods to evaluate expectations on the right hand side of (A.7). Additionally, we update the long run dividend growth, $g_d(s_{t+1}, s_t, X_t)$, which is in fact random. Consequently, the equilibrium recursion used to solve the model is:

$$
\begin{aligned}
& PD_t(s_t, X_t) \\
= \; & E_t\left[\left. \begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \ldots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right) \end{array} \right| s_t, X_t\right] \\
= \; & E_t\left[E_t\left[\left. \begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \ldots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right) \end{array} \right| s_{t+1}, s_t, X_t\right] \middle| s_t, X_t\right] \\
= \; & \sum_{s_{t+1}=1}^{2} \mathbb{P}(s_{t+1}|s_t, X_t^s) \quad \ldots \\
\times \; & E_t\left[\left. \begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \ldots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right) \end{array} \right| s_{t+1}, s_t, X_t\right] \\
= \; & \sum_{s_{t+1}=1}^{2} E_t(\pi_{s_{t+1},s_t}|s_t, X_t^s) \quad \ldots \\
\times \; & E_t\left[\left. \begin{array}{c} \beta e^{\left(\lambda-\frac{1}{\psi}\right)(\Delta\tilde{c}_{t+1}+\Delta a_{t+1})}\left(\frac{\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}}{\mathcal{R}_t\left(\tilde{U}_{t+1}\cdot e^{\Delta a_{t+1}}\right)}\right)^{\frac{1}{\psi}-\gamma} \ldots \\ \times \quad e^{g_d(s_{t+1},s_t,X_t)+0.5\sigma_d^2}\cdot\left(PD_{t+1}\left(s_{t+1}, s_t, X_t^s, \Delta a_{t+1}, X_t^{\Delta a}\right)+1\right) \end{array} \right| s_{t+1}, s_t, X_t\right]
\end{aligned}
$$

Again, the conditional expectation of transition probabilities under the household's filtration permits an analytical formula, while the inner expectation in the expression above can be evaluated using the quadrature-type integration methods.

C.3 Limiting Economies - Boundary Values for General Case

The key assumption of the numerical numerical is that the household eventually learns the true values of all uncertain parameters in productivity growth. Thus, the simplest limiting economy is one in which all parameters are known, including both transition probabilities π_{11} and π_{22}, mean growth rates μ_1 and μ_2, volatilities σ_1 and σ_2. In this case, s_t and K_t are the only state variables in the economy. We employ the numerical solution methodology outlined for all known parameters. Specifically, we find the continuation utility and the price-dividend ratio of the equity claim for a set of parameter values $\pi_{11}, \pi_{22}, \mu_1, \mu_2, \sigma_1$ and σ_2.

D Existence of Equilibrium

Similarly to Collin-Dufresne, Johannes, and Lochstoer (2016) and Johannes, Lochstoer, and Mou (2016), the existence of the equilibrium in our production-based economy relies on the fact that the value function is concave and finite for all economies in which the parameters are known. Therefore, we verify that these conditions are satisfied for all limiting boundary economies.