CERGE
Center for Economic Research and Graduate Education
Charles University

Cognitive Limitations and Behavioral Biases In The Asset Pricing Context

Giorgi Chavchanidze

Master's Thesis

Prague, July 2021

Acknowledgements

supported by

Visegrad Fund

The project is co-financed by the Governments of Czechia, Hungary, Poland and Slovakia through Visegrad Grants from International Visegrad Fund. The mission of the fund is to advance ideas for sustainable regional cooperation in Central Europe.

Special thanks to CERGE-EI for its financial support, world-class instruction and faculty expertise.

Abstract

I incorporate behavioral and bounded rationality elements into a single asset-pricing framework by setting up a two-period consumption-based portfolio selection problem in which a representative agent has biased priors, does not observe the current state and thus has incomplete information about future state probabilities. He forms posterior beliefs using signals that he selects according to the rational inattention discrete choice framework of Matějka and McKay (2015), where the precision of the beliefs depend intuitively on the priors and the cost of information λ. In the case of log-utility, the optimal portfolio is a convex combination of the N portfolios the investor would have selected in each of the N states if they were fully observable, where the weights reflect the subjective posterior likelihood of time-zero states. The posterior beliefs are induced by parsimonious reweighing of priors, where the weights depend on λ, discount factor β and the relative entropies of the future state distributions induced by different time-zero states.

Using a two-state example, I demonstrate how the cost of information and biases can be jointly analyzed in this framework and discuss implied deviations from fully rational behavior. The major advantage of the proposed model is its flexibility. When the cost of information λ is zero and the agent has correct priors, the model reduces to the standard neoclassical framework. When λ is non-zero and the agent has correct priors, it is a model of bounded rationality with endogenous signals and form of information, where the cost of information reflects the mental capacity of the agent. When λ is zero and the agent has biased priors, the model reduces to the behavioral framework with standard preferences. The proposed framework could lay the foundations for multi-periods heterogeneous-agents models in which the effects of biases and costly information can be jointly analyzed and its consumption-based formulation might render it useful well beyond the asset pricing context.

Keywords: Neoclassical Asset Pricing, Behavioral Asset Pricing, Rational Inattention, Cost of Information, Mental Capacity

Abstract

Zahrnuji prvky behaviorální a omezené racionality do modelu oceňování aktiv s reprezentativním aktérem řešícím problém volby portfolia ve dvou obdobích Aktér modelu má zkreslená apriorní očekávání a neúplnou informaci o rozdělení budoucích stavů. Aktér utváří své posteriorní očekávání dle signálů, které volí ve frameworku racionální nepozornosti s diskrétní volbou vytvořeném Matějkou a Mckayem (2015). Přesnost očekávání závisí na apriorním očekávání a nákladech na získání informace λ. V případě logaritmické užitkové funkce je optimální portfolio konvexní kombinací N portfolií, která by investor volil v některém z N možných stavů, pokud by byly stavy plně pozorovatelné. Váhy konvexní kombinace představují subjektivní posteriorní pravděpodobnosti stavů v čase nula. Posteriorní očekávání je dáno přehodnocením vah plynoucích z apriorního očekávání, kde váhy závisí na λ, diskontním faktoru β a relativní entropii rozdělení budoucích stavů daných rozdílnými stavy v čase nula.

S použitím příkladu o dvou stavech ukazuji, jak náklady na informace a zkreslení mohou být společně zkoumány v popsaném frameworku a diskutuji implikované rozdíly oproti zcela racionálnímu chování. Hlavní výhodou navrženého modelu je flexibilita. Pokud jsou náklady na získání informace λ nulové a aktér má správné apriorní očekávání, pak se model redukuje na standardní neoklasický framework. Pokud je λ nenulová a aktér má správné apriorní očekávání, pak se jedná o model s omezenou racionalitou s endogenními signály a náklady na informace odrážejícími mentální kapacitu aktéra. Pokud je λ nulová a aktér má zkreslené očekávání, pak se model redukuje na behaviorální model se standardními preferencemi. Navržený framework by mohl být rozšířen o vice období a heterogenní aktéry. Rozšíření by umožnilo zkoumat společný vliv zkreslení a nákladných informací a reprezentace modelu založená na spotřebě by mohla být užitečná i v aplikacích mimo oceňování aktiv.

Klíčová slova: neoklasické oceňování aktiv, behaviorální oceňování aktiv, racionální nepozornost, náklady na informace, mentální kapacita

Contents

1 Introduction **6**

2 Model Description **9**

2.1 General Framework . . . 9

2.2 Relationship to the Asset Pricing Framework . . . 10

2.2.1 Asset Pricing in Complete Markets Using the No Arbitrage Principle . . . 11

2.2.2 Consumption-Based Asset Pricing Equation . . . 12

2.2.3 Consumption-Based Asset Pricing Equation in a Portfolio Selection Setting . . . 14

2.2.4 Equivalence Between Consumption Allocation and Portfolio Selection in a Two-Period Model . . . 19

2.3 Optimal Signal Selection: Discrete Choice Problem in Matějka and McKay (2015) . . . 23

2.4 Optimal Signal Selection: Consumption-Based Framework . . . 24

3 Posterior Beliefs **26**

3.1 General Properties . . . 26

3.2 Posterior Beliefs With Log-Utility . . . 36

3.3 Optimal Portfolio Induced by Posterior Beliefs and Log-Utility . . . 40

4 Two-State Example With Log-Utility **41**

4.1 Portfolio Selection With Arbitrary Beliefs . . . 41

4.2 Posterior Beliefs and Their Properties . . . 44

4.3 Portfolio Selection With Incomplete Information and Correct Priors . . . 47

4.4 Portfolio Selection With Incomplete Information and Biased Beliefs . . . 50

4.4.1 Overconfidence . . . 50

4.4.2 Excess Optimism . . . 51

4.5 Implications For Asset Pricing . . . 52

5 Directions For Future Research **53**

5.1 Agent Preferences . . . 54

5.2 Belief Formation Using Representativeness Heuristic . . . 54

5.3 Heterogeneity and More Than Two Periods . . . 55

6 Summary **56**

7 References **58**

1 Introduction[1]

Standard neoclassical models of portfolio selection assume that investors are ultra-rational: they are fully informed about the state of the world today, hold correct beliefs about fundamental uncertainty tomorrow and always invest their wealth optimally by selecting the portfolio allocations that maximize their expected utility over a fixed time horizon. These assumptions, however, are unrealistic and the predictions generated by standard neoclassical models fail to account for several stylized features of financial data. Two of the perhaps most popular "puzzles" generated by the neoclassical approach are the risk premium puzzle and the low risk-free rate puzzle exposed in Mehra and Prescott (1985), who demonstrate that neoclassical models overpredict the risk-free rate and starkly underpredict the risk premia.

Two strands of literature - motivated, respectively, by bounded rationality and behavioral approaches - have emerged in response to the shortcomings of the neoclassical approach. Bounded rationality extends the standard models by imposing cognitive constraints on the decision-makers. In such models, the state of the world is not observable and thus investors do not necessarily hold correct beliefs about the future. They gather information to refine their beliefs and make better decisions, but information is costly to acquire and there is a limit on the amount that can be processed. Thus, investors choose signals with optimal cost-usefulness balance and select portfolios that maximize their utilities conditional on the information conveyed by their chosen signals.

Among models of bounded rationality, Rational Inattention (RI) of Sims (2003) is particularly relevant to the asset pricing context. It distinguishes itself from other models in this class through its feature known as the Endogenous Form of Information. Most models of bounded rationality impose exogenous restrictions on the form of signals a Decision-Maker (DM) can observe or choose. In RI, however, a decision-maker optimizes by choosing a joint distribution of a noisy signal s and the state of the world variable x it proxies. Because s is noisy, it is not perfectly informative, but the nature of signal errors is under the control of a decision-maker. Given that the DM can choose any joint distribution of s and x her constraints allow, it follows that there is no restriction on the nature of available signals in RI framework. Considering the abundance of information in the modern era and the monetary value an information edge can have in the finance sector, the Endogenous Form of Information makes RI much better suited than other models of bounded rationality for studying asset pricing problems (a reader might refer to Mackowiak et al. (2020) for an extensive review of Rational Inattention approach and to Gabaix (2014) for an example of a model of bounded rationality with exogenous form of information acquisition).

Another appealing feature of RI is that loss-minimizing strategies of rationally inattentive agents can generate behavioral biases akin to those uncovered in experimental settings (see

[1] Parts of this section have appeared in the final paper submitted to Combined Skills 1.

e.g. Mackowiak et al. (2020) and Miao and Xing (2020)). Most interestingly, Batchuluun et al. (2019) show that as their information capacity falls, RI agents save less of their wealth in the form of risky assets and act as if they were more risk-averse. Similarly, Luo (2006) shows that the risk premium implied by RI models is higher than the one implied by standard neoclassical models. This feature of RI is also in line with the *ambiguity aversion* uncovered by Kahneman and Tversky (1992) in an experimental setting - when people are presented with a choice between a certain payment and a lottery, they tend to accept the lower certain payment instead of the lottery the less information they are provided with about the latter.

Despite the relevance of RI to behavioral biases, it cannot explain all behavioral phenomena. For instance, many of the biases exposed in Kahneman and Tversky (1992) exhibit themselves in the cases in which respondents face simple problems and are provided with all relevant information about the context. More generally, Bordalo et al. (2019) argue that professional analysts form their beliefs using the representativeness heuristic, i.e. they form wrong expectations about particular stock returns not because they lack information or face a complicated choice problem, but because they misinterpret a particular feature of the data available to them. Similarly, Frydman et al. (2018) find that some traders exhibit suboptimal trading patterns because they trade as if their current mental account is a continuation of an older one. For this reason, RI alone is insufficient to bridge the gap between the standard neoclassical assumptions of ultra-rationality and evidence from psychology.

The essence of the behavioral approach is probably best described by Shefrin (2008), who exposes a general framework for the behavioral modification of neoclassical models. In this setting, cognitive limitations are not explicitly modeled, but investors are assumed to have erroneous beliefs and non-standard preferences that reflect behavioral tendencies, where the nature of modeled errors and behavioral biases are motivated by experimental or other empirical evidence and could depend on a specific context. Such an approach facilitates the modeling of market sentiment and allows us to identify and isolate from each other the fundamental and non-fundamental components of the asset pricing process. For this reason, behavioral models perform much better empirically than do standard neoclassical models (see Shefrin (2008) for various examples of these).

Despite its flexibility and better explanatory power, the behavioral approach is oblivious of the direct effects of cognitive limitations, which could be significant in some contexts. Consider the case of the risk premium puzzle and the low risk-free rate puzzle. Shefrin (2008) shows that in a behavioral setting, the risk premium is the sum of the fundamental risk premium (i.e. the premium implied by the standard neoclassical approach) and sentiment premium, where the latter arises because of the mispricing of both bonds and risky assets. Hence, the behavioral approach provides a parsimonious explanation of why neoclassical models might underpredict the return of the risky asset and overpredict the risk-free rate. However, for the behavioral argument alone to be fully convincing, the sentiment premium must persist in a positive direction and be very large (around 7% on average). This feature is somewhat difficult to reconcile with both the model and the empirical evidence, because the aggregate investor errors are time-varying and stochastic in both cases, reflecting the varying composition of the market by different types of investors. The introduction of RI in the behavioral framework could potentially remedy this issue by allowing us to model the effects of additional uncertainty brought forth by unobservable states of the world and the higher degree of risk aversion that it is likely to generate.

In short, both behavioral and RI approaches modify the neoclassical framework in ways that are in essence different yet complementary. The merger of the two approaches into a single modeling framework is thus expected to deliver better insights about financial phenomena than each can alone. Yet, to the best of my knowledge, no efforts have been made in this direction. I propose and study an extension of the standard two-period consumption-based portfolio selection problem in which a representative agent with CRRA preferences has potentially biased priors, does not observe the current state and thus has incomplete information about future state probabilities. He forms posterior beliefs using signals that he selects according to the rational inattention discrete choice framework of Matějka and McKay (2015), where the precision of the beliefs depend intuitively on the priors and the cost of information λ. In the case of log-utility, the optimal portfolio is a convex combination of the N portfolios the investor would have selected in each of the N states if they were fully observable, where the weights reflect the subjective posterior likelihood of time-zero states. The posterior beliefs are induced by parsimonious reweighinging of priors, where the weights depend on λ, discount factor β and the relative entropies of the future state distributions induced by different time-zero states.

I discuss a two-state example that illustrates how model implications differ from those of the standard neoclassical approach. I show that when the agent is assumed to have correct priors, incomplete information and the noisiness of signals induces him to be more risk-averse relative to the full rationality case in the good state and more risk-seeking in the bad one. I also demonstrate that when the investor is overconfident, the effects of incomplete information are mitigated when the signals are correct and exacerbated when the signals are incorrect.

It is hoped that the paper lays the foundations for more advanced models that could potentially outperform existing ones. The potential of the proposed model rests on its two main features. First, it is based on a more realistic general axiom. The axiom consists of two simple points: i) investors potentially have prior biases and face mental capacity constraints that prevent them from correctly identifying future state probabilities and ii) investors are not necessarily standard expected utility maximizers and their preferences might reflect individual dispositions and emotions (this paper focuses on i)). Second, the proposed model literally builds on and extends existing ones. When the cost of information λ is zero and the agent has correct priors, the model reduces to the standard neoclassical framework.

When λ is non-zero and the agent has correct priors, it is a model of bounded rationality with endogenous signals and form of information, where the cost of information reflects the mental capacity of the agent. When λ is zero and the agent has biased priors, the model reduces to the behavioral framework with standard preferences, which should be able to accommodate behavioral preferences too.

2 Model Description

2.1 General Framework

The economy is inhabited by a continuum of identical agents with CRRA utilities and an endowment process $\omega(x_t)$, $t = 0, 1$. x_1, the state of the world tomorrow, is uncertain and the nature of uncertainty depends on x_0, the state of the world today. The representative agent wants to select a feasible consumption plan that maximizes his two-period expected utility, i.e. the agent wants to solve the following Optimization Problem ($OP(x_0)$ henceforth)

$$\max_{c(x_0),\{c(x_1)\}_{x_1}} \left\{ u\left(c(x_0)\right) + \beta E_{P_{x_1|x_0}}\left[u\left(c(x_1)\right)\right] \right\}$$

subject to

$$\sum_{t,x_t} q(x_t)c(x_t) \leq \sum_{t,x_t} q(x_t)\omega(x_t),$$

where $q(x_1)$ denotes a state-price corresponding to x_1 and $E_{P_{x_1|x_0}}[.]$ denotes the expectation taken w.r.t $P_{x_1|x_0}$, the conditional distribution of x_1 given x_0. The agent knows $P_{x_1|x_0}$ for any realization of x_0, but since x_0 is unobservable, the agent does not know true $P_{x_1|x_0}$ and so solving $OP(x_0)$ is not feasible. Instead, the agent observes a realization of signal s that is informative about x_0, forms posterior $P_{x_1|s}$ and solves the choice problem using the posterior beliefs (call the resulting optimization problem $OP(s)$ henceforth). The agent chooses signal s according to the framework of Matějka and McKay (2015) and updates beliefs according to Bayes Law.

In general, $P_{x_1|x_0}$ need not be the true conditional distribution of x_1 given x_0 and might be thought of as the distribution the representative agent associates to x_1 conditional on state x_0. When that is the case, we say that the agent has biased beliefs, where the form of biases depends on underlying assumptions about the agent errors. For instance, the conditional distributions of an overconfident agent would have smaller variances than corresponding true distributions. The nature of true $P_{x_1|x_0}$ also depends on the underlying assumptions about the fundamental uncertainty. For instance, if we assume that there are N states and they are persistent, $P_{x_1|x_0^j}$ can be specified as being concentrated around state j, with probability of other states becoming smaller the further away they are from state j. Since $P_{x_1|s}$ is a convex combination of $\left\{P_{x_1|x_0^j}\right\}_{j=1}^N$ as shown in Section 3.1, such specification would lead to posterior beliefs that have greater variance than does the true distribution $P_{x_1|x_0}$, which could in turn induce greater risk aversion on the part of the agent relative to the case with fully observable x_0.

The assumption that the state of the world is unobservable reflects the fact that the real economy is something very fundamental that cannot be described by numbers, nor everything that can be expressed in numbers can be accounted for. It reflects the financial standing, intellectual capital, motives, aspirations and abilities of all economic agents in a given area. Moreover, even something as fundamental as macroeconomic or sector-level indicators are often released after delay and one often encounters news describing how the market or policy-makers have been encouraged or disappointed by the past developments that they have only recently become aware of. Posterior updating then reflects the attempts of the agents to identify the conditions of the present, on which future developments depend. The informativeness of the chosen signals depends on the cost of information parameter λ in the rational inattention framework of Matějka and McKay (2015), who show that when $\lambda = 0$, the signals are perfectly informative (for this model, the result is also proved in Section 3.1). Hence, the standard neoclassical framework is a special case of the proposed model.

The paper focuses on general CRRA and log preferences for simplicity, though the results of Sections 2 and 3.1 should be more widely applicable. In Section 5, I comment briefly about how the CRRA utility function can represent behavioral preferences. The optimization problem is formulated in terms of a consumption plan selection for its generality and greater tractability. I show throughout the coming sections that in the two-period model, the consumption plan can be interpreted as a portfolio of assets the pay-off of which coincides with the chosen consumption level for each state tomorrow. More formally, I show that in a complete markets setting with N arbitrary assets, the optimal consumption plan has, for any set of Arrow-Debreu prices, the same pay-off as the optimal portfolio induced by consumption-based portfolio optimization. I also show that pure consumption and portfolio selection frameworks induce the identical asset pricing equation.

2.2 Relationship to the Asset Pricing Framework

The optimization problem of the representative agent is posed as a consumption allocation problem for its relative simplicity and greater generality. In this section, I am going to show that the consumption allocation problem is in essence a portfolio selection problem, can explicitly be formulated as such and induces the same asset pricing equation. I proceed as follows: Firstly, I derive a general asset pricing equation using No Arbitrage Principle in a complete markets setting and derive its specific form implied by the consumption allocation problem. Secondly, I show that the consumption-based portfolio selection problem induces the same asset pricing equation and that consumption plan induced by the optimal portfolio coincides with the optimal consumption plan obtained directly from the consumption allocation problem.

2.2.1 Asset Pricing in Complete Markets Using the No Arbitrage Principle

Let x_1^j denote $j-$th state of the world tomorrow, $j = 1,2,...,N$, and let AD_j denote $j-$th state Arrow-Debreu security that pays 1 unit of consumption in state x_1^j and zero otherwise. Further, let $\tau_A(x_1^j)$ denote the pay-off of asset A in state x_1^j and assume complete markets with no frictions or transaction costs. By market completeness, AD_j exists for all j and so the pay-off of any asset can be replicated using the portfolio of Arrow-Debreu securities. To see this, define

$$\tau_A \equiv \begin{bmatrix} \tau_A(x_1^1) \\ \tau_A(x_1^2) \\ \vdots \\ \tau_A(x_1^N) \end{bmatrix}$$

and note that since $\tau_A \in \mathbb{R}^N$ and the pay-off vector of AD_j is $e_{j,N}$ (an $N-$dimensional vector with 1 in its $j-$th entry and zero elsewhere), τ_A can be expressed as a linear combination of the pay-offs of AD_j, $j = 1,2,...,N$. More specifically,

$$\tau_A = \sum_{j=1}^{N} \tau_A(x_1^j) e_{j,N}$$

and so holding an asset A is equivalent to holding a portfolio of Arrow-Debreu securities consisting of $\tau_A(x_1^j)$ units of AD_j, $j = 1,2,...,N$. The No Arbitrage Principle then implies that P_A, the price of A, coincides with the price of the corresponding Arrow-Debreu portfolio, i.e. if $q(x_1^j)$ denotes the price of AD_j,

$$P_A = \sum_{j=1}^{N} q(x_1^j)\tau_A(x_1^j).$$

Of course, the expression above is not very informative unless we make assumptions about how $q(x_1^j)$ is determined for each j. A common way to proceed is to specify the probability the market associates to x_1^j as $p_M(x_1^j)$, $j = 1,2,...,N$, and rewrite the price of A as

$$P_A = \sum_{j=1}^{N} p_M(x_1^j) \frac{q(x_1^j)}{p_M(x_1^j)} \tau_A(x_1^j) = E_{p_M}\left[m_M(x_1)\tau_A(x_1)\right],$$

where $E_{p_M}[.]$ denotes the expectation taken w.r.t. the market beliefs and $m_M(x_1^j) \equiv \frac{q(x_1^j)}{p_M(x_1^j)}$ is called the market stochastic discount factor. The intuition behind this formula is that market prices asset A by discounting its pay-off in each state tomorrow to the present and then taking the subjective expectation over the discounted pay-offs. $m_M(x_1^j)$ is a proxy for how much market values one unit of consumption/pay-off in state j tomorrow and in consumption-based structural models, it corresponds to the number of units of present-day consumption a representative investor is willing to give up in exchange for a claim for one unit of consumption in state j tomorrow.

Further analysis of the asset pricing equation is based on additional assumptions about how market beliefs and stochastic discount factor are determined. Market beliefs always arise from some sort of aggregation of the beliefs of individual investors. In a model with a continuum of identical investors, the market beliefs coincide with the beliefs of the representative investor, since every investor is a priori assumed to be identical. In a model with heterogeneous investors, aggregation is non-trivial and requires further assumptions and derivations. In models of full investor rationality, $p_M(x_1^j)$ coincides with the true probability of x_1^j, while in the bounded rationality and behavioral models, $p_M(x_1^j)$ deviates from the true probability and reflects the biases and errors of individual investors.

2.2.2 Consumption-Based Asset Pricing Equation

In this section, I derive the expressions for $p_M(x_1^j)$ and $m_M(x_1^j)$ in the framework of Section 2.1 and under the assumption of complete information and full rationality, i.e. the representative agent observes the state of the world x_0 today, has correct beliefs and solves $OP(x_0)$ (generalization to other cases is straightforward). Since all agents are identical, market beliefs coincide with the beliefs of the representative investor and so

$$p_M(x_1^j) = P_{x_1|x_0}\left(x_1^j|x_0\right), \tag{2.1}$$

where $P_{x_1|x_0}\left(x_1^j|x_0\right)$ is the true probability of x_1^j conditional on x_0. Since x_0 is for now assumed to be observable, denote $P_{x_1|x_0}\left(x_1^j|x_0\right)$ by p_j for simplicity of notation. $m_M(x_1^j)$ is then given by $\frac{q(x_1^j)}{p_j}$, which is determined by the solution of $OP(x_0)$. To see this, denote the Lagrangian of $OP(x_0)$ by $\mathcal{L}$ and note that

$$\mathcal{L}\left(c_0, \left(c(x_1^j)\right)_{j=1}^N, \mu\right) = u(c_0) + \beta E_{P_{x_1|x_0}}\left[u\left(c(x_1)\right)\right] + \mu\left(\omega_0 + \sum_{j=1}^N q(x_1^j)\omega(x_1^j) - c_0 - \sum_{j=1}^N q(x_1^j)c(x_1^j)\right),$$

where $u(c_0) = \frac{c_0^{1-\gamma}}{1-\gamma}$ and $\beta E_{P_{x_1|x_0}}[u(c(x_1))] = \beta \sum_{j=1}^{N} \frac{c(x_1^j)^{1-\gamma}}{1-\gamma} p_j$. The First Order Conditions of the Lagrangian w.r.t. c_0 and $c(x_1^j)$ then imply, respectively, that

$$c_0^{-\gamma} = \mu$$

and

$$\beta p_j c(x_1^j)^{-\gamma} = \mu q(x_1^j),$$

which together give

$$\frac{q(x_1^j)}{p_j} = \beta \left(\frac{c(x_1^j)}{c_0} \right)^{-\gamma},$$

where $\frac{c(x_1^j)}{c_0}$ is an aggregate consumption growth from period 0 to period 1 in the event that state j occurs tomorrow.

The preceding discussion implies the following asset pricing equation for A:

$$P_A = E_{P_{x_1|x_0}} \left[\beta \left(\frac{c(x_1)}{c_0} \right)^{-\gamma} \tau_A(x_1) \right]. \tag{2.2}$$

At a first glance, the expression might not appear intuitive, since it relies on x_1 - the abstract notion of the state of the world tomorrow. Note, however, that the distribution of x_1 can be defined w.l.g. so that there is a one-to-one correspondence between the realizations of x_1 and those of the consumption growth random variable $\frac{c(x_1)}{c_0}$. Thus, the states of the world can be defined in terms of aggregate consumption growth, which would make the asset pricing equation more tractable (one can think of this as a real-life "calibration" of x_1). To see that such a definition of x_1 is possible, note from the First Order Condition for $c(x_1^j)$ that

$$c(x_1^j) = \left(\frac{\beta p_j}{\mu q(x_1^j)} \right)^{\frac{1}{\gamma}}, \tag{2.3}$$

which implies that, for $j \neq j'$,

$$\frac{c(x_1^j)}{c_0} = \frac{c(x_1^{j'})}{c_0}$$

if and only if

$$\frac{p_j}{q(x_1^j)} = \frac{p_{j'}}{q(x_1^{j'})}. \tag{2.4}$$

If the equality above fails for all pairs j, j', then we have a one-to-one correspondence between the realizations of x_1 and those of $\frac{c(x_1)}{c_0}$. If the equality holds for some pairs, we can proceed as follows. Suppose first that the equality holds for one pair only and assume w.l.g. that the pair is given by the $(N-1)$−th and N−th states. We can then redefine states of tomorrow using the variable $\tilde{x}_1$ with $(N-1)$-sized support, where $\tilde{x}_1^j = x_1^j$ for all $j < N-1$ and $\tilde{x}_1^{N-1}$ denotes an event $x_1^{N-1} \cup x_1^N$. The modified optimization problem will then yield solutions $c^*(\tilde{x}_1^j)$ that satisfy

$$c^*(\tilde{x}_1^j) = \begin{cases} c^*(x_1^j), & j < N-1 \\ c^*(x_1^{N-1}) = c^*(x_1^N), & j = N-1 \end{cases}, \tag{2.5}$$

which will give us the desired one-to-one correspondence, since, by assumption, $(N-1), N$ is the only pair that violates (2.4). If we suppose now that (2.4) holds for more than one pair, we can iterate the procedure for each such pair until no remaining pairs satisfy (2.4).

To demonstrate that (2.5) indeed holds, note first that if we replace the choice variable $c(x_1^N)$ by choice variable $c(x_1^{N-1})$ in $\mathcal{L}$, we will get the modified Lagrangian

$$\mathcal{L}'\left(c_0, \left(c(x_1^j)\right)_{j=1}^{N-1}, \mu\right) = \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \sum_{j=1}^{N-2} \frac{c(x_1^j)^{1-\gamma}}{1-\gamma} p_j + \frac{c(x_1^{N-1})^{1-\gamma}}{1-\gamma} (p_{N-1} + p_N) +$$

$$+\mu \left(\omega_0 + \sum_{j=1}^{N} q(x_1^j)\omega(x_1^j) - c_0 - \sum_{j=1}^{N-2} q(x_1^j)c(x_1^j) - c(x_1^{N-1}) \left(q(x_1^{N-1}) + q(x_1^N)\right)\right),$$

which is the Lagrangian of $OP(x_0)$ under the additional constraint that $c(x_1^N) = c(x_1^{N-1})$. However, since at the optimum $c(x_1^N) = c(x_1^{N-1})$, the additional constraint is non-binding and so the solution of the modified $OP(x_0)$ coincides with the original solution. If we set up the Lagrangian for the consumption allocation problem using $\tilde{x}_1$ instead of x_1 as the state variable, it will coincide with the Lagrangian of the modified $OP(x_0)$ and so (2.5) follows.

Hence, the asset pricing equation $P_A = E_{P_{x_1|x_0}}\left[\beta \left(\frac{c(x_1)}{c_0}\right)^{-\gamma} \tau_A(x_1)\right]$ is now tractable, since it depends on aggregate consumption growth and its distribution, the time discount factor β, the coefficient of relative risk aversion γ and the pay-off structure of asset A.

2.2.3 Consumption-Based Asset Pricing Equation in a Portfolio Selection Setting

The derivation of market beliefs and the stochastic discount factor in the previous section might seem somewhat arbitrary - after all, the expressions for $p_M(x_1^j)$ and $m_M(x_1^j)$ come from a consumption allocation problem that makes no explicit reference to assets or their prices. Note, however, that the consumption allocation problem is in essence a portfolio selection problem - by selecting a consumption plan $\left(c_0, \left\{c(x_1^j)\right\}_{j=1}^N\right)$, the representative agent buys an asset (or a portfolio of assets) the pay-off of which is given by $c(x_1)$. To see this, note that by choosing $c(x_1^j) = c^*(x_1^j)$, the agent purchases a claim to $c^*(x_1^j)$ units of consumption in state x_1^j tomorrow, where a claim to one unit of consumption in x_1^j can be interpreted as AD_j, since the former pays one unit if state x_1^j occurs and zero otherwise. Purchasing a claim to $c^*(x_1^j)$ units of consumption in state x_1^j is then equivalent to buying $c^*(x_1^j)$ units of AD_j

and so selecting a consumption plan $\left(c_0, \left\{c(x_1^j)\right\}_{j=1}^N\right)$ can be interpreted as enjoying c_0 units of consumption today and purchasing a portfolio of Arrow-Debreu securities with the pay-off vector τ_{AD} that is given by

$$\tau_{AD} \equiv \begin{bmatrix} c(x_1^1) \\ c(x_1^2) \\ \vdots \\ c(x_1^N) \end{bmatrix}.$$

In Section 2.2.4, I demonstrate that the pay-off of the optimal portfolio of N arbitrary assets in a complete markets setting indeed coincides with τ_{AD} and throughout the rest of this section, I show that pure consumption and portfolio-based frameworks yield the same asset pricing equation.

Consider the following portfolio selection problem in a complete markets framework: the representative investor is endowed with wealth W and wants to allocate it optimally across consumption today c_0 and N assets with different pay-off structures tomorrow. His preferences are described by CRRA utility function $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and his beliefs are given by true state probabilities as before, since x_0 is observable. Let P_i and a_i denote the price and the number of purchased units of asset i, respectively, and let $\tau_i(x_1^j)$ be the pay-off of asset i in state x_1^j tomorrow. Then, the optimization problem of the representative investor (call it $\mathcal{P}(x_0)$ henceforth, or $\mathcal{P}$ for short) can be formulated as follows:

$$\max_{c_o, \{a_i\}_{i=1}^N} \left\{ \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \sum_{j=1}^N \frac{c(x_1^j)^{1-\gamma}}{1-\gamma} p_j \right\}$$

subject to

$$c_0 + \sum_{i=1}^N a_i P_i = W,$$

$$c(x_1^j) = \sum_{i=1}^N a_i \tau_i(x_1^j) \quad \forall j.$$

In words, at time 0, the representative investor decides how much to consume today and invests the rest of his wealth $W - c_0$ in a portfolio $\{a_i\}_{i=1}^N$ of assets i priced at P_i, $i = 1, 2, ..., N$, by choosing $c_o, \{a_i\}_{i=1}^N$ in a way that maximizes his two-period expected utility. At time 1, he receives a pay-off $a_i \tau_i(x_1^j)$ from his holdings of asset i and so his portfolio pays him a total of $\sum_{i=1}^N a_i \tau_i(x_1^j)$, all of which he consumes.

If we denote $c(x_1^j, a) \equiv \sum_{i=1}^N a_i \tau_i(x_1^j)$, $a \equiv (a_i)_{i=1}^N$, and replace $c(x_1^j)$ in the objective function by $c(x_1^j, a)$,then the Lagrangian of $\mathcal{P}$ can be written as

$$\mathcal{L}_{\mathcal{P}}\left(c_0, (a_i)_{i=1}^N, \mu_{\mathcal{P}}\right) = \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \sum_{j=1}^{N} \frac{c(x_1^j, a)^{1-\gamma}}{1-\gamma} p_j + \mu_{\mathcal{P}} \left(W - c_0 - \sum_{i=1}^{N} a_i P_i\right), \tag{2.6}$$

where the First Order Condition for c_0 and a_i are given, respectively, by

$$c_0^{-\gamma} = \mu_{\mathcal{P}}$$

and

$$\beta \sum_{j=1}^{N} c(x_1^j, a)^{-\gamma} \tau_i(x_1^j) p_j = \mu_{\mathcal{P}} P_i,$$

which together imply that

$$P_i = \sum_{j=1}^{N} \beta \left(\frac{c(x_1^j, a)}{c_0}\right)^{-\gamma} \tau_i(x_1^j) p_j = E\left[\beta \left(\frac{c(x_1, a)}{c_0}\right)^{-\gamma} \tau_i(x_1)\right].$$

Note that in this framework, it is always true that $c(x_1^j, a) = c(x_1^j)$: a as an argument in $c(x_1^j, a)$ merely indicates that $c(x_1^j)$ is not a free choice variable and is determined by the selected portfolio, which in turns makes the optimization problem more tractable. Hence, denoting $c(x_1^j)$ induced by the optimal portfolio as $c^*_{\mathcal{P}}(x_1^j)$ and recalling from (2.1) that $p_j \equiv P_{x_1|x_0}\left(x_1^j | x_0\right)$, we could rewrite the asset pricing equation above as

$$P_i = E_{P_{x_1|x_0}}\left[\beta \left(\frac{c^*_{\mathcal{P}}(x_1)}{c_0}\right)^{-\gamma} \tau_i(x_1)\right],$$

which is the same asset pricing expression derived in the previous section. Note, however, that this does not yet imply that pure consumption and consumption-based portfolio selection frameworks yield identical prices for any given asset - to establish the equivalence between the two asset pricing equations, we need to show that $P_i = P_A$ whenever $\tau_i = \tau_A$, where

$$P_A = E_{P_{x_1|x_0}}\left[\beta \left(\frac{c^*(x_1)}{c_0}\right)^{-\gamma} \tau_A(x_1)\right]$$

is the price of asset A as determined by the approach in the previous section. For this it is sufficient to show that

$$c^*_{\mathcal{P}}(x_1^j) = c^*(x_1^j) \quad \forall j. \tag{2.7}$$

I establish (2.7) in Section 2.2.4. In the framework of a continuum of identical agents, this can alternatively be demonstrated using the equilibrium conditions, which illustrates how the asset market and consumption economy equilibriums are related. The equilibrium of the economy underlying $OP(x_0)$ is given by

$$c_0 = \omega_0$$

$$c(x_1^j) = \omega(x_1^j) \quad \forall j,$$

and so c_0 and $c(x_1^j)$ are exogenously determined for all j. Note that this is not mathematically at odds with $OP(x_0)$, since by solving $OP(x_0)$ the agent solves for optimal consumption plan as a function of prices. The equilibrium conditions then indicate that in equilibrium prices adjust so that the representative agent wants to consume exactly her endowment. To see this, recall that by the First Order Conditions of $OP(x_0)$

$$c(x_1^j) = c_0 \left(\frac{\beta p_j}{q(x_1^j)} \right)^{\frac{1}{\gamma}},$$

which in equilibrium yields

$$\omega(x_1^j) = \omega_0 \left(\frac{\beta p_j}{q(x_1^j)} \right)^{\frac{1}{\gamma}}$$

and so the equilibrium prices are given by

$$q^*(x_1^j) = \beta p_j \left(\frac{\omega(x_1^j)}{\omega_0} \right)^{-\gamma}.$$

The intuition behind this result is as follows: suppose that initially prices are such that $c^*(x_1^j) > \omega(x_1^j)$ for some j and for some agent, i.e. some agent in the economy want to secure an amount of consumption in state x_1^j that exceeds his endowment in that state. To do that, he needs to buy additional consumption claims for state x_1^j, but since every agent is identical in this artificial economy, everyone else would want to do the same. With the supply of consumption claims fixed at $\omega(x_1^j)$ for each j, the corresponding market would experience an excess demand for state$-j$ consumption claims and so the price will increase until $c^*(x_1^j) = \omega(x_1^j)$.

To show that $\mathcal{P}$ induces the same consumption choices in equilibrium, we first need to make the economies underlying the two frameworks comparable. More specifically, we need to specify the asset supply in the portfolio selection setting exogenously and so that the total units of consumption available today and in each state tomorrow coincide with the endowments in the consumption economy. To do that, let a_i^s denote the number of units of asset i available in the economy and note that $a_i^s \tau_i(x_1^j)$ is the total amount of consumption asset i generates in state x_1^j. Hence, the total amount of consumption available for state x_1^j is given by $\sum_{i=1}^N a_i^s \tau_i(x_1^j)$ and the comparability of the consumption selection and asset allocation frameworks require that

$$\sum_{i=1}^{N} a_i^s \tau_i(x_1^j) = \omega(x_1^j), \tag{2.8}$$

where the induced system of equation can be written in matrix form as

$$\tau_{\mathcal{P}} a^s = \omega,$$

where $a^s$is an $N-$dimensional vector with a_i^s as its i-th entry, ω is an $N-$dimensional vector with $\omega(x_1^j)$ as its j-th entry and $\tau_{\mathcal{P}}$ is an $N \times N$ matrix with $\tau_i(x_1^j)$ as its (j,i)-th entry. The sought out value for a^s is then uniquely determined by

$$a^s = \tau_{\mathcal{P}}^{-1} \omega,$$

where $\tau_{\mathcal{P}}^{-1}$ exists, since its $i-$th column is a pay-off vector of asset i,and in complete markets with N assets and N states, the asset pay-off vectors are linearly independent.

Now that we have ensured that aggregate endowments in the two economies are the same, it remains to establish the equivalence of consumption allocations. In the framework of $\mathcal{P}$, the equilibrium conditions are described by the equality of each asset demand with its supply, i.e. in equilibrium

$$a_i = a_i^s \quad \forall i,$$

which implies that

$$\sum_{i=1}^{N} a_i \tau_i(x_1^j) = \sum_{i=1}^{N} a_i^s \tau_i(x_1^j) \equiv \omega(x_1^j) \quad \forall j.$$

Since $\sum_{i=1}^{N} a_i \tau_i(x_1^j)$ is the induced consumption plan, i.e. $c(x_1^j, a) \equiv \sum_{i=1}^{N} a_i \tau_i(x_1^j)$, it follows that in equilibrium

$$c(x_1^j, a) = \omega(x_1^j) \quad \forall j.$$

The intuition behind how the equilibrium and the optimization problem interact is analogous to the one in pure consumption framework. More specifically, suppose the price of asset i is such that $a_i > a_i^s$ for some investor and some i, i.e. some investor want to hold more asset i than what he is endowed with. With every other investor having the same endowment and preferences, and with the supply of asset i fixed at a_i^s, the market for asset i would experience an excess demand and its price would adjust until $a_i = a_i^s$.

2.2.4 Equivalence Between Consumption Allocation and Portfolio Selection in a Two-Period Model

In the previous section we claimed that the consumption allocation can be interpreted as a portfolio selection and showed that in the framework of continuum of identical agents, the two frameworks yield the same asset pricing equation for an arbitrary asset A. Indeed, consumption allocation and portfolio selection are inseparable notions: investing in a portfolio of assets is an indispensable vehicle for allocating consumption across time, and investment in any asset is always driven by future consumption motive. In this section, I am going to show that the two-period consumption allocation and a portfolio selection problems are equivalent - they induce the same optimal consumption plan, which can be described using (or interpreted as) the pay-off of the optimal portfolio of N arbitrary assets in a complete market setting. This will be accomplished as follows: first, I will show that $\mathcal{P}$ formulated in terms of Arrow-Debreu securities (call it $\mathcal{P}_{AD}$ henceforth) yields the same optimal consumption plan as $OP(x_0)$ and afterwards, I will demonstrate that $\mathcal{P}$ formulated in terms of N arbitrary securities in complete markets setting is equivalent to $\mathcal{P}_{AD}$.

To establish the equivalence between $\mathcal{P}_{AD}$ and $OP(x_0)$, assume that the representative investor in $\mathcal{P}$ is as wealthy as the representative agent in $OP(x_0)$, i.e. let $W = \omega_0 + \sum_{j=1}^{N} q(x_1^j)\omega(x_1^j)$. Note that this neither complicates nor restricts $\mathcal{P}$ in any way, since $q(x_1^j)$ is assumed to be fixed in the optimization problem and $q(x_1^j)$,$\omega(x_1^j) > 0$ are arbitrary for all j. Further, note that in the framework of $\mathcal{P}_{AD}$, asset i is given by AD_i and so $P_i = q(x_1^i)$ and

$$\tau_i(x_1^j) = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{otherwise} \end{cases}.$$

With this formulation, the constraint $c(x_1^j) = \sum_{i=1}^{N} a_i \tau_i(x_1^j)$ becomes

$$c(x_1^j) = a_j \tau_j(x_1^j) = a_j \equiv a_{j,AD} \tag{2.9}$$

and we get by (2.6) that the Lagrangian of $\mathcal{P}_{AD}$ is given by

$$\mathcal{L}_{\mathcal{P_{AD}}}\left(c_0, (a_{i,AD})_{i=1}^{N}, \mu_{\mathcal{P_{AD}}}\right) = \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \sum_{j=1}^{N} \frac{{a_{j,AD}}^{1-\gamma}}{1-\gamma} p_j + \mu_{\mathcal{P}_{AD}} \left(\omega_0 + \sum_{j=1}^{N} q(x_1^j)\omega(x_1^j) - c_0 - \sum_{j=1}^{N} a_{j,AD} q(x_1^j) \right).$$

Note from $\mathcal{L}_{\mathcal{P_{AD}}}\left(c_0, (a_{i,AD})_{i=1}^{N}, \mu_{\mathcal{P_{AD}}}\right)$ that the objective function and the constraint of $\mathcal{P}_{AD}$ are identical to those of $OP(x_0)$ (the only difference is notational - in $OP(x_0)$, we denote the choice variables by $c_0, \left(c(x_1^j)\right)_{j=1}^{N}$, while in $\mathcal{P}_{AD}$, we replace $\left(c(x_1^j)\right)_{j=1}^{N}$ as choice variables by

$(a_{j,AD})_{j=1}^{N})$. Hence, $\mu_{\mathcal{P}_{\mathcal{AD}}} = \mu$ and it follows that

$$\mathcal{L}_{\mathcal{P}_{\mathcal{AD}}}\left(c_0, (a_{i,AD})_{i=1}^{N}, \mu_{\mathcal{P}_{\mathcal{AD}}}\right) \equiv \mathcal{L}\left(c_0, \left(c(x_1^j)\right)_{j=1}^{N}, \mu\right),$$

where $\mathcal{L}$ and μ are the Lagrangian and the Lagrange multiplier of $OP(x_0)$, respectively. It is then immediate that $c^*(x_1^j) = a_{j,AD}^*$ for all j and since $c_{\mathcal{P}_{AD}}(x_1^j)$ also equals $a_{j,AD}^*$ by (2.9), we get that $c_{\mathcal{P}_{AD}}(x_1^j) = c^*(x_1^j)$.

It now remains to establish the equivalence between $\mathcal{P}_{AD}$ and $\mathcal{P}$. Consider $\mathcal{P}$ with N arbitrary assets in a complete markets setting and let the matrix of asset pay-offs be given by

$$\tau_{\mathcal{P}} \equiv \begin{bmatrix} \tau_1(x_1^1) & \tau_2(x_1^1) & \cdots & \tau_{N-1}(x_1^1) & \tau_N(x_1^1) \\ \tau_1(x_1^2) & \tau_2(x_1^2) & \cdots & \tau_{N-1}(x_1^2) & \tau_N(x_1^2) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \tau_1(x_1^{N-1}) & \tau_2(x_1^{N-1}) & \cdots & \tau_{N-1}(x_1^{N-1}) & \tau_N(x_1^{N-1}) \\ \tau_1(x_1^N) & \tau_2(x_1^N) & \cdots & \tau_{N-1}(x_1^N) & \tau_N(x_1^N) \end{bmatrix},$$

where (j,i)-th entry of $\tau_{\mathcal{P}}$ is the pay-off of asset i in state j. By market completeness, the columns of $\tau_{\mathcal{P}}$ are linearly independent and so for any $b \in \mathbb{R}^N$, there exists $\alpha \equiv (\alpha_i)_{i=1}^N$ such that $\tau_{\mathcal{P}}\alpha^T = b$ (i.e. we can replicate any pay-off vector in $\mathbb{R}^N$ by selecting an appropriate linear combination of the N assets). Hence, for each j, there exists unique $\alpha^j \equiv \left(\alpha_i^j\right)_{i=1}^N$ such that $\tau_{\mathcal{P}}\left(\alpha^j\right)^T = e_{j,N}$, where $e_{j,N}$ is an $N-$dimensional vector with 1 as its $j-$th entry and zero elsewhere. Note that the portfolio consisting of α_i^j units of asset i, $i = 1, 2, ..., N$, is equivalent to AD_j, since the two have identical pay-off vectors. Thus, define

$$q(x_1^j) \equiv \sum_{i=1}^{N} \alpha_i^j P_i. \tag{2.10}$$

To see that such definition is well-motivated, note first that the N assets in this framework may or may not include AD_j for any j. If AD_j is among the N assets, No-Arbitrage principle requires that the price of AD_j coincides with that of the replicating portfolio. If AD_j is not among the N assets, $q(x_1^j)$ can be freely specified, and such definition is justified, since holding the portfolio given by $\left(\alpha_i^j\right)_{i=1}^N$ in $\mathcal{P}$ is equivalent to holding AD_j in $\mathcal{P}_{AD}$, and so there is consistency in the way $q(x_1^j)$ is defined across the two settings. Moreover, in the optimization problem, the prices are assumed to be fixed from the perspective of the investor; hence, to see whether induced choices in $\mathcal{P}$ and $\mathcal{P}_{AD}$ are the same or not, we need to compare asset allocations across the two settings for the same arbitrary fixed set of prices. The definition above then ensures that the differences, if any, in portfolio selections across the two frameworks do not arise because the representative investor observes different prices for equivalent securities.

Next, let "Portfolio $\left(\alpha_i^j\right)_{i=1}^N$" denote the portfolio consisting of α_i^j units of asset i, $i = 1, 2, ..., N$, and note that we can replicate any pay-off vector using $\left\{\left(\alpha_i^j\right)_{i=1}^N\right\}_{j=1}^N$ since Portfolio $\left(\alpha_i^j\right)_{i=1}^N$ is equivalent to AD_j. Hence, analogously to the discussion in (2.2.1), No-Arbitrage principle implies that the price of asset i can be written as

$$P_i = \sum_{j=1}^{N} \tau_i(x_1^j) q(x_1^j)$$

and so

$$\sum_{i=1}^{N} a_i P_i = \sum_{i=1}^{N} a_i \sum_{j=1}^{N} \tau_i(x_1^j) q(x_1^j) = \sum_{j=1}^{N} \left(\sum_{i=1}^{N} a_i \tau_i(x_1^j)\right) q(x_1^j) \equiv \sum_{j=1}^{N} \bar{a}_j q(x_1^j),$$

where $\bar{a}_j \equiv \sum_{i=1}^N a_i \tau_i(x_1^j)$. Hence, the Lagrangian of $\mathcal{P}$ given in (2.6) can be rewritten as

$$\mathcal{L}_{\mathcal{P}}\left(c_0, (a_i)_{i=1}^N, \mu_{\mathcal{P}}\right) = \mathcal{L}_{\mathcal{P}}\left(c_0, (\bar{a}_i)_{i=1}^N, \mu_{\mathcal{P}}\right) = \frac{c_0^{1-\gamma}}{1-\gamma} + \beta \sum_{j=1}^{N} \frac{\bar{a}_j^{1-\gamma}}{1-\gamma} p_j +$$

$$+\mu_{\mathcal{P}} \left(\omega_0 + \sum_{j=1}^{N} q(x_1^j)\omega(x_1^j) - c_0 - \sum_{j=1}^{N} \bar{a}_j q(x_1^j)\right)$$

which is identical to $\mathcal{L}_{\mathcal{P_{AD}}}\left(c_0, (a_i)_{i=1}^N, \mu_{\mathcal{P_{AD}}}\right)$ except that in $\mathcal{P}_{AD}$, $\left\{q(x_1^j)\right\}_{j=1}^N$ is exogenously given within the optimization problem, while in $\mathcal{P}$, $\left\{q(x_1^j)\right\}_{j=1}^N$ is derived by (2.10) from exogenously given asset prices and pay-offs. Hence, to establish the equivalence between $\mathcal{L}_{\mathcal{P}}$ and $\mathcal{L}_{\mathcal{P_{AD}}}$, we need to show two things:

1. For any positive $\left\{\left(\tau_i(x_1^j)\right)_{j=1}^N\right\}_{j=1}^N$ and $\left\{q(x_1^j)\right\}_{j=1}^N$, there exist unique $\{P_i\}_{i=1}^N$ such that (2.10) holds for all j;
2. There is a one-to-one correspondence between $\bar{a} \equiv (\bar{a}_j)_{j=1}^N$ and $a \equiv (a_j)_{j=1}^N$.

1) ensures that $\mathcal{P}$ can be formulated in terms of Arrow-Debreu portfolios for any set of fixed Arrow-Debreu prices $\left\{q(x_1^j)\right\}_{j=1}^N$ (we have already shown above that the converse is true - for any fixed $\{P_i\}_{i=1}^N$, Arrow-Debreu prices can be uniquely constructed according to (2.10)). 2) ensures that we can recover optimal values of a by solving for optimal values of $\bar{a}$. This is essential for establishing equivalence, since the objective in the original formulation of $\mathcal{P}$ is to solve for optimal $\{a_i\}_{i=1}^N$ conditional on prices $\{P_i\}_{i=1}^N$.

To see 1), let α denote the $N \times N$ matrix the $(j,i)-$th entry of which is given by α_i^j from

(2.10) and note that by definition of α_i^j,

$$\tau_{\mathcal{P}}\alpha^T = \begin{bmatrix} \tau_{\mathcal{P}}\left(\alpha^1\right)^T & \tau_{\mathcal{P}}\left(\alpha^2\right)^T & \cdots & \tau_{\mathcal{P}}\left(\alpha^N\right)^T \end{bmatrix} = \begin{bmatrix} e_{1,N} & e_{2,N} & \cdots & e_{N,N} \end{bmatrix} = I_{N\times N},$$

where $I_{N\times N}$ is an $N \times N$ identity matrix. Hence, α^T is invertible and so is α. Further, let q denote the vector of state-prices with $q(x_1^j)$ as its $j-$th entry and let P denote the vector of asset prices with P_i as its $i-$th entry, $i, j = 1, 2, ..., N$. Then, by (2.10),

$$\alpha P = q$$

and since α is invertible, P is uniquely determined by

$$P = \alpha^{-1}q.$$

To see 2), it is sufficient to show that for any fixed $\bar{a}$, the system of equations in a

$$\bar{a}_j \equiv \sum_{i=1}^{N} a_i \tau_i(x_1^j) \quad \forall j$$

has a unique solution. To see this, write the system in the matrix form

$$\tau a = \bar{a}$$

and note that since τ is invertible, a is uniquely determined by

$$a = \tau^{-1}\bar{a}.$$

Hence, we have demonstrated the following:

1. For any set of fixed Arrow-Debreu prices $\left\{q(x_1^j)\right\}_{j=1}^N$, the portfolio selection problem with N arbitrary assets in complete markets setting can be posed as the portfolio selection problem with N Arrow-Debreu portfolios, and the optimal selection $\bar{a}^*$ of Arrow-Debreu portfolios induces the optimal selection a^* of the original N assets.

2. For any set of fixed Arrow-Debreu prices $\left\{q(x_1^j)\right\}_{j=1}^N$, $\bar{a}^*$ coincides with a_{AD}^*, the optimal selection of Arrow-Debreu securities in $\mathcal{P}_{AD}$.

3. The solution of $\mathcal{P}_{AD}$ induces the same consumption plan as that of $OP(x_0)$ and so $\mathcal{P}$, $\mathcal{P}_{AD}$ and $OP(x_0)$ are all equivalent. More specifically, for any set of fixed Arrow-Debreu prices $\left\{q(x_1^j)\right\}_{j=1}^N$, the three induce the same consumption plan, which is or can be interpreted as a portfolio of assets that deliver different pay-offs in different states of the world tomorrow.

2.3 Optimal Signal Selection: Discrete Choice Problem in Matějka and McKay (2015)

Matějka and McKay (2015) set up and solve a discrete choice problem in which a decision-maker (DM) chooses among N actions with a pay-off vector of $\boldsymbol{v} \in \mathbb{R}^N$, where v_i, $i-$th entry of $\boldsymbol{v}$, is a pay-off of $i-$th action. The DM does not observe $\boldsymbol{v}$ (also called a state of nature) but optimally chooses a costly information processing strategy that generates a signal $\boldsymbol{s} \in \mathbb{R}^N$ that proxies $\boldsymbol{v}$, where the cost increases with signal precision. Given the signal realization $\boldsymbol{s}$, the DM updates her prior beliefs $G(\boldsymbol{v})$ to her optimal posterior $F^*(\boldsymbol{v}|\boldsymbol{s})$ and chooses action i with the highest posterior expected pay-off $E_{F^*(\boldsymbol{v}|\boldsymbol{s})}(v_i)$. Hence, given the optimal information strategy and a signal realization $\boldsymbol{s}$,the pay-ff of the DM is

$$V\left(F^*(.|\boldsymbol{s})\right) = \max_i E_{F^*(\boldsymbol{v}|\boldsymbol{s})}(v_i),$$

where information strategy F^* is optimal in the sense that it maximizes ex ante expected posterior pay-off net of the cost of information strategy, i.e.

$$F^* = \arg\max_F \left\{ \sum_{\boldsymbol{v}} \sum_{\boldsymbol{s}} V\left(F(.|\boldsymbol{s})\right) F(\boldsymbol{s}|\boldsymbol{v}) G(\boldsymbol{v}) - c(F) \right\} \tag{2.11}$$

subject to

$$\sum_{\boldsymbol{s}} F(\boldsymbol{s}, \boldsymbol{v}) = G(\boldsymbol{v}),$$

where the cost of information strategy $c(F)$ is modeled as

$$c(F) = \lambda \left(H(G) - E_{\boldsymbol{s}}\left[H\left(F(.|\boldsymbol{s})\right)\right] \right).$$

$H(G)$, the entropy of G, measures the uncertainty inherent in prior beliefs, while $H\left(F(.|\boldsymbol{s})\right)$ - the uncertainty inherent in the posterior beliefs after observing $\boldsymbol{s}$. $E_{\boldsymbol{s}}\left[H\left(F(.|\boldsymbol{s})\right)\right]$ measures average entropy of posterior beliefs and hence $H(G) - E_{\boldsymbol{s}}\left[H\left(F(.|\boldsymbol{s})\right)\right]$ is the average reduction in the uncertainty about $\boldsymbol{v}$ as a result of knowing $\boldsymbol{s}$. Positive constant λ is the cost of uncertainty reduction by one unit and serves as an indirect proxy of the mental capacity of the agent. Matějka and McKay (2015) show that under F^* the probability of choosing action i (equivalently, of observing a signal that leads to action i) conditional on $\boldsymbol{v}$ is given by

$$\mathcal{P}_i(\boldsymbol{v}) = \frac{\mathcal{P}_i^0 e^{v_i/\lambda}}{\sum_{j=1}^N \mathcal{P}_j^0 e^{v_j/\lambda}},$$

where $\mathcal{P}_i^0 = \sum_{\boldsymbol{v}} \mathcal{P}_i(\boldsymbol{v}) G(\boldsymbol{v})$ is the unconditional probability of choosing action i. Matějka and McKay (2015) also show that F^* implies a one-to-one correspondence between actions and signals (no two distinct signals can lead to the same action if the information strategy is optimal), which allows us to index signal realizations using i and rewrite the $\mathcal{P}_i(\boldsymbol{v})$ in terms of signal probabilities as

$$\mathcal{P}_i(\boldsymbol{v}) = P_{s|\boldsymbol{v}}\left(s_i|\boldsymbol{v}\right) = \frac{P_s(s_i)e^{v_i/\lambda}}{\sum_{j=1}^{N} P_s(s_j)e^{v_j/\lambda}}$$

where $P_{s|\boldsymbol{v}}$ and P_s are conditional and unconditional probability mass functions of signal s.

2.4 Optimal Signal Selection: Consumption-Based Framework

To relate the framework of Matějka and McKay (2015) to the consumption plan choice problem, let $\{x_0^i\}_{i=1}^N$ be the support of x_0 and let action i correspond to the consumption plan optimization *as if* the state of the world today was x_0^i (i.e. action i corresponds to solving $OP(x_0^i)$, which results in the choice of the consumption plan $c_{i,0}, \{c_i(x_1)\}_{x_1}$). Let x_0^k denote the true state at time 0 and define the pay-off of action i conditional on x_0^k as

$$v_i \equiv V\left(i, x_0^k\right) = u\left(c_{i,0}\right) + \beta E_{P_{x_1|x_0^k}}\left[u\left(c_i(x_1)\right)\right],$$

which is the expected utility of consumption plan $c_{i,0}, \{c_i(x_1)\}_{x_1}$ in state x_0^k. The pay-off depends on x_0^k, because the latter determines the distribution of x_1. Since $OP(x_0)$ has a unique solution, $V\left(k, x_0^k\right) > V\left(i, x_0^k\right)$ $\forall i \neq k$ and so the support of $\boldsymbol{V}\left(\boldsymbol{x_0}\right) = \left(V\left(i, x_0\right)\right)_i'$ is $N-$sized and bijective to $\{x_0^i\}_{i=1}^N$. Hence, the state of nature $\boldsymbol{v}$ in Matějka and McKay (2015) can equivalently be described by the state of the world x_0 (i.e. a realization of x_0 fixes a value for $\boldsymbol{v}$ and different realizations of x_0 fix different values for $\boldsymbol{v}$) and any probability distribution over $\boldsymbol{V}\left(\boldsymbol{x_0}\right)$ is also a distribution over x_0 (and vice versa). This allows us to rewrite the problem in Matějka and McKay (2015) using x_0 as a random variable instead of $\boldsymbol{v}$ and obtain a distribution of s given x_0 induced by optimal information processing strategy as

$$P^*_{s|x_0^k}\left(s_i|x_0^k\right) = \frac{P_s(s_i)e^{V\left(i,x_0^k\right)/\lambda}}{\sum_{j=1}^{N} P_s(s_j)e^{V\left(j,x_0^k\right)/\lambda}}. \tag{2.12}$$

$P^*_{x_1|s}$ can then be derived using $P_{x_1|x_0}$, P_{x_0} and $P^*_{s|x_0}$ according to Bayes Law.

A clear motivation for the usage of $P^*_{x_1|s}$ in $OP(s)$ requires clarifying the relationship between the pay-off induced by the solution of $OP(s)$ and the pay-off w.r.t which the agent optimizes signals. According to the framework of Matějka and McKay (2015), the agent optimizes signals w.r.t. the posterior expected pay-off given by

$$V\left(P_{x_0|s}\left(.|s\right)\right) = \max_j E_{P_{x_0|s}}\left[V\left(j, x_0\right)\right],$$

i.e. $P^*_{x_0|s}\left(.|s\right)$ is induced by maximizing the ex ante expectation of $V\left(P_{x_0|s}\left(.|s\right)\right)$ net of cost of information w.r.t. $P_{x_0|s}(.|s)$. By model assumptions, s is informative about x_1 as much as it informs about x_0 and thus s provides no information about x_1 conditional on knowing x_0. Hence, $\Pr(x_1|x_0) = \Pr(x_1|x_0, s)$ and so

$$E_{P_{x_0|s}}[V(j, x_0)] = u(c_{j,0}) + \beta E_{P_{x_1|s}}[u(c_j(x_1))],$$

which implies that for any $P_{x_0|s}(.|s)$

$$V\left(P_{x_0|s}(.|s = s_i)\right) = \max_j E_{P_{x_0|s_i}}[V(j, x_0)] =$$

$$= u(c_{i,0}) + \beta E_{P_{x_1|s_i}}[u(c_i(x_1))].$$

In contrast, $\tilde{V}\left(P_{x_0|s}(.|s = s_i)\right)$, the pay-off given by the solution of $OP(s)$ with posterior beliefs induced by $P_{x_0|s}(.|s)$, is given by

$$\tilde{V}\left(P_{x_0|s}(.|s = s_i)\right) = u(c_{s_i,0}) + \beta E_{P_{x_1|s_i}}[u(c_{s_i}(x_1))],$$

where $c_{s_i,0}, \{c_{s_i}(x_1)\}_{x_1}$ is a solution of $OP(s_i)$. Note that $\tilde{V}\left(P_{x_0|s}(.|s = s_i)\right)$ and $V\left(P_{x_0|s}(.|s = s_i)\right)$ differ only up to the selected consumption plan and by definition of $OP(s_i)$,

$$\tilde{V}\left(P_{x_0|s}(.|s = s_i)\right) \geq V\left(P_{x_0|s}(.|s = s_i)\right).$$

Since the solution to $OP(s_i)$ is unique,

$$\tilde{V}\left(P_{x_0|s}(.|s = s_i)\right) > V\left(P_{x_0|s}(.|s = s_i)\right)$$

if and only if

$$\left(c_{s_i,0}, \{c_{s_i}(x_1)\}_{x_1}\right) \neq \left(c_{i,0}, \{c_i(x_1)\}_{x_1}\right).$$

It turns out that the two consumption allocations are indeed different and so the inequality is strict. To see this, assume first that the two allocations are equal. Under this assumption, $c_{s_i,0} = c_{i,0}$ and by (2.3),

$$c_{s_i}(x_1^j) = c_i(x_1^j)$$

implies

$$P_{x_1|s_i}(x_1|s_i) = P_{x_1|x_0}(x_1|x_0^i),$$

which is a contradiction, since

$$P_{x_1|s_i}(x_1|s_i) = \sum_{l=1}^{N} \Pr\left(x_1|x_0^l\right) \Pr\left(x_0^l\right) \frac{e^{V\left(i, x_0^l\right)/\lambda}}{E_{P_s}\left[e^{V\left(j, x_0^l\right)/\lambda}\right]},$$

as will be shown in the next section.

In words, the model design can be summarized as follows: the agent has prior beliefs about N possible states of nature and knows the portfolio she would select in each. She gathers information to refine her beliefs about present (and future) and selects one of the N portfolios she finds preferable to remaining $N-1$ given her information. The agent refines her information gathering strategy over time through sophisticated trial and error until she concludes that her information-action strategy induces optimal outcomes on average with respect to her initial crude optimizing behavior (selecting one of N portfolios). Once she evolves to the point at which her information strategy is in this way optimal, she starts rethinking her approach to portfolio selection and realizes that given her information strategy, the crude way of choosing is sub-optimal and starts selecting portfolios in a way that maximizes her posterior expected utility with beliefs induced by her information strategy. I show in Section 3.3 that in the case of log-utility, the posterior optimal portfolio is a convex combination of the N portfolios the investor would have selected in each of the N states if they were fully observable, where the weights reflect the subjective posterior likelihood of time-zero states. Hence, the agent optimizes signals with the aim of differentiating among N portfolios she has in mind and then selects a combination of them she finds optimal given her posterior beliefs.

3 Posterior Beliefs

3.1 General Properties

Posterior beliefs have three intuitive properties. First, they are expressed as the convex combination of future state distributions induced by different time-zero states, where the weights reflect the posterior likelihood of time-zero states. Second, as the cost of information goes to zero, the probability that the agent has posterior beliefs that coincide with true probabilities goes to one (assuming that the agent has correct priors). Third, when the cost of information is infinitely high, the posterior beliefs coincide with priors. I demonstrate these features formally next and provide intuition about how weights adjust with changes in the cost of information.

I begin with a preliminary lemma that will be useful throughout the paper.

Lemma 3.1. *For any* $\lambda > 0$,

$$\sum_{x_0} \frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} \Pr(x_0) = 1 \quad \forall i.$$

Proof. By Bayes Law,

$$P_{s_\lambda}(s_i) = \sum_{x_0} \Pr(s_i|x_0) \Pr(x_0)$$

and by (2.12),

$$\sum_{x_0} \Pr(s_i|x_0) \Pr(x_0) = P_{s_\lambda}(s_i) \sum_{x_0} \frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]} \Pr(x_0),$$

where $E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right] = \sum_{j=1}^{N} P_{s_\lambda}(s_j) e^{V(j,x_0)/\lambda}$. Hence,

$$P_{s_\lambda}(s_i) = P_{s_\lambda}(s_i) \sum_{x_0} \frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} \Pr(x_0)$$

and so

$$\sum_{x_0} \frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]} \Pr(x_0) = 1.$$

□

Alternatively, one could also observe that

$$\Pr(x_0|s_i) = \frac{\Pr(s_i|x_0) \Pr(x_0)}{P_{s_\lambda}(s_i)} = \frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]} \Pr(x_0)$$

and since by definition of a probability distribution

$$\sum_{x_0} \Pr(x_0|s_i) = 1,$$

Lemma 3.1 follows.

Proposition 3.1. *Let $P_{x_1|s_\lambda}(x_1|s_i)$ denote the distribution of x_1 conditional on $s_\lambda = s_i$. Then,*

$$P_{x_1|s_\lambda}(x_1|s_i) = \sum_{x_0} \Pr(x_0, x_1) \frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]}.$$

.

Proof. Note first that

$$P_{x_1|s_i}(x_1|s_i) = \sum_{x_0} \Pr(x_0, x_1|s_i) = \sum_{x_0} \Pr(x_1|x_0, s_i) \Pr(x_0|s_i).$$

By model assumptions, $\Pr(x_1|x_0, s_i) = \Pr(x_1|x_0)$, and by Bayes Law and (2.12),

$$\Pr(x_0|s_i) = \frac{\Pr(s_i|x_0) \Pr(x_0)}{\Pr(s_i)} = \Pr(x_0) \frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}, \tag{3.1}$$

where $E_{P_s}\left[e^{V(j,x_0)/\lambda}\right] = \sum_{j=1}^{N} P_s(s_j)e^{V(j,x_0)/\lambda}$. Hence,

$$P_{x_1|s_\lambda}(x_1|s_i) = \sum_{x_0} \Pr(x_1|x_0)\Pr(x_0)\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} =$$

$$= \sum_{x_0} \Pr(x_0,x_1)\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}.$$

It now remains to show that $P_{x_1|s_i}(x_1|s_i)$ is indeed a probability mass function, i.e. want to prove that $P_{x_1|s_i}(x_1|s_i) \geq 0\ \forall x_1$ and $\sum_{x_1} P_{x_1|s_i}(x_1|s_i) = 1$. The first part is obvious. For the second part, note that

$$\sum_{x_1} P_{x_1|s_i}(x_1|s_i) = \sum_{x_1}\sum_{x_0} \Pr(x_0,x_1)\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} =$$

$$= \sum_{x_0}\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}\sum_{x_1}\Pr(x_0,x_1) = \sum_{x_0}\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}\Pr(x_0) = 1,$$

where the last equality follows from Lemma 3.1. □

As an additional sanity check, note also that by Bayes Law,

$$\Pr(x_1) = \sum_i P_{x_1|s_\lambda}(x_1|s_i)P_{s_\lambda}(s_i).$$

If we replace $P_{x_1|s_\lambda}(x_1|s_i)$ on the RHS with the expression from Proposition 3.1, we get

$$\Pr(x_1) = \sum_i\sum_{x_0}\Pr(x_0,x_1)\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}P_s(s_i) =$$

$$= \sum_{x_0}\Pr(x_0,x_1)\sum_i\frac{e^{V(i,x_0)/\lambda}P_{s_\lambda}(s_i)}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} = \sum_{x_0}\Pr(x_0,x_1) = \Pr(x_1).$$

The result of Proposition 3.1 has an intuitive interpretation: upon observing a signal s_i, the agent re-weights the joint probability of x_0 and x_1 by $\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}$ for any given x_0 and x_1. $e^{V(i,x_0)/\lambda}$ is a monotone-increasing transformation of the pay-off of action i in state x_0, while $E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]$ is the average transformed pay-off in state x_0. If the pay-off of action i in state x_0 is above the average pay-off in state x_0, then for any x_1, $\Pr(x_0,x_1)$ receives a weight greater than 1 and vice versa. Moreover, the greater the difference between $e^{V(i,x_0)/\lambda}$ and $E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]$, the higher the weight $\Pr(x_0,x_1)$ receives for any x_1. This is the consequence of optimal signal selection - upon observing a signal under which action i is optimal, the states under which action i has relatively high pay-off become more likely and vice versa. Moreover, we will see in Section 3.2 that the weights $\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}$ are particularly simple and intuitive when the agent preferences are described by log-utility.

Alternatively, the re-weighting could be interpreted as follows: prior to observing any information, the probability the agent associates to date-event pair x_1 is given by

$$P_{x_1}(x_1) = \sum_{x_0} \Pr(x_1|x_0)\Pr(x_0),$$

which is the convex combination of conditional probabilities $\Pr(x_1|x_0)$ taken over x_0, with weights given by $\Pr(x_0)$. Upon observing the signal realization s_i, the agent updates his beliefs about x_0 from $\Pr(x_0)$ to $\Pr(x_0|s_i)$ and adjusts his belief about x_1 by replacing the weights $\{\Pr(x_0)\}_{x_0}$ in the convex combination above by weights $\{\Pr(x_0|s_i)\}_{x_0}$, where

$$\Pr(x_0|s_i) = \Pr(x_0)\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]},$$

as shown in (3.1). The intuition about $\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}$ can then be applied to the prior probabilities of $\{x_0\}_{x_0}$ instead of the joint probability $\Pr(x_0, x_1)$.

The precision of posterior beliefs is intuitively related to λ. When λ is high,

$$\frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]} \approx 1 \quad \forall i,$$

and so

$$P_{x_1|s_\lambda}(x_1|s_i) = \sum_{x_0} \Pr(x_1|x_0)\Pr(x_0)\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} \approx$$
$$\approx \sum_{x_0} \Pr(x_1|x_0)\Pr(x_0) = \Pr(x_1).$$

When the cost of information λ is close to zero and the current state is x_0^k, the probability of an incorrect signal is very low and

$$\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} \approx 0 \quad \forall i \neq k,$$

$$\frac{e^{V(k,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} \approx \frac{1}{\Pr(x_0^k)},$$

i.e. with probability close to 1,

$$P_{x_1|s_i}(x_1|s_\lambda) = P_{x_1|s_i}(x_1|s_k) = \sum_{x_0} \Pr(x_1|x_0)\Pr(x_0)\frac{e^{V(k,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} \approx$$

$$\approx \Pr\left(x_1|x_0^k\right)\Pr\left(x_0^k\right)\frac{1}{\Pr\left(x_0^k\right)} = \Pr\left(x_1|x_0^k\right).$$

Hence, the signals are perfectly informative in the limit as $\lambda \to 0$ and perfectly uninformative as $\lambda \to \infty$. This will be demonstrated formally using Propositions 3.2 and 3.3 that rely on Lemmas which are proved first.

Lemma 3.2. *Let i and l index signal realizations and states, respectively, and define*

$$D_\lambda(i,l) \equiv \frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^N P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}. \tag{3.2}$$

Then, the following are true

i) for any i and l such that $i \neq l$,

$$\lim_{\lambda\to 0} D_\lambda(i,l) = 0.$$

ii) for any i and l such that $i = l$,

$$\lim_{\lambda\to 0} D_\lambda(i,l) = \frac{1}{P_{x_0}(x_0^i)}.$$

Proof. To prove i), take arbitrary i and note that if we divide both the numerator and denominator of $D_\lambda(i,l)$ by $e^{V\left(l,x_0^l\right)/\lambda}$, we get

$$D_\lambda(i,l) = \frac{e^{\left(V\left(i,x_0^l\right)-V\left(l,x_0^l\right)\right)/\lambda}}{P_{s_\lambda}(s_l)+\sum_{j\neq l}P_{s_\lambda}(s_j)e^{\left(V\left(j,x_0^l\right)-V\left(l,x_0^l\right)\right)/\lambda}}, \tag{3.3}$$

where $P_{s_\lambda}(s_i) > P_{x_0}(x_0^i) - \sum_{j\neq i}P_{s_\lambda}(s_j)e^{\left(V\left(j,x_0^i\right)-V\left(i,x_0^i\right)\right)/\lambda}$ by (**??**). Thus,

$$0 < D_\lambda(i,l) < \frac{e^{\left(V\left(i,x_0^l\right)-V\left(l,x_0^l\right)\right)/\lambda}}{P_{x_0}(x_0^i)},$$

where, for $l \neq i$,

$$\lim_{\lambda\to 0}\frac{e^{\left(V\left(i,x_0^l\right)-V\left(l,x_0^l\right)\right)/\lambda}}{P_{x_0}(x_0^i)} = 0,$$

since $V\left(j,x_0^l\right) - V\left(l,x_0^l\right) < 0$ (see Section 2.4). Hence, we get by Squeeze Lemma from Calculus that

$$\lim_{\lambda\to 0} D_\lambda(i,l) = 0 \quad \forall l \neq i, \tag{3.4}$$

as desired.

To prove ii), take arbitrary i and note by Lemma 3.1 that for any $\lambda > 0$,

$$\sum_{l=1}^{N} D_\lambda(i,l)\, P_{x_0}(x_0^l) = 1$$

and so

$$\lim_{\lambda\to 0} \sum_{l=1}^{N} D_\lambda(i,l)\, P_{x_0}(x_0^l) = \sum_{l=1}^{N} P_{x_0}(x_0^l) \lim_{\lambda\to 0} D_\lambda(i,l) = 1.$$

By (3.4), we further get

$$\sum_{l=1}^{N} P_{x_0}(x_0^l) \lim_{\lambda\to 0} D_\lambda(i,l) = \Pr\left(x_0^i\right) \lim_{\lambda\to 0} D_\lambda(i,i) + \sum_{l\neq i} P_{x_0}(x_0^l) \lim_{\lambda\to 0} D_\lambda(i,l) = \Pr\left(x_0^i\right) \lim_{\lambda\to 0} D_\lambda(i,i)$$

and so

$$P_{x_0}(x_0^i) \lim_{\lambda\to 0} D_\lambda(i,i) = 1, \tag{3.5}$$

or

$$\lim_{\lambda\to 0} D_\lambda(i,i) = \frac{1}{P_{x_0}(x_0^i)}.$$

□

Lemma 3.3. *Let $P_{x_0}(x_0)$ and $P_{s_\lambda}(s)$ denote the prior probability mass functions of x_0 and s_λ, respectively, and let s_i denote the realization of s_λ which signals that the state of the world is x_0^i. Then, for any i,*

$$\lim_{\lambda\to 0} P_{s_\lambda}(s_i) = P_{x_0}(x_0^i).$$

Proof. Take arbitrary i and note by (3.3) that

$$D_\lambda(i,i) = \frac{1}{P_{s_\lambda}(s_i) + \sum_{j\neq i} P_{s_\lambda}(s_j) e^{\left(V\left(j,x_0^i\right) - V\left(i,x_0^i\right)\right)/\lambda}}, \tag{3.6}$$

where

$$0 \leq P_{s_\lambda}(s_j) e^{\left(V\left(j,x_0^i\right) - V\left(i,x_0^i\right)\right)/\lambda} \leq e^{\left(V\left(j,x_0^i\right) - V\left(i,x_0^i\right)\right)/\lambda}.$$

Since $V\left(j,x_0^i\right) - V\left(i,x_0^i\right) < 0$,

$$\lim_{\lambda\to 0} e^{\left(V\left(j,x_0^i\right) - V\left(i,x_0^i\right)\right)/\lambda} = 0 \quad \forall j \neq i,$$

and so, by Squeeze Lemma from Calculus,

$$\lim_{\lambda\to 0} P_{s_\lambda}(s_j) e^{\left(V\left(j,x_0^i\right) - V\left(i,x_0^i\right)\right)/\lambda} = 0 \quad \forall j \neq i$$

and

$$\lim_{\lambda\to 0}\sum_{j\neq l} P_{s_\lambda}(s_j) e^{\left(V\left(j,x_0^i\right)-V\left(i,x_0^i\right)\right)/\lambda} = \sum_{j\neq l}\lim_{\lambda\to 0} P_{s_\lambda}(s_j) e^{\left(V\left(j,x_0^i\right)-V\left(i,x_0^i\right)\right)/\lambda} = 0. \tag{3.7}$$

Putting (3.5), (3.6) and (3.7) together, we get

$$P_{x_0}(x_0^i)\lim_{\lambda\to 0} D_\lambda(i,i) = P_{x_0}(x_0^i)\frac{1}{\lim_{\lambda\to 0} P_{s_\lambda}(s_i)} = 1 \tag{3.8}$$

and so

$$\lim_{\lambda\to 0} P_{s_\lambda}(s_i) = P_{x_0}(x_0^i).$$

□

Lemma 3.4. *Let s_j denote the realization of s_λ which signals that the state of the world is x_0^j and let x_0^k denote the true state of the world at time* 0*. Then, as $\lambda \to 0$,*

$$s_\lambda | x_0^k \xrightarrow{p} s_k.$$

Proof. To prove the lemma, it is sufficient to show that

$$\lim_{\lambda\to 0} P_{s_\lambda|x_0^k}\left(s_k|x_0^k\right) = 1.$$

By (2.12) and (3.2),

$$P_{s_\lambda|x_0^k}\left(s_k|x_0^k\right) = P_{s_\lambda}(s_k) D_\lambda(k,k)$$

and by Lemma 3.3,

$$\lim_{\lambda\to 0} P_{s_\lambda|x_0^k}\left(s_k|x_0^k\right) = \lim_{\lambda\to 0} P_{s_\lambda}(s_k)\lim_{\lambda\to 0} D_\lambda(k,k) = P_{x_0}\left(x_0^k\right)\lim_{\lambda\to 0} D_\lambda(k,k).$$

Noting that $P_{x_0}\left(x_0^k\right)\lim_{\lambda\to 0} D_\lambda(k,k) = 1$ by (3.8), we get that

$$\lim_{\lambda\to 0} P_{s_\lambda|x_0^k}\left(s_k|x_0^k\right) = 1,$$

as desired. □

Simply put, Lemma 3.4 indicates that the probability of s_λ generating a wrong signal vanishes as the cost of information goes to zero.

Proposition 3.2. *Let x_0^k be the true state of the world at time 0. Then, as $\lambda \to 0$,*

$$P_{x_1|s_\lambda}(x_1|s_\lambda) \xrightarrow{p} P_{x_1|x_0^k}\left(x_1|x_0^k\right).$$

Proof. Note first that

$$P_{x_1|s_\lambda}(x_1|s_\lambda) = P_{x_1|s_\lambda}(x_1|s_\lambda) - P_{x_1|s_\lambda}(x_1|s_k) + P_{x_1|s_\lambda}(x_1|s_k)$$

and so

$$\text{plim}_{\lambda\to 0} P_{x_1|s_\lambda}(x_1|s_\lambda) = \text{plim}_{\lambda\to 0}\left(P_{x_1|s_\lambda}(x_1|s_\lambda) - P_{x_1|s_\lambda}(x_1|s_k)\right) + \text{plim}_{\lambda\to 0} P_{x_1|s_\lambda}(x_1|s_k).$$

Hence, to prove the proposition, it is sufficient to show that

$$\text{plim}_{\lambda\to 0}\left(P_{x_1|s_\lambda}(x_1|s_\lambda) - P_{x_1|s_\lambda}(x_1|s_k)\right) = 0 \quad (*)$$

and

$$\lim_{\lambda\to 0} P_{x_1|s_\lambda}(x_1|s_k) = P_{x_1|x_0^k}\left(x_1|x_0^k\right) \quad (**).$$

To show $(*)$, note that $P_{x_1|s_\lambda}(x_1|.)$ is a composition of continuous functions and thus is itself continuous. Hence, because $s_\lambda|x_0^k \xrightarrow{p} s_k$ by Lemma 3.4, we have by the Continuous Mapping Theorem of Mann-Wald that

$$P_{x_1|s_\lambda}(x_1|s_\lambda) \xrightarrow{p} P_{x_1|s_\lambda}(x_1|\text{plim}_{\lambda\to 0} s_\lambda) = P_{x_1|s_\lambda}(x_1|s_k).$$

To show (**), note that

$$\lim_{\lambda\to 0} P_{x_1|s_\lambda}(x_1|s_k) = \lim_{\lambda\to 0}\sum_{x_0}\Pr(x_1|x_0)\Pr(x_0)\frac{e^{V(k,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} =$$

$$= \Pr\left(x_1|x_0^k\right)\Pr\left(x_0^k\right)\lim_{\lambda\to 0}\frac{e^{V\left(k,x_0^k\right)/\lambda}}{E_{P_s}\left[e^{V\left(j,x_0^k\right)/\lambda}\right]} + \sum_{l\neq k}\Pr(x_1|x_0)\Pr(x_0)\lim_{\lambda\to 0}\frac{e^{V\left(k,x_0^l\right)/\lambda}}{E_{P_s}\left[e^{V\left(j,x_0^l\right)/\lambda}\right]} =$$

$$= \Pr\left(x_1|x_0^k\right)\Pr\left(x_0^k\right)\frac{1}{\Pr\left(x_0^k\right)} + \sum_{l\neq k}\Pr(x_1|x_0)\Pr(x_0)\times 0 = \Pr\left(x_1|x_0^k\right),$$

where the last line follows from Lemma 3.2. □

Proposition 3.3. *Let s_i denote the realization of s_λ which signals that the state of the world is x_0^i and let $P_{x_1}(x_1)$ denote prior beliefs about x_1. Then, for any i,*

$$\lim_{\lambda\to\infty} P_{x_1|s_\lambda}(x_1|s_i) = P_{x_1}(x_1).$$

Proof. Note first that by limit laws,

$$\lim_{\lambda\to\infty} P_{x_1|s_\lambda}(x_1|s_i) = \lim_{\lambda\to\infty}\sum_{x_0}\Pr(x_0,x_1)\frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]} =$$

$$= \sum_{x_0} \Pr(x_0, x_1) \lim_{\lambda \to \infty} \frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]},$$

where $\frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]}$ is a continuous function of $\frac{1}{\lambda}$, since it is a composition of continuous functions in $\frac{1}{\lambda}$. Hence, for any x_0,

$$\lim_{\lambda \to \infty} \frac{e^{V(i,x_0)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)/\lambda}\right]} = \lim_{1/\lambda \to 0} \frac{e^{V(i,x_0)\frac{1}{\lambda}}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)\frac{1}{\lambda}}\right]} = \frac{e^{V(i,x_0)\times 0}}{E_{P_{s_\lambda}}\left[e^{V(j,x_0)\times 0}\right]} = 1$$

and so

$$\lim_{\lambda \to \infty} P_{x_1|s_\lambda}(x_1|s_i) = \sum_{x_0} \Pr(x_0, x_1) = P_{x_1}(x_1).$$

□

Lastly, I prove a proposition that provides an insight regarding how the weights evolve with marginal changes in λ at any given λ.

Proposition 3.4. *Let* $D_\lambda(i,l) \equiv \frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}}$. *Then, for any* i *and* l*,*

$$\frac{\partial}{\partial \lambda} D_\lambda(i,l) = \lambda^{-2} D_\lambda(i,l) \times E_{P_{s_\lambda}}\left[D_\lambda(j,l)\left(\lambda^2 \frac{\partial \log P_{s_\lambda}(s_j)}{\partial \lambda} + V\left(j,x_0^l\right) - V\left(i,x_0^l\right)\right)\right].$$

Proof. Note first that for any i and l,

$$\frac{\partial}{\partial \lambda}\left(e^{V\left(i,x_0^l\right)/\lambda}\right) = -\lambda^{-2} V\left(i,x_0^l\right) e^{V\left(i,x_0^l\right)/\lambda}$$

and by the Chain Rule of differentiation,

$$\frac{\partial}{\partial \lambda}\left(\sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}\right) = \sum_{j=1}^N \left(-\lambda^{-2} V\left(j,x_0^l\right) e^{V\left(j,x_0^l\right)/\lambda} P_{s_\lambda}(s_j) + e^{V\left(j,x_0^l\right)/\lambda} \frac{\partial P_{s_\lambda}(s_j)}{\partial \lambda}\right).$$

Applying again the Chain Rule to $e^{V\left(i,x_0^l\right)/\lambda}\left(\sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}\right)^{-1}$, we get

$$\frac{\partial}{\partial \lambda}\left(\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}}\right) = \frac{\frac{\partial}{\partial \lambda}\left(e^{V\left(i,x_0^l\right)/\lambda}\right) \sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}}{\left(\sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}\right)^2} -$$

$$- \frac{e^{V\left(i,x_0^l\right)/\lambda} \frac{\partial}{\partial \lambda}\left(\sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}\right)}{\left(\sum_{j=1}^N P_{s_\lambda}(s_j) e^{V\left(j,x_0^l\right)/\lambda}\right)^2}$$

and inserting the expressions for partial derivatives yields

$$\frac{\partial}{\partial\lambda}\left(\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}\right)=\frac{-\lambda^{-2}V\left(i,x_0^l\right)e^{V\left(i,x_0^l\right)/\lambda}\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}{\left(\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}\right)^2}-$$

$$-\frac{e^{V\left(i,x_0^l\right)/\lambda}\sum_{j=1}^{N}\left(-\lambda^{-2}V\left(j,x_0^l\right)e^{V\left(j,x_0^l\right)/\lambda}P_{s_\lambda}(s_j)+e^{V\left(j,x_0^l\right)/\lambda}\frac{\partial P_{s_\lambda}(s_j)}{\partial\lambda}\right)}{\left(\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}\right)^2}.$$

If we factor out $\lambda^{-2}\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}$ from the RHS, we further get

$$\frac{\partial}{\partial\lambda}\left(\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}\right)=\lambda^{-2}\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}\times$$

$$\times\sum_{j=1}^{N}\frac{-V\left(i,x_0^l\right)e^{V\left(j,x_0^l\right)/\lambda}P_{s_\lambda}(s_j)+V\left(j,x_0^l\right)e^{V\left(j,x_0^l\right)/\lambda}P_{s_\lambda}(s_j)+\lambda^2e^{V\left(j,x_0^l\right)/\lambda}\frac{\partial P_{s_\lambda}(s_j)}{\partial\lambda}\frac{1}{P_{s_\lambda}(s_j)}P_{s_\lambda}(s_j)}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}},$$

while factoring out $\frac{e^{V\left(j,x_0^l\right)/\lambda}P_{s_\lambda}(s_j)}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}$ from the ratio inside the summand on the last line gives

$$\frac{\partial}{\partial\lambda}\left(\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}\right)=\lambda^{-2}\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}\times$$

$$\times\sum_{j=1}^{N}\left(\lambda^2\frac{\partial P_{s_\lambda}(s_j)}{\partial\lambda}\frac{1}{P_{s_\lambda}(s_j)}+V\left(j,x_0^l\right)-V\left(i,x_0^l\right)\right)\frac{e^{V\left(j,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}P_{s_\lambda}(s_j).$$

Finally, setting $D_\lambda(i,l)\equiv\frac{e^{V\left(i,x_0^l\right)/\lambda}}{\sum_{j=1}^{N}P_{s_\lambda}(s_j)e^{V\left(j,x_0^l\right)/\lambda}}$ and observing that $\frac{\partial P_{s_\lambda}(s_j)}{\partial\lambda}\frac{1}{P_{s_\lambda}(s_j)}=\frac{\partial\log P_{s_\lambda}(s_j)}{\partial\lambda}$, we get

$$\frac{\partial}{\partial\lambda}D_\lambda(i,l)=\lambda^{-2}D_\lambda(i,l)\times\sum_{j=1}^{N}D_\lambda(j,l)\left(\lambda^2\frac{\partial\log P_{s_\lambda}(s_j)}{\partial\lambda}+V\left(j,x_0^l\right)-V\left(i,x_0^l\right)\right)P_{s_\lambda}(s_j)=$$

$$=\lambda^{-2}D_\lambda(i,l)\times E_{P_{s_\lambda}}\left[D_\lambda(j,l)\left(\lambda^2\frac{\partial\log P_{s_\lambda}(s_j)}{\partial\lambda}+V\left(j,x_0^l\right)-V\left(i,x_0^l\right)\right)\right].$$

□

Hence, since $\lambda^{-2}D_\lambda(i,l)>0$, the proposition above implies that for a given value of λ, a marginal decrease in the cost of information will induce the agent to assign a higher weight to state x_0^l upon observing a signal realization s_i if and only if

$$E_{P_{s_\lambda}}\left[D_\lambda(j,l)\left(\lambda^2\frac{\partial\log P_{s_\lambda}(s_j)}{\partial\lambda}+V\left(j,x_0^l\right)-V\left(i,x_0^l\right)\right)\right]<0.$$

3.2 Posterior Beliefs With Log-Utility

In the case of log-utility, which is a special case of the CRRA utility $u(c)=\frac{c^{1-\gamma}-1}{1-\gamma}$ corresponding to $\gamma=1$ (i.e. $\lim_{\gamma\to 1}u(c)=\log c$), the posterior beliefs are particularly intuitive, since they are described solely in terms of cost of information λ, discount factor β and the relative entropies of the future state distributions induced by different time-zero states. Relative Entropy from distribution Q to distribution P, also known as the Kullback-Leibler Divergence of Q from P and denoted by $KL(P||Q)$, is a well-known mathematical object that measures how much distribution Q differs from distribution P. Assuming the two distributions have the same support and letting x denote an arbitrary element in the common support, $KL(P||Q)$ is defined as

$$KL(P||Q)\equiv\sum_x P(x)\log\frac{P(x)}{Q(x)}=E_P\left[\log\frac{P(x)}{Q(x)}\right],$$

i.e. $KL(P||Q)$ is defined as the expected log-difference between the probabilities implied by the two distributions, where the expectation is taken w.r.t P. $KL(P||Q)$ is always non-negative and equals zero if and only if $P=Q$.

I start with deriving the solution of $OP(x_0^i)$, which will be used for deriving a specific algebraic form of posterior beliefs implied by log-utility. Let $\mathcal{L}_{x_0^i,\log}\left(c_0,\left(c(x_1^j)\right)_{j=1}^N,\mu_{x_0^i,\log}\right)$ denote the Lagrangian of the optimization problem of the agent with log-utility in fully observed state x_0^i and note that

$$\mathcal{L}_{x_0^i,\log}\left(c_{0,i},\left(c_i(x_1^j)\right)_{j=1}^N,\mu_{x_0^i,\log}\right)=\log c_{0,i}+\beta\sum_{j=1}^N\log c_i(x_1^j)p_i\left(x_1^j\right)+$$

$$\mu_{x_0^i,\log}\left(\omega_0+\sum_{j=1}^N q(x_1^j)\omega(x_1^j)-c_{0,i}-\sum_{j=1}^N q(x_1^j)c_i(x_1^j)\right),$$

where $p_i\left(x_1^j\right)\equiv P_{x_1|x_0}\left(x_1^j|x_0^i\right)$ and ω_0 and $\omega(x_1^j)$ are the endowments of the representative agent at time 0 and in state j tomorrow, respectively. Consumption choice variables $c_{0,i},\left(c_i(x_1^j)\right)_{j=1}^N$ are indexed by i to explicitly indicate the dependence of choices on the current state (or on the beliefs $P_{x_1|x_0}(.|x_0^i)$ induced by current state x_0^i). By the First Order Conditions of the optimization problem w.r.t $c_{0,i}$ and $c_i(x_1^j)$, respectively, we have

$$\frac{1}{c_{0,i}}=\mu_{x_0^i,\log}$$

and

$$\beta\frac{1}{c_i(x_1^j)}p_i\left(x_1^j\right)=\mu_{x_0^i,\log}q(x_1^j),$$

which together imply that

$$c_i(x_1^j)=c_{0,i}\beta\frac{p_i\left(x_1^j\right)}{q(x_1^j)}. \tag{3.9}$$

Next, I use the First Order Conditions to obtain a specific expression for $e^{V\left(i,x_0^l\right)/\lambda}$. Recall from Section 2.4 that the pay-ff of action i in state x_0^l is given by

$$V\left(i,x_0^l\right)=u\left(c_{i,0}\right)+\beta E_{P_{x_1|x_0^l}}\left[u\left(c_i(x_1)\right)\right],$$

where action i corresponds to selecting $c_{0,i},\left(c_i(x_1^j)\right)_{j=1}^N$ (optimizing as if the state of the world was x_0^i) and the pay-off corresponds to the expected utility of $c_{0,i},\left(c_i(x_1^j)\right)_{j=1}^N$. In the case of log-utility and by (3.9),

$$V\left(i,x_0^l\right)=\log c_{0,i}+\beta\sum_{j=1}^N\log c_i(x_1^j)p_l\left(x_1^j\right)=\log c_{0,i}+\beta\sum_{j=1}^N\log\left(c_{0,i}\beta\frac{p_i\left(x_1^j\right)}{q(x_1^j)}\right)p_l\left(x_1^j\right),$$

where, by the properties of the logarithmic function,

$$\sum_{j=1}^N\log\left(c_{0,i}\beta\frac{p_i\left(x_1^j\right)}{q(x_1^j)}\right)p_l\left(x_1^j\right)=\sum_{j=1}^N\log\left(\left(c_{0,i}\beta\frac{p_i\left(x_1^j\right)}{q(x_1^j)}\right)^{p_l\left(x_1^j\right)}\right)=\log\prod_{j=1}^N\left(c_{0,i}\beta\frac{p_i\left(x_1^j\right)}{q(x_1^j)}\right)^{p_l\left(x_1^j\right)}.$$

Note that by a trivial rearrangement of the terms in the product,

$$\prod_{j=1}^N\left(c_{0,i}\beta\frac{p_i\left(x_1^j\right)}{q(x_1^j)}\right)^{p_l\left(x_1^j\right)}=(c_{0,i}\beta)^{\sum_j p_l\left(x_1^j\right)}\prod_{j=1}^N\left(\frac{1}{q(x_1^j)}\right)^{p_l\left(x_1^j\right)}\prod_{j=1}^N p_i\left(x_1^j\right)^{p_l\left(x_1^j\right)}=$$

$$=Ac_{0,i}\prod_{j=1}^N p_i\left(x_1^j\right)^{p_l\left(x_1^j\right)}, \tag{3.10}$$

where the last equation follows by $\sum_j p_l\left(x_1^j\right)=1$ and $A\equiv\beta\prod_{j=1}^N\left(\frac{1}{q(x_1^j)}\right)^{p_l\left(x_1^j\right)}$. Hence, $V\left(i,x_0^l\right)$ can be written as

$$V\left(i,x_0^l\right)=\log c_{0,i}+\beta\log\left(Ac_{0,i}\prod_{j=1}^N p_i\left(x_1^j\right)^{p_l\left(x_1^j\right)}\right)=\log c_{0,i}+\log\left(A^\beta c_{0,i}^\beta\prod_{j=1}^N p_i\left(x_1^j\right)^{\beta p_l\left(x_1^j\right)}\right)=$$

$$= \log\left(A^{\beta}c_{0,i}^{1+\beta}\prod_{j=1}^{N}p_i\left(x_1^j\right)^{\beta p_l\left(x_1^j\right)}\right)$$

and so

$$e^{V\left(i,x_0^l\right)/\lambda} = e^{\log\left(\left(A^{\beta}c_{0,i}^{1+\beta}\prod_{j=1}^{N}p_i\left(x_1^j\right)^{\beta p_l\left(x_1^j\right)}\right)^{\frac{1}{\lambda}}\right)} = \left(A^{\beta}c_{0,i}^{1+\beta}\prod_{j=1}^{N}p_i\left(x_1^j\right)^{\beta p_l\left(x_1^j\right)}\right)^{\frac{1}{\lambda}}. \qquad (3.11)$$

Next, I derive $\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]}$, where $E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right] \equiv \sum_{h=1}^{N} e^{V\left(h,x_0^l\right)/\lambda}P_{s_\lambda}\left(s_h\right)$. Recall that s_h denotes a signal realization that suggests that the state of the world is x_0^h and that leads to action h in the signal selection stage described in Section 2.4. Note further that $\prod_{j=1}^{N}p_i\left(x_1^j\right)^{\beta p_l\left(x_1^j\right)}$ in (3.11) can be written as

$$\prod_{j=1}^{N}p_i\left(x_1^j\right)^{\beta p_l\left(x_1^j\right)} = e^{\log\left(\prod_{j=1}^{N}p_i\left(x_1^j\right)^{\beta p_l\left(x_1^j\right)}\right)} = e^{\beta\sum_{j=1}^{N}p_l\left(x_1^j\right)\log p_i\left(x_1^j\right)} =$$

$$= e^{\beta E_l\left[\log p_i(x_1)\right]},$$

where $E_l\left[.\right]$ denotes the expectation object induced by $P_{x_1|x_0}\left(.|x_0^l\right)$. Hence, (3.11) can be written as

$$e^{V\left(i,x_0^l\right)/\lambda} = \left(A^{\beta}c_{0,i}^{1+\beta}e^{\beta E_l\left[\log p_i(x_1)\right]}\right)^{\frac{1}{\lambda}} = A^{\beta/\lambda}c_{0,i}^{(1+\beta)/\lambda}e^{\frac{\beta}{\lambda}E_l\left[\log p_i(x_1)\right]}$$

and observing that A does not depend on the signal realization, we get

$$\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]} = \frac{A^{\beta/\lambda}c_{0,i}^{(1+\beta)/\lambda}e^{\frac{\beta}{\lambda}E_l\left[\log p_i(x_1)\right]}}{A^{\beta/\lambda}E_s\left[c_{0,h}^{(1+\beta)/\lambda}e^{\frac{\beta}{\lambda}E_l\left[\log p_h(x_1)\right]}\right]} = \frac{c_{0,i}^{(1+\beta)/\lambda}e^{\frac{\beta}{\lambda}E_l\left[\log p_i(x_1)\right]}}{E_s\left[c_{0,h}^{(1+\beta)/\lambda}e^{\frac{\beta}{\lambda}E_l\left[\log p_h(x_1)\right]}\right]}. \qquad (3.12)$$

Further, I am going to demonstrate that $c_{0,h}$ above is same across all h and hence, the consumption terms can be dropped too from (3.12). To see this, note that for any given prices $\left\{q(x_1^j)\right\}_{j=1}^{N}$ and a fixed budget, the budget constraint and (3.9) imply that the consumption plan $c_{0,h}, \left(c_h(x_1^j)\right)_{j=1}^{N}$ satisfies

$$\omega_0 + \sum_{j=1}^{N}q(x_1^j)\omega(x_1^j) = c_{0,h} + \sum_{j=1}^{N}q(x_1^j)\left(c_{0,h}\beta\frac{p_h\left(x_1^j\right)}{q(x_1^j)}\right)$$

and so

$$\omega_0 + \sum_{j=1}^{N} q(x_1^j)\omega(x_1^j) = c_{0,h} + \beta c_{0,h} \sum_{j=1}^{N} p_h\left(x_1^j\right) = c_{0,h}(1+\beta).$$

Hence, for given state prices $\left\{q(x_1^j)\right\}_{j=1}^{N}$ and for any h,

$$c_{0,h} = \frac{\omega_0 + \sum_{j=1}^{N} q(x_1^j)\omega(x_1^j)}{1+\beta} \tag{3.13}$$

and so

$$\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]} = \frac{c_{0,i}^{(1+\beta)/\lambda} e^{\frac{\beta}{\lambda}E_l[\log p_i(x_1)]}}{c_{0,h}^{(1+\beta)/\lambda} E_s\left[e^{\frac{\beta}{\lambda}E_l[\log p_h(x_1)]}\right]} = \frac{e^{\frac{\beta}{\lambda}E_l[\log p_i(x_1)]}}{E_s\left[e^{\frac{\beta}{\lambda}E_l[\log p_h(x_1)]}\right]}.$$

Finally, I obtain $\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]}$ in terms of relative entropies $KL(P_l(.)||P_h(.)), h = 1,2,...,N$, where $P_h(.) \equiv P_{x_1|x_0}\left(.|x_0^h\right)$. Note that if we multiply both the numerator and the denominator on the RHS of (3.12) by $e^{-\frac{\beta}{\lambda}E_l[\log p_l(x_1)]}$, we get

$$\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]} = \frac{e^{\frac{\beta}{\lambda}E_l[\log p_i(x_1)]}e^{-\frac{\beta}{\lambda}E_l[\log p_l(x_1)]}}{E_s\left[e^{\frac{\beta}{\lambda}E_l[\log p_h(x_1)]}e^{-\frac{\beta}{\lambda}E_l[\log p_l(x_1)]}\right]} =$$

$$= \frac{e^{\frac{\beta}{\lambda}E_l[\log p_i(x_1) - \log p_l(x_1)]}}{E_s\left[e^{\frac{\beta}{\lambda}E_l[\log p_h(x_1) - \log p_l(x_1)]}\right]},$$

where

$$E_l[\log p_h(x_1) - \log p_l(x_1)] = E_{P_l}\left[\log \frac{p_h(x_1)}{p_l(x_1)}\right] = -E_{P_l}\left[\log \frac{p_l(x_1)}{p_h(x_1)}\right] = -KL(P_l(.)||P_h(.))$$

and so

$$\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]} = \frac{e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_h(.))}\right]}. \tag{3.14}$$

Hence, by Proposition 3.1, the probability of x_1^j given a signal realization s_i can be written as

$$P_{x_1|s_\lambda}(x_1^j|s_i) = \sum_l \Pr\left(x_1^j|x_0^l\right)\Pr\left(x_0^l\right) \frac{e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_h(.))}\right]}. \tag{3.15}$$

Using and following Proposition 3.1 in Section 3.1, we showed that the agent updates his beliefs about x_1^j by replacing the weights $\Pr\left(x_0^l\right)$ in the convex combination defining the prior belief about x_1^j,

$$P_{x_1}(x_1^j) = \sum_l \Pr\left(x_1^j | x_0^l\right) \Pr\left(x_0^l\right),$$

by weights $\Pr\left(x_0^l\right) \frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]}$ that represent the updated probabilities of x_0^l based on a signal realization s_i. We also provided intuition behind re-weighting prior beliefs about x_0^l by $\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_s\left[e^{V\left(h,x_0^l\right)/\lambda}\right]}$. In the case of log-utility, the intuition is particularly sharp. Note that $e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}$ is a monotone-decreasing transformation of $KL(P_l(.)||P_i(.))$ and hence the closer $P_l(.)$ is to $P_i(.)$, the larger the $e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}$ term. Hence, $e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}$ can be interpreted as a measure of how close the two distributions are on a scale from zero to 1, with 1 corresponding to perfect proximity, since $KL(P_l(.)||P_i(.)) = 0$ if and only if $P_l(.) = P_i(.)$ and so $e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))} = 1$ if and only if $P_l(.) = P_i(.)$. $\frac{e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_h(.))}\right]}$ can then be interpreted as a measure of the proximity of $P_l(.)$ to $P_i(.)$ relative to the signal-probability-weighted average proximity of $P_l(.)$ to all other distributions induced by $\left\{x_0^h\right\}_{h=1}^N$. If the proximity of $P_l(.)$ to $P_i(.)$ is above the average, then the updated probability of x_0^l is greater than its prior probability and vice versa. In other words, upon observing a signal realization that suggests the state of the world is x_0^i, the distribution over x_1 induced by x_0^i and other similar distributions receive greater weight in the determination of the updated probabilities of x_1^j, $j = 1,2,...,N$. For instance, the weight x_0^i receives in the event of s_i being observed is always greater than 1, since $e^{-\frac{\beta}{\lambda}KL(P_i(.)||P_i(.))} = 1 > e^{-\frac{\beta}{\lambda}KL(P_i(.)||P_h(.))}$ for all h.

3.3 Optimal Portfolio Induced by Posterior Beliefs and Log-Utility

Interestingly, with log-utility, the optimal portfolio is a convex combination of the N portfolios the investor would have selected in each of the N states if they were fully observable, where the weights reflect the subjective posterior likelihood of time-zero states. To see this, recall by (3.9) that for a fixed set of Arrow-Debreu prices, the optimal consumption plan in state x_0^i, if it were fully observable, is given by

$$c_i(x_1^j) = c_0 \beta \frac{p_i\left(x_1^j\right)}{q(x_1^j)},$$

where c_0 is invariant across i by (3.13). Recall also by Section 2.2.4 that in state x_1^j, the pay-off of the optimal portfolio selected in complete markets setting in fully observable x_0^i coincides with $c_i(x_1^j)$, assuming the same wealth endowments in the two cases. Further, it is straightforward to show that the optimal consumption plan $c_{s_h}(x_1^j)$ induced by $OP(s_h)$ with log-utility satisfies

$$c_{s_h}(x_1^j) = c_{0,s_h}\beta\frac{p_{s_h}\left(x_1^j\right)}{q(x_1^j)},$$

where $p_{s_h}\left(x_1^j\right) = P_{x_1|s_\lambda}\left(x_1^j|s_h\right)$ and $c_{0,s_h} = c_0$, where the latter can be seen from (3.13). Thus, since

$$P_{x_1|s_\lambda}(x_1^j|s_h) = \sum_i \Pr\left(x_1^j|x_0^i\right)\Pr\left(x_0^i|s_h\right) = \sum_i p_i\left(x_1^j\right)\Pr\left(x_0^i|s_h\right),$$

we get that

$$c_{s_h}(x_1^j) = c_0\beta\frac{1}{q(x_1^j)}\sum_i p_i\left(x_1^j\right)\Pr\left(x_0^i|s_h\right) = \sum_i \Pr\left(x_0^i|s_h\right)c_0\beta\frac{p_i\left(x_1^j\right)}{q(x_1^j)} =$$

$$= \sum_i \Pr\left(x_0^i|s_h\right)c_i(x_1^j).$$

Since $c_{s_h}(x_1^j)$ is the pay-off of the optimal portfolio induced by $OP(s_h)$ and since all portfolios with identical pay-offs in all states are equivalent, the optimal portfolio in the case of incomplete information can be interpreted as the convex combination of the optimal portfolios induced by x_0^i, $i = 1, 2, ..., N$.

4 Two-State Example With Log-Utility

The following two-state example with log-utility illustrates how incomplete information distorts choice and how the distortions exhibit when the investor also has prior biases.

4.1 Portfolio Selection With Arbitrary Beliefs

I begin by solving a simple portfolio selection problem with arbitrary beliefs p_1 and p_2 about the two states of the world tomorrow, $0 < p_1 < 1$ and $p_2 = 1 - p_1$ (i.e. the representative investor associates probabilities p_1 and p_2 to the occurrences of states 1 and 2, respectively). Consider a two-period setting with two states in each period and two assets - a safe asset and a risky asset. The safe asset has a gross rate of return R^f in each state of the world tomorrow, while the risky asset has gross returns R_1 and R_2 in x_1^1 and x_1^2, respectively, $R_2 > R^f > R_1$ (i.e. think of x_1^2 as the good state and x_1^1 as the bad one). The representative investor is endowed with wealth W, consumes c_0 units of consumption at time zero and allocates the rest of his wealth between the two assets by investing $w_s(W - c_0)$ in the safe asset and $w_r(W - c_0)$ in the risky asset, $w_s + w_r = 1$. His portfolio defined by w_s and w_r pays $w_s(W - c_0)R^f + w_r(W - c_0)R_1$ in x_1^1 and $w_s(W - c_0)R^f + w_r(W - c_0)R_2$ in x_1^2, all of which he consumes at time 1. The objective of the investor is to select c_0, w_s, w_r such that the triple maximizes his two-period expected utility subject to his wealth. Hence, the optimization problem of the representative investor can be written as follows:

$$\max_{c_0, w_s, w_r} \left\{ \log c_0 + \beta \sum_{j=1}^{2} p_j \log c(x_1^j) \right\}$$

subject to

$$c(x_1^j) = w_s (W - c_0) R^f + w_r (W - c_0) R_j$$

$$w_s + w_r = 1.$$

If we plug in the first constraint into the objective function, the Lagrangian of the optimization problem can be written as

$$\mathcal{L}(c_0, w_s, w_r, \mu) = \log c_0 + \beta \sum_{j=1}^{2} p_j \log \Big(w_s (W - c_0) R^f + w_r (W - c_0) R_j \Big) + \mu [1 - w_s - w_r],$$

where the First Order Conditions w.r.t w_s and w_r yield, respectively,

$$\beta \sum_{j=1}^{2} p_j \frac{(W - c_0) R^f}{w_s (W - c_0) R^f + w_r (W - c_0) R_j} = \mu$$

and

$$\beta \sum_{j=1}^{2} p_j \frac{(W - c_0) R_j}{w_s (W - c_0) R^f + w_r (W - c_0) R_j} = \mu.$$

Note that $W - c_0$ can be factored out from the denominator in both expressions and so the First Order Conditions can be rewritten as

$$\sum_{j=1}^{2} p_j R^f \frac{1}{w_s R^f + w_r R_j} = \frac{\mu}{\beta}$$

and

$$\sum_{j=1}^{2} p_j R_j \frac{1}{w_s R^f + w_r R_j} = \frac{\mu}{\beta}.$$

Note that the derivations could perhaps be simplified if we replaced w_r by $1 - w_s$. However, because the following approach is generalizable to the case of N states and N assets, I proceed as follows: denote

$$z_j \equiv \frac{1}{w_s R^f + w_r R_j}, \quad j = 1, 2$$

and note that the first order conditions can be written in matrix form as

$$Az = b,$$

where $A = \begin{bmatrix} p_1 R^f & p_2 R^f \\ p_1 R_1 & p_2 R_2 \end{bmatrix}$, $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and $b = \begin{bmatrix} \mu/\beta \\ \mu/\beta \end{bmatrix}$. Hence, observing that the discriminant of A is given by

$$D_A = p_1 p_2 R^f R_2 - p_1 p_2 R^f R_1 = p_1 p_2 R^f (R_2 - R_1) > 0,$$

the inverse of A can be written as

$$A^{-1} = \frac{1}{D_A} \begin{bmatrix} p_2 R_2 & -p_2 R^f \\ -p_1 R_1 & p_1 R^f \end{bmatrix}$$

and so

$$z = A^{-1} b = \frac{\mu}{\beta D_A} \begin{bmatrix} p_2 R_2 - p_2 R^f \\ -p_1 R_1 + p_1 R^f \end{bmatrix} = \frac{\mu}{\beta D_A} \begin{bmatrix} p_2 \left(R_2 - R^f \right) \\ p_1 \left(R^f - R_1 \right) \end{bmatrix}.$$

Next, assume for simplicity that $R_2 - R^f = R^f - R_1 \equiv ER$ and note that

$$z = \frac{\mu ER}{\beta R^f (R_2 - R_1)} \begin{bmatrix} \frac{1}{p_1} \\ \frac{1}{p_2} \end{bmatrix} = \frac{1}{K} \mu \begin{bmatrix} \frac{1}{p_1} \\ \frac{1}{p_2} \end{bmatrix},$$

where $K \equiv \frac{\beta R^f (R_2 - R_1)}{ER}$. Thus, by definition of z,

$$w_s R^f + w_r R_j = \frac{K}{\mu} p_j \quad j = 1, 2,$$

where the pair of equations can be written in matrix form as

$$\tilde{A} w = \tilde{b},$$

where $\tilde{A} = \begin{bmatrix} R^f & R_1 \\ R^f & R_2 \end{bmatrix}$, $w = \begin{bmatrix} w_s \\ w_r \end{bmatrix}$ and $\tilde{b} = \begin{bmatrix} \frac{K}{\mu} p_1 \\ \frac{K}{\mu} p_2 \end{bmatrix}$. Hence,

$$w = \tilde{A}^{-1} \tilde{b} = \frac{1}{R^f (R_2 - R_1)} \begin{bmatrix} R_2 & -R_1 \\ -R^f & R^f \end{bmatrix} \begin{bmatrix} \frac{K}{\mu} p_1 \\ \frac{K}{\mu} p_2 \end{bmatrix} =$$

$$= \frac{1}{R^f (R_2 - R_1)} \frac{K}{\mu} \begin{bmatrix} p_1 R_2 - p_2 R_1 \\ R^f (p_2 - p_1) \end{bmatrix} = \frac{\beta}{\mu ER} \begin{bmatrix} p_1 R_2 - p_2 R_1 \\ R^f (p_2 - p_1) \end{bmatrix}$$

and since $w_s + w_r = 1$,

$$\frac{\beta}{\mu ER} (p_1 ER + p_2 ER) = 1$$

and

$$\mu = \beta.$$

Thus, the optimal weights are given by

$$w = \frac{1}{ER} \begin{bmatrix} p_1 R_2 - p_2 R_1 \\ R^f (p_2 - p_1) \end{bmatrix}.$$

The solution above provides a simple and intuitive insight about how investor beliefs affect portfolio selection: the higher the likelihood the investor associates with the good state (relative to the bad one), the higher the $p_2 - p_1$ term and so investment in the risky asset increases. Since the weights on the risky and safe assets add up to one, investment in the safe asset decreases (this can also be seen by direct inspection). Hence, to understand how incomplete information and prior biases affect investment in this simple setting, it is sufficient to analyze how investor beliefs deviate from true probabilities as we introduce biases and the scarcity of information into the analysis. This will be accomplished in the sections that follow.

4.2 Posterior Beliefs and Their Properties

We know by (3.15) that the posterior beliefs in the case of log-utility and N states are given by

$$P_{x_1|s_\lambda}(x_1^j|s_i) = \sum_{l=1}^{N} \Pr\left(x_1^j|x_0^l\right) \Pr\left(x_0^l\right) \frac{e^{-\frac{\beta}{\lambda} KL(P_l(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda} KL(P_l(.)||P_h(.))}\right]}.$$

In our example, $N = 2$ and recall from the previous section that state 2 denotes a good state and state 1 denotes a bad state. For simplicity and greater tractability, I make three additional assumptions:

1. *Uniform Priors:* The representative investor finds both states equally likely before observing any information; $\Pr(x_0^1) = \Pr(x_0^2) = \frac{1}{2}$.

2. *Persistent States:* Given x_0^j, x_1^j is more likely than x_1^{3-j}, $j = 1, 2$, i.e.

$$\Pr\left(x_1^1|x_0^1\right) > \Pr\left(x_1^2|x_0^1\right)$$

   and

$$\Pr\left(x_1^2|x_0^2\right) > \Pr\left(x_1^1|x_0^2\right).$$

3. *Symmetric Conditional Distributions:* given $p > \frac{1}{2}$,

$$\Pr\left(x_1^1|x_0^1\right) = \Pr\left(x_1^2|x_0^2\right) = p$$

and

$$\Pr\left(x_1^2|x_0^1\right) = \Pr\left(x_1^1|x_0^2\right) = 1-p.$$

(1) and (3) together imply that unconditional signal probabilities are also uniform. To see this, note first that the symmetry of the conditional distributions implies the symmetry of the relative entropies: with $P_i\left(x_1^j\right) \equiv \Pr\left(x_1^j|x_0^i\right)$, $i=1,2$, we have

$$KL\left(P_1\left(.\right)||P_2\left(.\right)\right) = P_1\left(x_1^1\right)\log\frac{P_1\left(x_1^1\right)}{P_2\left(x_1^1\right)} + P_1\left(x_1^2\right)\log\frac{P_1\left(x_1^2\right)}{P_2\left(x_1^2\right)} =$$

$$= p\log\frac{p}{q} + q\log\frac{q}{p} = P_2\left(x_1^2\right)\log\frac{P_2\left(x_1^2\right)}{P_1\left(x_1^2\right)} + P_2\left(x_1^1\right)\log\frac{P_2\left(x_1^1\right)}{P_1\left(x_1^1\right)} = KL\left(P_2\left(.\right)||P_1\left(.\right)\right).$$

Since $KL\left(P_2\left(.\right)||P_2\left(.\right)\right) = KL\left(P_1\left(.\right)||P_1\left(.\right)\right) = 0$, denote $KL\left(P_1\left(.\right)||P_2\left(.\right)\right) = KL\left(P_2\left(.\right)||P_1\left(.\right)\right) \equiv KL$. Next, recall by Lemma 3.1 that for any $\lambda > 0$ and $i = 1, 2, ..., N$,

$$\sum_{x_0} \frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]} \Pr\left(x_0\right) = 1.$$

In our example, this condition translates into

$$\sum_{l=1}^{2} \frac{e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_h(.))}\right]}\frac{1}{2} = 1, \tag{4.1}$$

which can be expanded as

$$\frac{e^{-\frac{\beta}{\lambda}KL(P_1(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda}KL(P_1(.)||P_h(.))}\right]}\frac{1}{2} + \frac{e^{-\frac{\beta}{\lambda}KL(P_2(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda}KL(P_2(.)||P_h(.))}\right]}\frac{1}{2} = 1,$$

where

$$E_s\left[e^{-\frac{\beta}{\lambda}KL(P_1(.)||P_h(.))}\right] \equiv \sum_{h=1}^{2} P_{s_\lambda}\left(s_h\right)e^{-\frac{\beta}{\lambda}KL(P_1(.)||P_h(.))} = P_{s_\lambda}\left(s_1\right) + P_{s_\lambda}\left(s_2\right)e^{-\frac{\beta}{\lambda}KL}$$

and

$$E_s\left[e^{-\frac{\beta}{\lambda}KL(P_2(.)||P_h(.))}\right] \equiv \sum_{h=1}^{2} P_{s_\lambda}\left(s_h\right)e^{-\frac{\beta}{\lambda}KL(P_2(.)||P_h(.))} = P_{s_\lambda}\left(s_1\right)e^{-\frac{\beta}{\lambda}KL} + P_{s_\lambda}\left(s_2\right).$$

Hence, assuming for now that $i=1$,[2] condition (4.1) can be written as

$$\frac{1}{P_{s_\lambda}(s_1)+P_{s_\lambda}(s_2)e^{-\frac{\beta}{\lambda}KL}}\frac{1}{2}+\frac{e^{-\frac{\beta}{\lambda}KL}}{P_{s_\lambda}(s_1)e^{-\frac{\beta}{\lambda}KL}+P_{s_\lambda}(s_2)}\frac{1}{2}=1. \tag{4.2}$$

It is straightforward to see that if we set $P_{s_\lambda}(s_1)=P_{s_\lambda}(s_2)=\frac{1}{2}$, the condition above is satisfied and it can be shown using Lemma 2 in Matějka and McKay (2015) that the solution of (4.2) is unique.[3] Hence, the posterior beliefs in our example can be written as

$$P_{x_1|s_\lambda}(x_1^j|s_1)=\Pr\left(x_1^j|x_0^1\right)\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}+\Pr\left(x_1^j|x_0^2\right)\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}\quad j=1,2 \tag{4.3}$$

and

$$P_{x_1|s_\lambda}(x_1^j|s_2)=\Pr\left(x_1^j|x_0^1\right)\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}+\Pr\left(x_1^j|x_0^2\right)\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}\quad j=1,2. \tag{4.4}$$

Note that because of uniform prior beliefs about x_0, the prior probability assigned to x_1^j is a simple average of $\Pr\left(x_1^j|x_0^1\right)$ and $\Pr\left(x_1^j|x_0^2\right)$ and so

$$P_{x_1}(x_1^j)=\Pr\left(x_1^j|x_0^1\right)\frac{1}{2}+\Pr\left(x_1^j|x_0^2\right)\frac{1}{2}=\frac{1}{2},$$

where the last equality follows from $\Pr\left(x_1^j|x_0^2\right)=1-\Pr\left(x_1^j|x_0^1\right)$, which in turn holds by Assumption 3. The posterior probability assigned to x_1^j is also a convex combination of $\Pr\left(x_1^j|x_0^1\right)$ and $\Pr\left(x_1^j|x_0^2\right)$, but now the weights are uneven: $\Pr\left(x_1^j|x_0^1\right)$ receives a higher weight if s_1 is observed and $\Pr\left(x_1^j|x_0^2\right)$ receives a higher weight if s_2 is observed. Moreover, the smaller the cost of information λ, the more acute the re-weighting is. To see this, note that

$$\frac{\partial}{\partial\lambda}\left(\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}\right)=-\frac{\beta KLe^{-\frac{\beta}{\lambda}KL}}{\lambda^2\left(1+e^{-\frac{\beta}{\lambda}KL}\right)^2}<0, \tag{4.5}$$

which implies that a decrease in λ increases the weight $\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}$ at all values of λ. Since the two weights $\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}$ and $\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}$ add up to one for any λ, a decrease in λ always decreases $\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}$. In line with the results of Section 3.1, when λ is very close to zero, re-weighting is very sharp and so posterior belief $P_{x_1|s_\lambda}(x_1^j|s_i)$ is almost the same as $\Pr\left(x_1^j|x_0^i\right)$, while when λ is very large, there is almost no re-weighting at all and so $P_{x_1|s_\lambda}(x_1^j|s_i)$ is almost identical

[2] $i=2$ yields the same solution.

[3] Using Lemma 2 in Matějka and McKay (2015), condition (4.2) can equivalently be obtained as a linear equation in $P_{s_\lambda}(s_1)$ that has $\frac{1}{2}$ as a root.

to priors:

$$\lim_{\lambda\to 0}\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}=\frac{1}{1+\lim_{\lambda\to 0}e^{-\frac{\beta}{\lambda}KL}}=1,$$

$$\lim_{\lambda\to 0}\left(\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}\right)=1-\lim_{\lambda\to 0}\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}=0$$

and

$$\lim_{\lambda\to\infty}\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}=\frac{1}{1+\lim_{\lambda\to\infty}e^{-\frac{\beta}{\lambda}KL}}=\frac{1}{2},$$

$$\lim_{\lambda\to\infty}\left(\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}\right)=1-\lim_{\lambda\to\infty}\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}=\frac{1}{2}.$$

4.3 Portfolio Selection With Incomplete Information and Correct Priors

We showed in Section 4.1 that when the investor assigns probability p_j to state j tomorrow, $j=1,2$, the optimal weights on the safe and risky assets are given, respectively, by

$$w=\frac{1}{R_2-R^f}\begin{bmatrix} p_1R_2-p_2R_1 \\ R^f\left(p_2-p_1\right) \end{bmatrix},$$

where $w=(w_s,w_r)^T$. When the investor has correct prior beliefs and fully observes the true state of the world x_0^k, his beliefs about tomorrow coincide with true probabilities: $p_j=\Pr\left(x_1^j|x_0^k\right)$, $j=1,2$. When the state of the world x_0^k is not fully observable, the investor observes a signal realization s_i instead, i not necessarily equal to k, and updates his prior beliefs based on s_i : $p_j=P_{x_1|s_\lambda}(x_1^j|s_i)$, $j=1,2$. Since

$$w_r=\frac{R^f\left(p_2-p_1\right)}{R_2-R^f},$$

$$w_s=1-\frac{R^f\left(p_2-p_1\right)}{R_2-R^f}$$

and

$$p_2-p_1=p_2-(1-p_2)=2p_2-1$$

the differences in portfolio allocations across the complete and incomplete information cases can be fully described by analyzing how p_2 differs across the two cases. Since there are two possible states at time 0 and two possible signal realizations for each state, there are a total of four possible scenarios that could occur. For each case, let w_s^c and w_s denote the weights on the safe asset in the cases of complete and incomplete information, respectively, and recall that state 2 denotes the good state and state 1 denotes the bad state. I discuss each of the four scenarios separately.

- $(\boldsymbol{x_0^2, s_2})$: Suppose state 2 occurs at time zero and the investor observes a correct signal realization s_2. The true probability of state 2 tomorrow is thus given by $\Pr\left(x_1^2|x_0^2\right)$, while the posterior formed based on s_2 is described according to (4.4) as

$$p_2 = P_{x_1|s_\lambda}(x_1^2|s_2) = \Pr\left(x_1^2|x_0^1\right) \frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}} + \Pr\left(x_1^2|x_0^2\right) \frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}.$$

Since, $\Pr\left(x_1^2|x_0^2\right) > \Pr\left(x_1^2|x_0^1\right)$ by Assumption 2 in Section 4.2, $\Pr\left(x_1^2|x_0^2\right) > P_{x_1|s_\lambda}(x_1^2|s_2)$ and so $w_s > w_s^c$. The intuition behind this result is that the investor is aware of the possibility that the signal might wrongly indicate that the state of the world today is good and decides to invest more in the safe asset relative to the complete information case to mitigate the additional source of risk arising from imperfect signals. Moreover, by (4.5),

$$\frac{\partial}{\partial\lambda} P_{x_1|s_\lambda}(x_1^2|s_2) < 0$$

and so the additional risk aversion induced by signal noisiness increases as the cost of information λ rises.

- $(\boldsymbol{x_0^2, s_1})$: Suppose state 2 occurs at time zero and the investor observes an incorrect signal realization s_1. The true probability of state 2 tomorrow is thus given by $\Pr\left(x_1^2|x_0^2\right)$, while the posterior formed based on s_1 is described according to (4.3) as

$$p_2 = P_{x_1|s_\lambda}(x_1^2|s_1) = \Pr\left(x_1^2|x_0^1\right) \frac{1}{1+e^{-\frac{\beta}{\lambda}KL}} + \Pr\left(x_1^2|x_0^2\right) \frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}.$$

Since, $\Pr\left(x_1^2|x_0^2\right) > \Pr\left(x_1^2|x_0^1\right)$ by Assumption 2 in Section 4.2, $\Pr\left(x_1^2|x_0^2\right) > P_{x_1|s_\lambda}(x_1^2|s_1)$ and so $w_s > w_s^c$. The intuition here is straightforward: s_1 wrongly suggests that the bad state is more likely tomorrow than the good state and so the investor buys more of the safe asset relative to the full information case. However, since by (4.5),

$$\frac{\partial}{\partial\lambda} P_{x_1|s_\lambda}(x_1^2|s_1) > 0,$$

the additional investment in the safe asset falls with increasing cost of information λ. This is because with higher λ signals are less reliable and so the investor gives relatively higher weight to his prior beliefs when choosing a portfolio allocation.

Note that since $P_{x_1|s_\lambda}(x_1^2|s_2) > P_{x_1|s_\lambda}(x_1^2|s_1)$, the investment in the safe asset following the bad signal is higher than the investment following the good signal. However, in good state of the world x_0^2, the investment in the safe asset is higher relative to the full information case for any signal realization: the incorrect signal encourages greater investment by wrongly suggesting that the good state is unlikely tomorrow, while the correct signal induces additional investment in the safe asset because it carries the risk of being wrong (the investor then insures himself against the possibility that the state of the world is not as favorable as suggested by the signal). Hence, incomplete information induces greater risk aversion relative to the complete information case.

- $\boldsymbol{(x_0^1, s_1)}$: Suppose state 1 occurs at time zero and the investor observes a correct signal realization s_1. The true probability of state 2 tomorrow is thus given by $\Pr\left(x_1^2|x_0^1\right)$, while the posterior formed based on s_1 is described according to (4.3) as

$$p_2 = P_{x_1|s_\lambda}(x_1^2|s_1) = \Pr\left(x_1^2|x_0^1\right)\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}} + \Pr\left(x_1^2|x_0^2\right)\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}}.$$

  Since, $\Pr\left(x_1^2|x_0^2\right) > \Pr\left(x_1^2|x_0^1\right)$ by Assumption 2 in Section 4.2, $\Pr\left(x_1^2|x_0^1\right) < P_{x_1|s_\lambda}(x_1^2|s_1)$ and so $w_s < w_s^c$. The intuition is as follows: the investor is aware of the possibility that the signal might wrongly indicate that the current state is bad and the good state tomorrow - unlikely. This possibility induces him to be more hopeful of the good state tomorrow than he would have been if the state of the world was fully observable. Since, the risky asset pays more in the good state, the investor purchases more of it. When the cost of information is high, the signals are less accurate and so the investor is relatively more hopeful of the good state given the bad signal realization. As a result, he buys more of the risky asset. This can be seen formally by

$$\frac{\partial}{\partial \lambda} P_{x_1|s_\lambda}(x_1^2|s_1) > 0,$$

  which in turn follows from (4.5).

- $\boldsymbol{(x_0^1, s_2)}$: Suppose state 1 occurs at time zero and the investor observes an incorrect signal realization s_2. The true probability of state 2 tomorrow is thus given by $\Pr\left(x_1^2|x_0^1\right)$, while the posterior formed based on s_2 is described according to (4.4) as

$$p_2 = P_{x_1|s_\lambda}(x_1^2|s_2) = \Pr\left(x_1^2|x_0^1\right)\frac{e^{-\frac{\beta}{\lambda}KL}}{1+e^{-\frac{\beta}{\lambda}KL}} + \Pr\left(x_1^2|x_0^2\right)\frac{1}{1+e^{-\frac{\beta}{\lambda}KL}}.$$

  Since, $\Pr\left(x_1^2|x_0^2\right) > \Pr\left(x_1^2|x_0^1\right)$ by Assumption 2 in Section 4.2, $\Pr\left(x_1^2|x_0^1\right) < P_{x_1|s_\lambda}(x_1^2|s_2)$ and so $w_s < w_s^c$. The incorrect signal induces excess optimism and the investment in the risky asset increases. When the cost of information is high, the signals are less reliable and hence induced excess optimism is weaker. As a result, the excess investment in the risky asset falls with increasing λ. This can be seen formally by

$$\frac{\partial}{\partial \lambda} P_{x_1|s_\lambda}(x_1^2|s_2) < 0,$$

which in turn follows from (4.5).

To summarize, when the signals are not perfectly accurate, the investor factors in signal reliability into the decision-making. In the good state, this induces risk aversion relative to the complete information case, while in the bad state - risk-seeking.

4.4 Portfolio Selection With Incomplete Information and Biased Beliefs

So far we have assumed that the representative investor has correct prior beliefs and have analyzed the effects of incomplete information only. Below, I discuss two cases in which the investor has incorrect prior beliefs and analyze how the prior biases affect belief formation and investment behavior in the presence of incomplete information.

4.4.1 Overconfidence

The previous sections assumed that the investor beliefs conditional on fully observing the state of the world at time zero were given by true probabilities

$$\Pr\left(x_1^1|x_0^1\right) = \Pr\left(x_1^2|x_0^2\right) = p,$$

$$\Pr\left(x_1^2|x_0^1\right) = \Pr\left(x_1^1|x_0^2\right) = 1-p,$$

$p > \frac{1}{2}$. Assume now that the investor is overconfident, i.e. the conditional variance of x_1 implied by his beliefs is smaller than the true conditional variance. If we let $\tilde{p} \neq p$ describe investor beliefs and recall that the variance of Bernoulli distribution with parameter p is $p(1-p)$, the overconfidence condition can be written as

$$p(1-p) > \tilde{p}(1-\tilde{p}).$$

In the case of two states and under the assumption that $\tilde{p} > \frac{1}{2}$, the overconfidence implies $\tilde{p} > p$. Hence, an overconfident investor inflates the probability of the good state tomorrow when the current state is good and exaggerates the likelihood of the bad state tomorrow when the current state is bad, i.e. he is excessively optimistic in the good state and excessively pessimistic in the bad state. Let $\tilde{P}_{x_1|s_\lambda}(.|s_i)$ denote the posterior beliefs of the overconfident investor and note that by $\tilde{p} > p$,

$$\tilde{P}_{x_1|s_\lambda}(x_1^2|s_2) > P_{x_1|s_\lambda}(x_1^2|s_2)$$

and

$$\tilde{P}_{x_1|s_\lambda}(x_1^2|s_1) < P_{x_1|s_\lambda}(x_1^2|s_1).$$

i.e. since the weights in the posterior do not depend on the conditional probabilities, prior overconfidence translates into the posterior overconfidence. Thus, relative to the unbiased investor with incomplete information, the biased imperfectly informed investor is risk-seeking when he observes a good signal and risk-averse when he observes a bad signal.

To see how the overconfidence bias interacts with the cost of information, note first that by $\tilde{p} > p$ and the discussion in Section 4.3, overconfidence in the case of complete information induces risk-seeking in the good state and risk-aversion in the bad state relative to the fully rational behavior. Since the non-zero cost of information has an exactly opposite effect when the signal is correct, the two effects mitigate the deviations of each other from the full rationality case. When the signal is incorrect, however, the effects of incomplete information and overconfidence act in the same direction and so the deviation from optimal behavior is more severe.

4.4.2 Excess Optimism

Recall that the prior distribution of the states is uniform and suppose now that the investor is excessively optimistic: he finds the good state more likely at time zero than the bad state. If we let $\tilde{\Pr}\left(x_0^2\right)$ denote the probability the investor associates to the good state at time zero, we then have that $\tilde{\Pr}\left(x_0^2\right) > \Pr\left(x_0^2\right) = \frac{1}{2}$ and

$$\tilde{\Pr}\left(x_1^2\right) = \Pr\left(x_1^2 | x_0^1\right) \tilde{\Pr}\left(x_0^1\right) + \Pr\left(x_1^2 | x_0^2\right) \tilde{\Pr}\left(x_0^2\right) > \Pr\left(x_1^2\right) = \frac{1}{2}.$$

Note that with the biased priors, condition (4.2) given by

$$\frac{1}{P_{s_\lambda}(s_1) + P_{s_\lambda}(s_2) e^{-\frac{\beta}{\lambda}KL}} \frac{1}{2} + \frac{e^{-\frac{\beta}{\lambda}KL}}{P_{s_\lambda}(s_1) e^{-\frac{\beta}{\lambda}KL} + P_{s_\lambda}(s_2)} \frac{1}{2} = 1$$

is replaced by

$$\frac{1}{P_{s_\lambda}(s_1) + P_{s_\lambda}(s_2) e^{-\frac{\beta}{\lambda}KL}} \left(1 - \tilde{\Pr}\left(x_0^2\right)\right) + \frac{e^{-\frac{\beta}{\lambda}KL}}{P_{s_\lambda}(s_1) e^{-\frac{\beta}{\lambda}KL} + P_{s_\lambda}(s_2)} \tilde{\Pr}\left(x_0^2\right) = 1. \tag{4.6}$$

Since $\frac{1}{P_{s_\lambda}(s_1) + P_{s_\lambda}(s_2) e^{-\frac{\beta}{\lambda}KL}}$ and $\frac{e^{-\frac{\beta}{\lambda}KL}}{P_{s_\lambda}(s_1) + P_{s_\lambda}(s_2) e^{-\frac{\beta}{\lambda}KL}}$ now have uneven weights, $P_{s_\lambda}(s_1) = P_{s_\lambda}(s_2) = \frac{1}{2}$ does not solve (4.6) any more, i.e.

$$\frac{1}{\frac{1}{2} + \frac{1}{2} e^{-\frac{\beta}{\lambda}KL}} \left(1 - \tilde{\Pr}\left(x_0^2\right)\right) + \frac{e^{-\frac{\beta}{\lambda}KL}}{\frac{1}{2} e^{-\frac{\beta}{\lambda}KL} + \frac{1}{2}} \tilde{\Pr}\left(x_0^2\right) =$$

$$= \frac{1}{1+e^{-\frac{\beta}{\lambda}KL}} 2\left(1-\tilde{\Pr}\left(x_0^2\right)\right)+\frac{e^{-\frac{\beta}{\lambda}KL}}{e^{-\frac{\beta}{\lambda}KL}+1} 2\tilde{\Pr}\left(x_0^2\right)<1.$$

It can be seen using Lemma 2 in Matějka and McKay (2015) that (4.6) implies $P_{s_\lambda}(s_2) > \frac{1}{2}$ whenever $\tilde{\Pr}\left(x_0^2\right) > \frac{1}{2}$. Hence, biases in prior beliefs induces biased signals, which in turn enter into the posterior beliefs that are given by

$$\tilde{P}_{x_1|s_\lambda}(x_1^2|s_2) = \Pr\left(x_1^2|x_0^1\right)\tilde{\Pr}\left(x_0^1\right)\frac{e^{-\frac{\beta}{\lambda}KL}}{P_{s_\lambda}(s_1)+P_{s_\lambda}(s_2)e^{-\frac{\beta}{\lambda}KL}}+\Pr\left(x_1^2|x_0^2\right)\tilde{\Pr}\left(x_0^2\right)\frac{1}{P_{s_\lambda}(s_1)e^{-\frac{\beta}{\lambda}KL}+P_{s_\lambda}(s_2)}$$

and

$$\tilde{P}_{x_1|s_\lambda}(x_1^2|s_1) = \Pr\left(x_1^2|x_0^1\right)\tilde{\Pr}\left(x_0^1\right)\frac{1}{P_{s_\lambda}(s_1)+P_{s_\lambda}(s_2)e^{-\frac{\beta}{\lambda}KL}}+\Pr\left(x_1^2|x_0^2\right)\tilde{\Pr}\left(x_0^2\right)\frac{e^{-\frac{\beta}{\lambda}KL}}{P_{s_\lambda}(s_1)e^{-\frac{\beta}{\lambda}KL}+P_{s_\lambda}(s_2)}.$$

Note that

$$P_{s_\lambda}(s_1)e^{-\frac{\beta}{\lambda}KL}+P_{s_\lambda}(s_2) > \frac{1}{2}e^{-\frac{\beta}{\lambda}KL}+\frac{1}{2}$$

and

$$P_{s_\lambda}(s_1)+P_{s_\lambda}(s_2)e^{-\frac{\beta}{\lambda}KL} < \frac{1}{2}e^{-\frac{\beta}{\lambda}KL}+\frac{1}{2}.$$

Hence, for any signal realization, $\Pr\left(x_1^2|x_0^2\right)$ receives a lower posterior weight relative to the incomplete information case without prior biases and $\Pr\left(x_1^2|x_0^1\right)$ - higher one. Hence, the distortion in prior beliefs about x_0 induces a distortion in the unconditional signal probabilities and the two counteract each other when posterior beliefs are formed.

4.5 Implications For Asset Pricing

Note that for any asset A with pay-off $\tau_A(x_1)$, the return is given by

$$\frac{\tau_A(x_1)}{P_A},$$

where P_A is the price of asset A. Thus, by assuming so far in Section 4 that the returns of the assets are predetermined, we have assumed that prices are fixed and have analyzed how the demand for the risky and safe assets change as we introduce biases and incomplete information. In equilibrium, however, the demand for either asset equals its exogenously specified supply (see (2.8) for details) and so prices adjust to reflect the preferences of the representative agent. To see how prices reflect the agent preferences in our example, recall by (2.2) that the price of asset A in fully observable state x_0 is given by

$$P_A = E_{P_{x_1|x_0}}\left[\beta\left(\frac{c(x_1)}{c_0}\right)^{-\gamma}\tau_A(x_1)\right].$$

When we replace the general CRRA utility function and $P_{x_1|x_0}$ by log-utility and $P_{x_1|s_\lambda}$, respectively, in the derivations in Section 2.2.2, the asset pricing equation becomes

$$P_A = E_{P_{x_1|s_\lambda}}\left[\beta g(x_1)^{-1}\tau_A(x_1)\right],$$

where $g(x_1) \equiv \frac{c(x_1)}{c_0}$ denotes the aggregate consumption growth exogenously determined in equilibrium, since market clearing in the endowment economy requires that $c(x_t) = \omega(x_t) \quad \forall x_t$, $t = 1, 2$. Hence, in our two-state example,

$$P_A = \beta g(x_1^1)^{-1}\tau_A(x_1^1)P_{x_1|s_\lambda}\left(x_1^1|s_i\right) + \beta g(x_1^2)^{-1}\tau_A(x_1^2)P_{x_1|s_\lambda}\left(x_1^2|s_i\right)$$

and so the price of the safe asset A_s is given by

$$P_{A_s} = \beta g(x_1^1)^{-1}P_{x_1|s_\lambda}\left(x_1^1|s_i\right) + \beta g(x_1^2)^{-1}P_{x_1|s_\lambda}\left(x_1^2|s_i\right).$$

Recall that in our example, state 2 denotes the good state ($g(x_1^1)^{-1} > g(x_1^2)^{-1}$) and that the demand for the safe asset decreases whenever the probability of the good state increases. Note that P_{A_s} above is a convex combination of $\beta g(x_1^1)^{-1}$ and $\beta g(x_1^2)^{-1}$ and so whenever the probability of the good state increases, $\beta g(x_1^2)^{-1}$ receives a higher weight and so P_{A_s} drops. Hence, P_{A_s} depends on the probability assigned to the good state in the same way the demand for A_s does. Analyzing the price of the risky asset in a similar way shows that whenever the probability of the good state increases, the price differential between the risky and the safe asset widens. Since we have discussed the implications of prior biases and incomplete information in terms of the effect they have on the probability assigned to the good state, the discussion in previous sections readily applies to the prices.

5 Directions For Future Research

The long-term goal of the research undertaken in this paper is to create a realistic model which builds on the merits of its predecessors and facilitates a joint analysis of the effects of behavioral biases and incompleteness of information on asset prices. If such a generalization is accomplished without model tractability being sacrificed, the framework should ultimately be able to provide better insights into the financial phenomena by providing a more holistic description of human decision-making. Previous sections laid out simple first steps towards that goal. Next, I discuss the generalizations and modifications of the model that could push it closer towards the aim.

5.1 Agent Preferences

We saw in Section 3.2 that in the case of the log utility, the weights using which the agent forms posterior beliefs upon observing a signal realization i is given by

$$\frac{e^{V\left(i,x_0^l\right)/\lambda}}{E_{P_{s_\lambda}}\left[e^{V\left(j,x_0^l\right)/\lambda}\right]} = \frac{e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_i(.))}}{E_s\left[e^{-\frac{\beta}{\lambda}KL(P_l(.)||P_h(.))}\right]}.$$

In general, however, the weights depend on preferences since

$$V\left(i,x_0^l\right) = u\left(c_{i,0}\right) + \beta E_{P_{x_1|x_0^l}}\left[u\left(c_i(x_1)\right)\right].$$

This is an interesting feature, since it implies that the information strategy and the posterior beliefs of the agent depends on his preferences and so the signals as well as reactions to them reflect the attitudes of the agent. In this light, deriving the weights for general CRRA utility would be a fruitful next step, since it has two major advantages over log-utility. Firstly, log-utility is a special case of the general CRRA utility $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ corresponding to $\gamma = 1$. Deriving the weights using $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ is thus expected to illustrate how signal probabilities and posterior weights depend on the risk aversion parameter γ. For instance, a risk averse agent might optimize signals in a way that make them relatively more precise at correctly identifying bad states of the world.

Secondly, Shefrin (2008) shows that a behavioral framework of choice based on Securities, Potential and Aspirations (SP/A) theory of Lopes (1987) can be posed in terms of CRRA expected utility maximization with modifications to agent endowments and beliefs. Therefore, the CRRA-based framework might also facilitate the analysis of how individual dispositions affect information strategy and portfolio selection in the context of incomplete information.

5.2 Belief Formation Using Representativeness Heuristic

Section 4.4 illustrates how prior biases and incomplete information can be jointly analyzed by considering two simple cases, while Section 4.5 illustrates the effects on asset prices. While the framework should facilitate analysis of various biases, it seems particularly well-suited for incorporating the representativeness heuristic, a behavioral tendency professional investors seem to be exhibiting (see e.g. Shefrin (2008) and Bordalo et al. (2019)). Formally, the representativeness heuristic is a distortion of Bayesian updating.[4] More specifically, for any events A and B, the posterior of A conditional on B is given by

$$\Pr(A|B) = \frac{\Pr(B|A)}{\Pr(B)}\Pr(A),$$

[4]The discussion that follows builds on the basic ideas in Shefrin (2008) and Bordalo et al. (2019) about modeling representativeness heuristic.

where $\frac{\Pr(B|A)}{\Pr(B)}$ denotes representativeness, since it indicates the degree by which event A is representative of event B, or how much the occurrence of event A makes event B more likely relative to the unconditional probability of the latter. We say that the agent exhibits the representativeness heuristic if he judges relative likelihood of events A_1 and A_2 conditional on event B by comparing how representative they are of event B, or more generally, by discounting the differences between base probabilities $\Pr(A_1)$ and $\Pr(A_2)$ when comparing $\Pr(A_1|B)$ and $\Pr(A_2|B)$.

The framework proposed in this paper is particularly well suited for analyzing representativeness heuristic, since the agent reweighs his beliefs upon observing a signal s_i using representativeness, i.e.

$$P_{x_1|s_\lambda}(x_1|s_i) = \sum_{x_0} \Pr(x_1|x_0) \Pr(x_0) \frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]},$$

where $\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}$ is the representativeness of s_i by x_0, since

$$\Pr(x_0|s_i) = \Pr(x_0) \frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}.$$

The representativeness heuristic can then be modeled using posterior beliefs in which $\Pr(x_0)$ receives less weight relative to $\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}$. For instance, in the extreme case, the representativeness-biased agent can be modeled as reweighing $\Pr(x_1|x_0)$ using $\frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]}$ alone so that

$$P_{x_1|s_\lambda}(x_1|s_i) = C \sum_{x_0} \Pr(x_1|x_0) \frac{e^{V(i,x_0)/\lambda}}{E_{P_s}\left[e^{V(j,x_0)/\lambda}\right]},$$

where C is a positive constant that ensures posterior probabilities sum up to 1. Hence, replacing the posterior beliefs of the agent using the representativeness-induced analogue should provide insights about how representativeness heuristic might affect investment decisions in the presence of incomplete information.

5.3 Heterogeneity and More Than Two Periods

So far we have assumed that the economy is inhabited by a continuum of identical investors and so the discussion about the behavior of the representative agent was implicitly assumed to apply to the aggregate economy (or the whole market). Clearly, a continuum of identical agents is a simplifying assumption that does not hold in reality. This, however, does not make the representative agent model useless, since the latter constitutes a building block of more realistic heterogeneous agents models in which the behavior of individual investor is modeled analogously to that of the representative investor (except agents now potentially differ in their beliefs, preferences, endowments and the cost of information). The natural first step towards introducing heterogeneity in this framework would be analyzing the economy inhabited by two types of agents who differ, among other things, in their cost of information. It would be particularly interesting to analyze how market beliefs are formed in this case and what happens when the realizations of their private signals do not agree.

Like the homogeneity assumption, the two-period time frame of choice is too simplistic, since most investment decisions have a larger horizon and are not governed by the simple motive of allocating consumption between today and the next period only. In this light, it would be insightful to analyze whether the incompleteness of information induces the agent (relative to the full rationality case) to have a different attitude about distant future and thus select a different portfolio of assets.

6 Summary

Behavioral and bounded rationality approaches to asset pricing attempt to account for essential features of human decision-making absent in standard neoclassical models, which assume full investor rationality. The bounded rationality approach studies the direct effects of cognitive limitations and costly information acquisition without modeling biases of judgement explicitly, while the behavioral school of thought directly models the biases of judgement often unrelated to the limited information or mental capacity. Even though the models of bounded rationality can explain some behavioral biases, the two frameworks in general have different assumptions and implications, yet those assumptions are only complementary and together provide a more holistic axiom underlying investor behavior: investors have limited mental capacity that prevent them from always correctly identifying the probabilities of future states *and* they potentially exhibit prior biases in beliefs and behavioral tendencies in preferences.

I incorporate behavioral and bounded rationality elements into a single framework by studying an extension of the standard two-period consumption-based portfolio selection problem in which a representative agent with CRRA preferences has potentially biased priors, does not observe the current state and is not certain about future state probabilities. He selects signals in the rational inattention discrete choice framework of Matějka and McKay (2015) in order to learn about the current state and upon observing a signal realization, forms posterior beliefs about the likelihood of future states. The generated posterior is a convex combination of the future state distributions induced by different time-zero states, where the weights reflect the subjective posterior likelihood of time-zero states. In the case of log-utility, the posterior beliefs are induced by parsimonious reweighing of priors, where the weights depend on λ, discount factor β and the relative entropies of the conditional future state distributions.

Moreover, the optimal portfolio induced by the log-utility is a convex combination of the N portfolios the investor would have selected in each of the N states if they were fully observable, where the weights coincide with those in the posterior.

The precision of the posterior beliefs in general depend on the priors and the cost of information λ. When λ is very high, posterior beliefs are almost identical to priors, while as λ gets arbitrarily small and when the agent has correct priors, the posterior beliefs converge in probability to the true distribution. The implications for behavior depend on the underlying assumptions about the nature of investor beliefs. Using a two-state example with log-utility, I demonstrate that when the agent is assumed to have correct priors, incomplete information and the noisiness of signals induces him to be more risk-averse relative to the full rationality case in the good state and more risk-seeking in the bad one. I also show that when the investor is overconfident, the effects of incomplete information are mitigated when the signals are correct and exacerbated when the signals are incorrect.

The proposed framework is promising since it facilitates a holistic modeling of human decision-making and is a direct generalization of standard neoclassical and behavioral approaches, which are special cases of the general model. Moreover, its usefulness is not limited to asset pricing context, since the model is originally formulated in terms of a consumption allocation problem. The extended framework also seems particularly well-suited for incorporating the representativeness heuristic and CRRA-based behavioral preferences into analysis, and could lay the foundation for multi-period heterogeneous agents model that outperform existing ones by allowing us to analyze the effects of costs of information and behavioral biases into a single framework.

7 References

Batchuluun, Altantsetseg, Yulei Luo, and Eric R Young (2019). "Portfolio Choice with Information-Processing Limits". In: *Annals of Economics and Finance* 20 (1), pp. 137–162.

Bordalo, Pedro, Nicola Gennaioli, Rafael La Porta, and Andrei Shleifer (2019). "Diagnostic Expectations and Stock Returns". In: *Journal of Finance* 74 (6), pp. 2839–2874.

Frydman, Cary, Samuel M. Hartzmark, and David H. Solomon (2018). "Rolling Mental Accounts". In: *The Review of Financial Studies* 31 (1), pp. 362–397.

Gabaix, Xavier (2014). "A Sparsity-Based Model of Bounded Rationality". In: *The Quarterly Journal of Economics* 4, pp. 1661–1710.

Kahneman, Daniel and Amos Tversky (1992). "Advances in Prospect Theory: Cumulative Representation of Uncertainty". In: *Journal of Risk and Uncertainty* 5 (4), pp. 297–323.

Lopes, Lola L. (1987). "Between Hope and Fear: The Psychology of Risk". In: *Advances in Experimental Social Psychology* 20 (C), pp. 255–295.

Luo, Yulei (2006). "Rational Inattention, Portfolio Choice, and the Equity Premium". Job Market Paper.

Mackowiak, Bartosz, Filip Matějka, and Mirko Wiederholt (2020). "Rational Inattention: A Review". CEPR Discussion Paper 15408.

Matějka, Filip and Alisdair McKay (2015). "Rational Inattention to Discrete Choices: A New Foundation for the Multinomial Logit Model". In: *American Economic Review* 105 (1), pp. 272–298.

Mehra, Rajnish and Edward C. Prescott (1985). "The Equity Premium: A Puzzle". In: *Journal of Monetary Economics* 15 (2), pp. 145–161.

Miao, Jianjun and Hao Xing (2020). "Dynamic Discrete Choice under Rational Inattention". Working Paper.

Shefrin, Hersh (2008). *Behavioral Approach To Asset Pricing.* Academic Press.

Sims, Christopher A (2003). "Implications of Rational Inattention". In: *Journal of Monetary Economics* 50, pp. 665–690.